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A CLOSED FORM ANALYTIC SOLUTION FOR COUPLED NONUNIFORM TRANSMISSION LINES

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Abstract—A closed form analytic solution is introduced for arbitrary Coupled Nonuniform Transmission Lines (CNTLs). First, the differential equations of CNTLs are written as a suitable matrix differential equation. Then, the matrix differential equation is solved to obtain the chain parameter matrix of CNTLs. Afterward, the voltage and current of lines are obtained at any point using the chain parameter matrix. The validation of the introduced solution is studied, finally.

1. INTRODUCTION

Coupled Nonuniform Transmission Lines (CNTLs) are widely used in microwave circuits [1, 2]. The differential equations describing CNTLs have non-constant matrices, so except for a few special cases no analytical solution exists for them. Several methods have been introduced to analyze arbitrary CNTLs [3–11]. These methods can be classified to two groups. In the methods of the first group, all of the elements of the per-unit-length matrices of CNTLs must be vary similarly [3–6], in contrary with the methods of the second group [7–11]. There are some methods to analyze CNTLs such as cascading many short sections [7], finite difference [8], Taylor's series expansion [9], Fourier series expansion [10] and the equivalent sources method [11]. These methods are numerical and do not yield a closed form analytic solution. Closed form analytic solution has been yielded only for a few special types of CNTLs such as exponential one [9–Appendix]. In this paper, a closed form analytic solution is introduced for arbitrary CNTLs. First, the differential equations of CNTLs are written as a suitable matrix differential equation. Then the matrix differential equation is solved to obtain the chain parameter matrix of CNTLs.

Finally, the validation of the introduced solution is studied using two comprehensive examples.

2. THE EQUATIONS OF CNTLS

The equations related to CNTLs are reviewed in this section. It is assumed that the principal propagation mode of the lines is TEM or quasi-TEM. This assumption is valid when the widths in the cross section are small enough compared to the wavelength. Figure 1 shows typical CNTLs consisting of M lines with length d and with arbitrary terminal loads $Z_{S,m}(\omega)$ and $Z_{L,m}(\omega)$, where $m = 1, 2, \ldots, M$. The differential equations describing lossy and dispersive CNTLs in the frequency domain are given by

$$\frac{d\mathbf{V}(z)}{dz} = -\mathbf{Z}(z)\mathbf{I}(z) \tag{1}$$

$$\frac{d\mathbf{I}(z)}{dz} = -\mathbf{Y}(z)\mathbf{V}(z) \tag{2}$$

in which $\mathbf{V}$ and $\mathbf{I}$ are $M \times 1$ voltage and current vectors, respectively. Also we have

$$\mathbf{Z}(z) = \mathbf{R}(z) + j\omega\mathbf{L}(z) \tag{3}$$

$$\mathbf{Y}(z) = \mathbf{G}(z) + j\omega\mathbf{C}(z) \tag{4}$$

In (3)–(4), $\mathbf{R}$, $\mathbf{L}$, $\mathbf{G}$ and $\mathbf{C}$ are the per-unit-length matrices, whose dimensions are $M \times M$. It is known that the voltage and current of any arbitrary point z can be related to the voltage and current of the

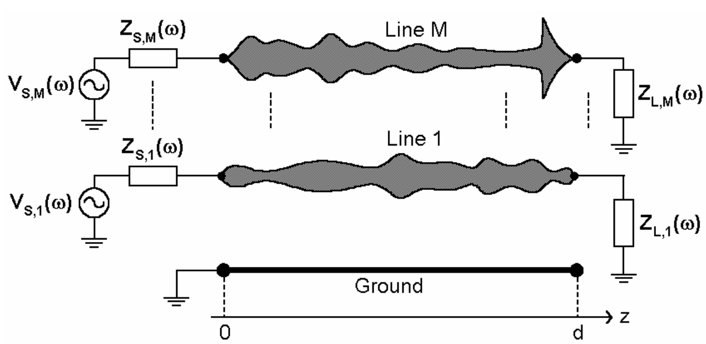

Figure 1. A typical CNTL consisting of M lines of length d and terminated by arbitrary loads.

point $z = 0$ by a chain parameter matrix as follows

$$\begin{bmatrix} \mathbf{V}(z) \\ \mathbf{I}(z) \end{bmatrix} = \mathbf{\Phi}(z) \begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix} \tag{5}$$

3. ANALYTIC SOLUTION OF CNTLS

In this section an approach is proposed to solve the differential equations of CNTLs, analytically. The set of differential Equations (1)–(2), can be written as the following matrix differential equation

$$\frac{d}{dz}\mathbf{X}(z) = -\mathbf{A}(z)\mathbf{X}(z) \tag{6}$$

where

$$\mathbf{X}(z) = \begin{bmatrix} \mathbf{V}(z) & \mathbf{V}(z) \\ \mathbf{I}(z) & \mathbf{I}(z) \end{bmatrix} \tag{7}$$

and

$$\mathbf{A}(z) = \begin{bmatrix} \mathbf{0} & \mathbf{Z}(z) \\ \mathbf{Y}(z) & \mathbf{0} \end{bmatrix} \tag{8}$$

