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A UNIQUENESS THEOREM FOR INITIAL-BOUNDARY VALUE PROBLEMS IN TELLEGEN MEDIUM

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Abstract—Uniqueness of the initial-boundary value problems for a Tellegen media is studied. Initial-boundary value problems for a Tellegen Medium are summarized for the sake completeness; the conservation of energy for such a medium is reviewed. Sufficient conditions are given for the positive definiteness of the energy stored in a Tellegen media due to the fields. It is shown that the if the energy stored in a Tellegen medium is positive definite, then an initial-boundary value problem for a Tellegen media has only one solution, if it exists.

1. INTRODUCTION

Lightening, one of the most common form electricity existed on earth even before any kind of life form appeared, and has been one the major factor affecting almost every aspect in nature. Starting with fear, followed by spiritual approaches, human kind spent significant effort to understand it. Observations and experimentations with the electrostatic properties of "amber" and the magnetic properties of "magnetite" started in ancient times. Although the similarity between electrostatic forces, magnetic forces, and gravitation was well understood by 18th century, the relation between electric and magnetic fields was fully explored later.

It was a revolutionary step in science when James Clerk Maxwell presented nine equations summarizing all known laws on electricity and magnetism in his famous memoir [1] in 1864. The most important feature of this memoir was the introduction of a postulate needed for an additional term to make the set of electromagnetic equations self-consistent. After Maxwell, the scientists have spent significant effort to find the best way to present Maxwell's theory. Among other forms, we will follow the "point form" of the Maxwell's equations in this paper.

Maxwell's equations lead to many directions for research to understand nature; some of them actively continue today. Bi-Isotropic (in short BI-) media is one of the most important directions for generalization of the electromagnetic theory [2] in order to understand the phenomena like optical activity (rotation of the polarization plane of polarized light), electromagnetic activity (rotation of the polarization plane of polarized electromagnetic waves at large wave lengths), magneto-electric activity (magnetic field created by an electric field). Various forms of constitutive relations for BI material are proposed, and this field is an active research area today.

After Lindman [4] demonstrated electromagnetic activity, Tellegen suggested various ways to generate macroscopical magneto-electric effect. He created a new passive circuit element which he called *gyrator* for this purpose [5] and suggested a constitutive relation to explain the magneto-electric effect in materials. Since then, Tellegen media has attracted attention of the scientists (see, for example [6] and [7]).

It is difficult to trace back the origin of the BI-media because many researchers have contributed to the subject and this subject has been known for more than two hundred years. The multi-disciplinary nature of the subject also makes difficult to indicate one or few sources for the origin of the BI-media. An extensive account for the development of the BI-media Lindell et al. [3] can be recommended.

The constitutive equations for BI-media is given by

$$\vec{D}(\vec{r},\, t) = \varepsilon \vec{E}(\vec{r},\, t) + \xi \vec{H}(\vec{r},\, t) - \kappa \frac{\partial \vec{H}(\vec{r},\, t)}{\partial t} \tag{1}$$

$$\vec{B}(\vec{r},\, t) = \mu \vec{H}(\vec{r},\, t) + \xi \vec{E}(\vec{r},\, t) + \kappa \frac{\partial \vec{E}(\vec{r},\, t)}{\partial t} \tag{2}$$

As it will be stated later, t and $\vec{r}$ represent the time and the position vector. $\vec{E}(\vec{r},\, t)$, $\vec{D}(\vec{r},\, t)$, $\vec{B}(\vec{r},\, t)$, $\vec{H}(\vec{r},\, t)$ are used to denote the *electric field*, the *electric flux density*, the *magnetic flux density*, and the *magnetic field*, respectively. ε is the permittivity, μ is the permeability, as. The constant κ is called the "*chirality parameter*", and ξ describes the "*magnetoelectric effect*" in the medium (we believe that Tellegen parameter would be an appropriate name for this material constant). The constitutive equations for BI-medium and various types of BI-media are introduced in [3]. In this paper we will consider the uniqueness of the solution of the initial-boundary value problems for a *Tellegen Medium* for which $\kappa = 0$.