From the matrix algebra and using (6), we can write the following

$$d\mathbf{X}(z)\mathbf{X}^{-1}(z) \cong d\left(\ln(\mathbf{X}(z)\right) = -\mathbf{A}(z)dz \tag{9}$$

Integrating (9) gives us the following

$$\ln(\mathbf{X}(z)) - \ln(\mathbf{X}(0)) \cong \ln\left(\mathbf{X}(z)\mathbf{X}^{-1}(0)\right) = -\int_0^z \mathbf{A}(z')dz' \tag{10}$$

Therefore, one can determine the following solution for the unknown matrix in (6)

$$\mathbf{X}(z) = \exp\left(-\int_0^z \mathbf{A}(z')dz'\right)\mathbf{X}(0) = \mathbf{\Phi}(z)\mathbf{X}(0) \tag{11}$$

The approximations used in (9) and (10) are dependent to the matrix $\mathbf{A}(z)$ and its integral. From (5), (7) and (11), the following closed form analytic solution is obtained as the chain parameter matrix of CNTLs.

$$\mathbf{\Phi}(z) = \exp\left(-\int_0^z \mathbf{A}(z')dz'\right) = \exp\left(-\begin{bmatrix} \mathbf{0} & \int_0^z \mathbf{Z}(z')dz' \\ \int_0^z \mathbf{Y}(z')dz' & \mathbf{0} \end{bmatrix}\right) \tag{12}$$

The voltage and current at any point z can be obtained after finding those at two terminals using the chain parameter matrix of the point $z = d$ and the terminal conditions.

The solution (12) is a generalized formula of the known solution for uniform coupled transmission lines [7], indeed. Moreover, the proposed approach to solve the matrix differential Equation (6) is inspired of the approach that is usually used to solve the ordinary scalar differential equations, indeed.

4. EXAMPLES AND RESULTS

In this section, the introduced closed form solution is validated using two comprehensive examples. Firstly, consider a lossless microstrip CNTL with $M = 2$ strips, whose substrate is the air. This homogeneous structure has the following per-unit-length matrices.

$$\mathbf{L}(z) = \mathbf{L}_0 \exp(kz/d) \tag{13}$$
$$\mathbf{C}(z) = \mathbf{C}_0 \exp(kz/d) \tag{14}$$
$$\mathbf{R}(z) = \mathbf{G}(z) = 0 \tag{15}$$

Now, assume that $k = 2$, $d = 10\,\text{cm}$, $f = 1.0\,\text{GHz}$, $Z_{S,1} = Z_{S,2} = 50\,\Omega$, $Z_{L,1} = Z_{L,2} = 100\,\Omega$, $V_{S,1} = 1\,\text{V}$, $V_{S,2} = 0$ and also

$$\mathbf{C}_0 = \begin{bmatrix} 65.7 & -7.15 \\ -7.15 & 65.7 \end{bmatrix} \text{pF/m} \tag{16}$$
$$\mathbf{L}_0 = \frac{1}{c^2}\mathbf{C}_0^{-1} = \begin{bmatrix} 171.1 & 18.62 \\ 18.62 & 171.1 \end{bmatrix} \text{nH/m} \tag{17}$$

in which c is the velocity of the light. The matrix $\mathbf{A}(z)$ of this CNTL will be as follows

$$\mathbf{A}(z) = \begin{bmatrix} \mathbf{0} & j\omega\mathbf{L}_0 \exp(kz/d) \\ j\omega\mathbf{C}_0 \exp(kz/d) & \mathbf{0} \end{bmatrix} \tag{18}$$

Therefore, the chain parameter matrix of this CNTL will be given by

$$\mathbf{\Phi}(z) = \exp\left(-j\omega\frac{d}{k}\begin{bmatrix} \mathbf{0} & \mathbf{L}_0\left(\exp(kz/d) - 1\right) \\ \mathbf{C}_0\left(\exp(kz/d) - 1\right) & \mathbf{0} \end{bmatrix}\right) \tag{19}$$

Figures 2–3 compare the amplitude and phase of the lines voltages, obtained from the introduced solution with the exact one [7]. One sees that the introduced solution has perfectly given exact results.