BI media (Eqs. (1) and (2)) has much potential to explore for a better understanding of the electromagnetic materials. That's why; BI media is an active research field. We would like to give a brief review on the current research activities in BI media. Nair and Choudhury [8] introduced a rigorous analytical approach to investigate patterns of a field propagating in a simple bi-isotropic fiber with circular cross-section. Panin et al. [9] investigated wave polarization conversion that occurred when an electromagnetic wave with elliptical polarization is incident upon a layered structure comprising a strip grating, magneto dielectric layer, and chiral layer. Hussain et al. [10] studied fractional curl operator to study the wave propagation through a fractional parallel plate filled with homogeneous, isotropic chiral material. Fractional dual solutions have been established and several applications of these solutions were introduced. Khatir et al. [11] gave the exact solution for the scattering of transverse magnetic (TM) wave by an elliptic bi-isotropic cylinder. Lindell and Sihvola [12], using differential-form representation of the Maxwell equations, interpreted the classical Gibbsian vectors as a bi-isotropic medium with certain values of the constitutive parameters. It was also shown that this medium is a generalization of both PEC (perfect electric conductor) and PMC (perfect magnetic conductor) media, with similar properties. Plane-wave reflection from a planar interface of air and perfect electromagnetic medium was given as an application. Sheen [13] developed the scalar wave equations of auxiliary vector potentials for time harmonic electric and magnetic fields on transverse electric, transverse magnetic and transverse electromagnetic modes in an biaxial anisotropic medium. Zheng et al. [14] developed conformal FDTD approach to analyze scattering from conducting bodies coated with bi-isotropic materials. Zhang et al. [15] studied electromagnetic scattering from arbitrary three-dimensional bi-anisotropic objects using hybrid finite element-boundary integral method where analytical expressions of finite element matrices for tetrahedrons were derived with the aid of Schur product of matrices. Lindell and Wallén [16] presented the constitutive equation for the bi-anisotropic medium as a relation between the two electromagnetic two-forms through a medium dyadic of the Hodge type using concepts of Gibbsian vector analysis. As a continuation of their previous work, Lindell and Wallén [17] discussed the solution of Green's dyadic in analytical form for a specific class of bi-isotropic media. Li et al. [18] presented an eigenfunction expansion of the electric-type dyadic Green's function (DGF) for unbounded gyrotropic bi-anisotropic media in terms of cylindrical vector wave functions. Zhang and Zheng [19] established an algebraic eigenvalue problem for analyzing the propagation characteristics of electromagnetic waves inside the PBG Structure consisting of complex medium by using Bloch theorem and plane wave expansion. Naqvi et al. [20] studied the dual solutions of Maxwell equations with the fractional calculus in bi-isotropic using the field decomposition approach. Wave propagation with both negative and positive phase velocity propagation was considered.

It is an important task to investigate if the initial-boundary value problems for Tellegen medium are well-posed. It is well-known that an initial boundary-value problem defined by a mathematical model for a class of physical problems is said to be well-posed if there exists a solution to a initial boundary-value problem, if the initial boundary-value problem has only one solution, and the solution is stable (i.e., is continuously dependent on the data). The objective of this paper is to give a uniqueness theorem for the solution of an initial-boundary value problem for Tellegen medium.

For the sake of completeness, the following section is devoted to the definition of initial-boundary value problems in Tellegen media. In the subsequent section, the conditions under which the energy stored due to fields is positive definite is give, and it is followed by a uniqueness theorem for the initial-boundary value problem. A summary of the results and a conclusion are given in the last section.

2. INITIAL-BOUNDARY VALUE PROBLEMS IN TELLEGEN MEDIA

Let V be a bounded region in a 3D Euclidean space with a boundary S. The boundary is assumed to be smooth enough such that it is possible to define a unique normal at every point on the surface. It is also assumed that the region V is occupied by a linear electromagnetic Tellegen material.

An initial-boundary value problem defined for a linear, homogeneous, Tellegen medium is to find a set of vector fields (the *electromagnetic field*)

$$\left\{\vec{E}(\vec{r},\,t),\,\vec{D}(\vec{r},\,t),\,\vec{B}(\vec{r},\,t),\,\vec{H}(\vec{r},\,t)\right\} \quad \text{for } t \geq 0 \text{ and } \vec{r} \in V \tag{3}$$

where t and $\vec{r}$ represent the time and the position vector, respectively, and

$$\begin{aligned}
&\vec{E}(\vec{r},\,t) : \text{the } \textit{electric field}\\
&\vec{D}(\vec{r},\,t) : \text{the } \textit{electric flux density}\\
&\vec{B}(\vec{r},\,t) : \text{the } \textit{magnetic flux density}\\
&\vec{H}(\vec{r},\,t) : \text{the } \textit{magnetic field}
\end{aligned} \tag{4}$$

that satisfy the *Maxwell-Minkowski equations*

the *Faraday's law* :

$$\nabla\times\vec{E}(\vec{r},\, t)=-\frac{\partial}{\partial t}\vec{B}(\vec{r},\, t)\quad t\geq 0 \text{ and } \vec{r}\in V \tag{5}$$

the *Ampere's law* :

$$\nabla\times\vec{H}(\vec{r},\, t)=\vec{J}(\vec{r},\, t)+\frac{\partial}{\partial t}\vec{D}(\vec{r},\, t)\quad t\geq 0 \text{ and } \vec{r}\in V \tag{6}$$

the *Gauss'slaw* :

$$\nabla\cdot\vec{D}(\vec{r},\, t)=\rho(\vec{r},\, t)\quad t\geq 0 \text{ and } \vec{r}\in V \tag{7}$$

the *magnetic Gauss law* :

$$\nabla\cdot\vec{B}(\vec{r},\, t)=0\quad t\geq 0 \text{ and } \vec{r}\in V \tag{8}$$

and

$$\text{the } \textit{continuity equation}: \nabla\cdot\vec{J}(\vec{r},\, t) = -\frac{\partial}{\partial t}\rho(\vec{r},\, t) \qquad t\geq 0 \text{ and } \vec{r}\in V \tag{9}$$

where

$$\vec{J}(\vec{r},\, t) \text{ is the } \textit{current density}, \text{ and} \tag{10}$$

$$\rho(\vec{r},\, t) \text{ is the } \textit{charge density}, \tag{11}$$

are the source field, and, the boundary conditions

$$\vec{n}\times\vec{E}(\vec{r},\, t)=\vec{E}^{*}(\vec{r},\, t)\quad t\geq 0 \text{ and } \vec{r}\in S_E \tag{12}$$

and

$$\vec{n}\times\vec{H}(\vec{r},\, t)=\vec{H}^{*}(\vec{r},\, t)\quad t\geq 0 \text{ and } \vec{r}\in S_H \tag{13}$$

where S_E and S_H are the complimentary parts of the boundary S, i.e., and $\vec{E}^{*}(\vec{r},\, t)$ and $\vec{H}^{*}(\vec{r},\, t)$ are given functions, and the initial conditions

$$\vec{E}_o(\vec{r})=\vec{E}(\vec{r},\, 0),\quad \vec{H}_o(\vec{r})=\vec{H}(\vec{r},\, 0) \tag{14}$$

the constitutive equations for homogeneous and isotropic Tellegen materials

$$\vec{D}(\vec{r},\, t) = \varepsilon\vec{E}(\vec{r},\, t)+\xi\vec{H}(\vec{r},\, t) \tag{15}$$

$$\vec{B}(\vec{r},\, t) = \xi\vec{E}(\vec{r},\, t)+\mu\vec{H}(\vec{r},\, t) \tag{16}$$

for the given *source fields* (10), (11), initial conditions (14) and, the boundary values $\vec{E}^{*}(\vec{r},\, t)$ for $t\geq 0$ and $\forall\vec{r}\in S_E$ and $\vec{H}^{*}(\vec{r},\, t)$ for $t\geq 0$ and $\forall\vec{r}\in S_H$.

3. CONSERVATION OF ENERGY FOR TELLEGEN MEDIA

In this section, we will obtain the energy balance for a Tellegen medium and give a theorem for the positive definiteness of the energy stored in a Tellegen medium in a electromagnetic process.

One of the common ways to establish the energy balance for a electromagnetic process is to multiply the Ampere's law (6) by $\vec{E}$ and the Faraday's law (5) by $\vec{H}$ (both dot product) and subtract from each other

$$\overset{\frown}{\vec{E}} \cdot \left(\nabla \times \overset{\frown}{\vec{H}}\right) - \overset{\frown}{\vec{H}} \cdot \left(\nabla \times \overset{\frown}{\vec{E}}\right) = \overset{\frown}{\vec{J}} \cdot \overset{\frown}{\vec{E}} + \overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{B}}}{\partial t} + \overset{\frown}{\vec{E}} \cdot \frac{\partial \overset{\frown}{\vec{D}}}{\partial t} \tag{17}$$