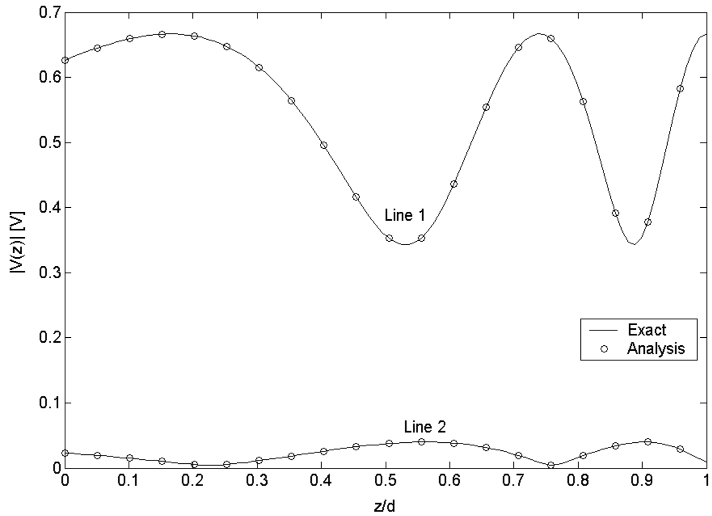

Figure 2. The amplitude of the voltage for CNTL having the same variations $k = 2$ for $\mathbf{L}$ and $\mathbf{C}$ at $f = 1\,\mathrm{GHz}$.

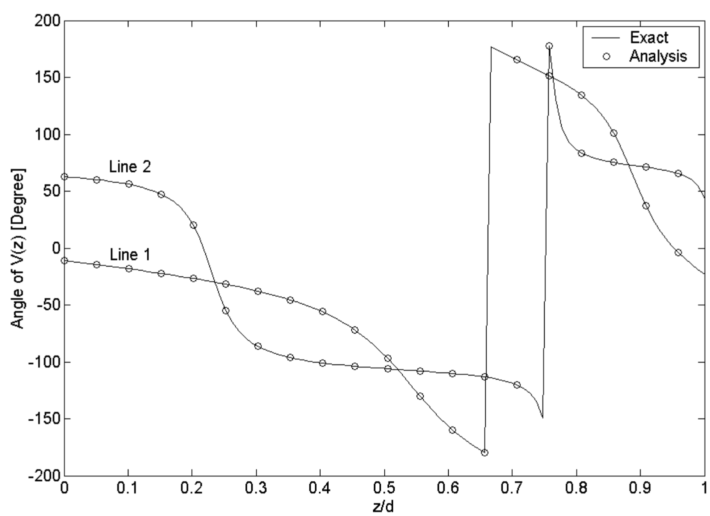

Figure 3. The phase of the voltage for CNTL having the same variations $k = 2$ for $\mathbf{L}$ and $\mathbf{C}$ at $f = 1\,\mathrm{GHz}$.

Secondly, consider the above CNTL but its capacitance matrix is considered as follows instead of (14)

$$\mathbf{C}(z) = \mathbf{C}_0 \exp(-kz/d) \tag{20}$$

Now, the matrix $\mathbf{A}(z)$ of this CNTL will be as follows

$$\mathbf{A}(z) = \begin{bmatrix} \mathbf{0} & j\omega\mathbf{L}_0 \exp(kz/d) \\ j\omega\mathbf{C}_0 \exp(-kz/d) & \mathbf{0} \end{bmatrix} \tag{21}$$

Therefore, the chain parameter matrix of this CNTL will be given by

$$\mathbf{\Phi}(z) = \exp\left(-j\omega\frac{d}{k}\begin{bmatrix} \mathbf{0} & \mathbf{L}_0(\exp(kz/d) - 1) \\ \mathbf{C}_0\,(1 - \exp(-kz/d)) & \mathbf{0} \end{bmatrix}\right) \tag{22}$$

Figures 4–5 compare the amplitude and phase of the lines voltages, considering $k = 0.1$, obtained from the introduced solution with the exact one [9–Appendix]. One sees that the introduced solution has not perfectly given exact results but it has a good agreement with exact solution. Also, Figure 6 illustrates the relative error corresponding to the lines voltages versus frequency, considering $k = 0.1$ and 0.2. It is seen that as k decreases, the accuracy of the solutions increases. Also, the better accuracy is existed at very low frequencies. It is worth to

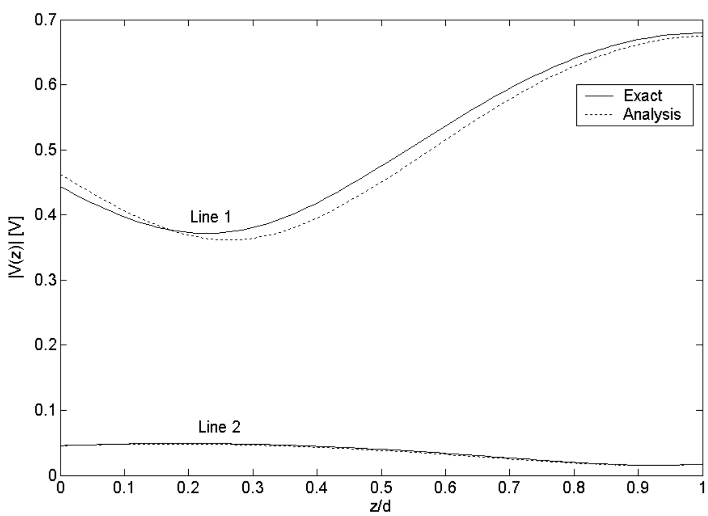

Figure 4. The amplitude of the voltage for CNTL having different variations $k = \pm 0.1$ at $f = 1\,\text{GHz}$.