Considering the vector identity

$$\overset{\frown}{\vec{E}} \cdot \left(\nabla \times \overset{\frown}{\vec{H}}\right) - \overset{\frown}{\vec{H}} \cdot \left(\nabla \times \overset{\frown}{\vec{E}}\right) = -\nabla \cdot \left(\overset{\frown}{\vec{E}} \times \overset{\frown}{\vec{H}}\right)$$

and the constitutive relations (15) and (16) for Tellegen media

$$\begin{aligned}
\overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{B}}}{\partial t} + \overset{\frown}{\vec{E}} \cdot \frac{\partial \overset{\frown}{\vec{D}}}{\partial t} \;=\;& \overset{\frown}{\vec{H}} \cdot \frac{\partial \left[\xi \vec{E}(\vec{r}, t) + \mu \vec{H}(\vec{r}, t)\right]}{\partial t} \\
& + \overset{\frown}{\vec{E}} \cdot \frac{\partial \left[\varepsilon \vec{E}(\vec{r}, t) + \xi \vec{H}(\vec{r}, t)\right]}{\partial t} \\
=\;& \mu \vec{H} \cdot \frac{\partial \vec{H}(\vec{r}, t)}{\partial t} + \varepsilon \vec{E} \cdot \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} \\
& + \xi \vec{H} \cdot \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} + \xi \vec{E} \cdot \frac{\partial \vec{H}(\vec{r}, t)}{\partial t} \\
=\;& \frac{1}{2} \frac{\partial}{\partial t} \left\{\mu \vec{H} \cdot \vec{H} + \varepsilon \vec{E} \cdot \vec{E} + 2\xi \vec{H} \cdot \vec{E}\right\}
\end{aligned}$$

we have from (17)

$$\vec{J} \cdot \vec{E} + \nabla \cdot \left(\vec{E} \times \vec{H}\right) + \frac{1}{2} \frac{\partial}{\partial t} \left\{\mu \vec{H} \cdot \vec{H} + \varepsilon \vec{E} \cdot \vec{E} + 2\xi \vec{H} \cdot \vec{E}\right\} = 0 \tag{18}$$

Integrating this equation over an arbitrary volume V with a boundary surface S of a Tellegen media, and taking into account the divergence theorem for the second term (Poynting vector), we arrive at

$$\int_V \vec{J} \cdot \vec{E} dv + \int_S \left(\vec{E} \times \vec{H}\right) da + \frac{1}{2} \frac{\partial}{\partial t} \int_V \left\{\mu \vec{H} \cdot \vec{H} + \varepsilon \vec{E} \cdot \vec{E} + 2\xi \vec{H} \cdot \vec{E}\right\} = 0 \tag{19}$$

As is well-known the first term represents the power lost from the field (to increase the kinetic energy of the charges, in other words the temperature of the media), the second term (the surface integral) is the Poynting vector representing the energy flux to the region through the surface, and the last term is the rate of the change in the energy stored in the volume occupied by a Tellegen medium. It is clear that the energy stored in a Tellegen medium has an extra term ($2\xi\vec{H}\cdot\vec{E}$) in addition to the usual terms.

The following theorem gives a sufficient condition for the positive definiteness of the energy stored in a Tellegen medium.

THEOREM: The energy stored in a Tellegen medium ($\mu\vec{H}\cdot\vec{H}+\varepsilon\vec{E}\cdot\vec{E}+2\xi\vec{E}\cdot\vec{H}$) is positive definite if $\mu>0$, $\varepsilon>0$, $\xi\leq\sqrt{\mu\varepsilon}$, i.e.,

$$\mu\vec{H}\cdot\vec{H}+\varepsilon\vec{E}\cdot\vec{E}+2\xi\vec{E}\cdot\vec{H}\geq 0 \text{ for all } \vec{H} \text{ and } \vec{E} \text{ for } t\geq 0 \text{ and } \vec{r}\in V. \tag{20}$$

if

$$\mu>0,\quad \varepsilon>0,\quad \xi\leq\sqrt{\mu\varepsilon} \tag{21}$$

PROOF:
Since,

$$\vec{E}\cdot\vec{H}=\left|\vec{E}\right|\left|\vec{H}\right|\alpha$$

where

$$-1\leq\alpha\leq 1$$

we have

$$\vec{E}\cdot\vec{H}\geq-\left|\vec{E}\right|\left|\vec{H}\right|$$

and therefore

$$\mu\vec{H}\cdot\vec{H}+\varepsilon\vec{E}\cdot\vec{E}+2\xi\vec{E}\cdot\vec{H}\geq\mu\left|\vec{H}\right|^2+\varepsilon\left|\vec{E}\right|^2-2\xi\left|\vec{E}\right|\left|\vec{H}\right| \tag{22}$$