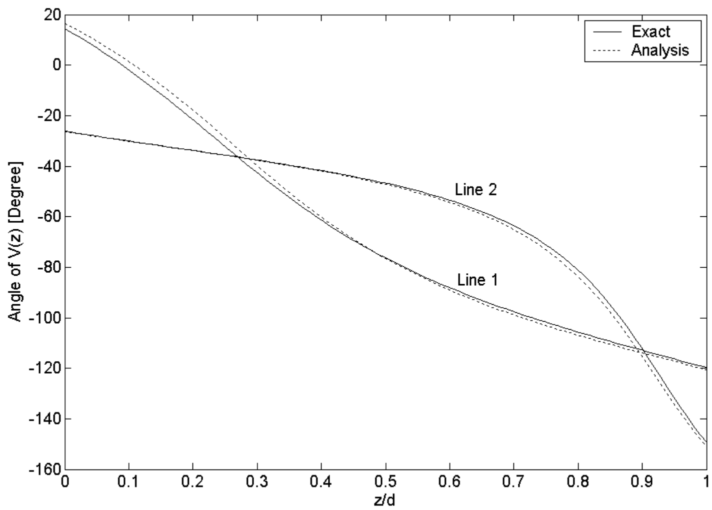

Figure 5. The phase of the voltage for CNTL having different variations $k = \pm 0.1$ at $f = 1\,\text{GHz}$.

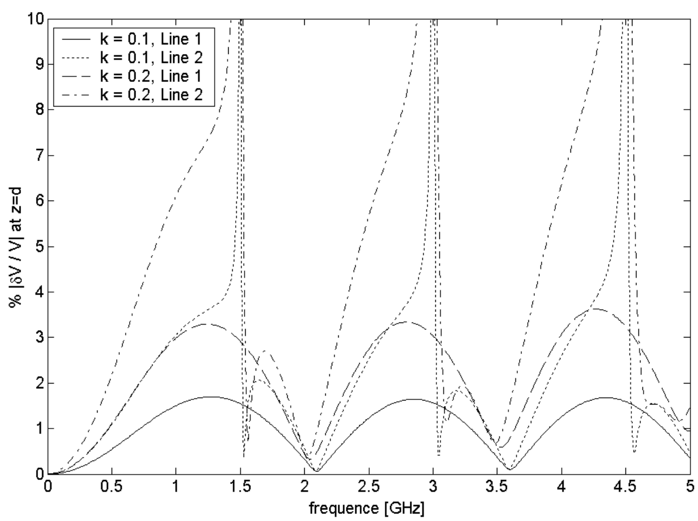

Figure 6. The relative error corresponding to the far-end voltages with respect to frequency.

mention that the determinant of the chain parameter matrices obtained in the above examples were exactly equal to one at all frequencies.

According to the above examples, one may be conclude the following results:

1. The introduced solution gives exact results at all frequencies for arbitrary CNTLs, whose the per-unit-length matrices have the same variations.
2. The accuracy of the introduced solution is acceptable for arbitrary CNTLs especially at very low frequencies.
3. The accuracy of the introduced solution is increased as the variations of the per-unit-length matrices (k/d in the examples) are decreased.

From mathematics point of view and according to the results 2 and 3 and also paying attention to the matrices existed in the exponential functions of (19) and (22), we can conclude that the accuracy of the introduced solution is increased in the case of the anti-diagonal elements of the integral of the matrix $\mathbf{A}(z)$ tend to zero (as in low frequencies) or have the same variations.

5. CONCLUSION

A closed form analytic solution was introduced for arbitrary Coupled Nonuniform Transmission Lines (CNTLs). First, the differential equations of CNTLs are written as a suitable matrix differential equation. Then the matrix differential equation is solved to obtain the chain parameter matrix of CNTLs. The validation of the introduced solution was studied using a comprehensive example. It was seen that the introduced solution gives exact results at all frequencies for CNTLs, whose the per-unit-length matrices have the same variations. Also, the accuracy of the introduced solution is acceptable for arbitrary CNTLs especially at very low frequencies. Moreover, the accuracy of the introduced solution is increased as the variations of the per-unit-length matrices are decreased.

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