On the other hand if the conditions (21) are satisfied, we have

$$\mu\vec{H}\cdot\vec{H}+\varepsilon\vec{E}\cdot\vec{E}-2\xi\vec{E}\cdot\vec{H}\geq\mu\left|\vec{H}\right|^2+\varepsilon\left|\vec{E}\right|^2-2\sqrt{\mu\varepsilon}\left|\vec{E}\right|\left|\vec{H}\right| \tag{23}$$

Since,

$$\mu\left|\vec{H}\right|^2+\varepsilon\left|\vec{E}\right|^2-2\sqrt{\mu\varepsilon}\left|\vec{E}\right|\left|\vec{H}\right|=\left(\sqrt{\mu}\vec{H}-\sqrt{\varepsilon}\vec{E}\right)\cdot\left(\sqrt{\mu}\vec{H}-\sqrt{\varepsilon}\vec{E}\right)\geq 0 \tag{24}$$

combining (22)–(24), we have

$$\mu\vec{H}\cdot\vec{H}+\varepsilon\vec{E}\cdot\vec{E}+2\xi\vec{E}\cdot\vec{H}\geq\left(\sqrt{\mu}\vec{H}-\sqrt{\varepsilon}\vec{E}\right)\cdot\left(\sqrt{\mu}\vec{H}-\sqrt{\varepsilon}\vec{E}\right)\geq 0 \tag{25}$$

for all $\overset{\frown}{\vec{H}}$ and $\overset{\frown}{\vec{E}}$ and for all $t \geq 0$ and $\vec{r} \in V$. Q.E.D.
REMARK: The right-hand side of (25) is zero either $\sqrt{\mu}\vec{H} = \sqrt{\varepsilon}\vec{E}$ or $\vec{H} = \vec{0}$ and $\vec{E} = \vec{0}$. In the case of $\sqrt{\mu}\vec{H} = \sqrt{\varepsilon}\vec{E}$ it is clear that the left hand side is positive as long as $\mu > 0$ or $\varepsilon > 0$, and $\xi > 0$. Therefore the first condition does not violate the validity of the theorem.

4. A UNIQUE THEOREM FOR INITIAL-BOUNDARY VALUE PROBLEMS IN TELLEGEN MEDIUM

THEOREM: The initial-boundary value problem (IBVP) defined above possesses a unique solution if $\mu > 0$, $\varepsilon > 0$ and $\xi \leq \sqrt{\mu\varepsilon}$.
PROOF: Let us assume that there are two solutions to the IBVP defined above for given the charge density $\rho(\vec{r}, t)$, the current density $\vec{J}(\vec{r}, t)$, the initial conditions $\vec{E}_o(\vec{r}) = \vec{E}(\vec{r}, 0)$, $\vec{H}_o(\vec{r}) = \vec{H}(\vec{r}, 0)$, and the boundary condition $\vec{E}^*(\vec{r}, t)$ and $\vec{H}^*(\vec{r}, t)$

$$\left\{\vec{E}_1(\vec{r}, t),\, \vec{D}_1(\vec{r}, t),\, \vec{B}_1(\vec{r}, t),\, \vec{H}_1(\vec{r}, t)\right\}$$

and

$$\left\{\vec{E}_2(\vec{r}, t),\, \vec{D}_2(\vec{r}, t),\, \vec{B}_2(\vec{r}, t),\, \vec{H}_2(\vec{r}, t)\right\} \quad \text{for } t \geq 0 \text{ and } \vec{r} \in V \tag{26}$$

Since the initial-boundary value problem is linear, the difference solution defined by

$$\overset{\frown}{\vec{E}}(\vec{r}, t) = \vec{E}_1(\vec{r}, t) - \vec{E}_2(\vec{r}, t) \quad \text{for } t \geq 0 \text{ and } \forall\vec{r} \in V \tag{27}$$

$$\overset{\frown}{\vec{D}}(\vec{r}, t) = \vec{D}_1(\vec{r}, t) - \vec{D}_2(\vec{r}, t) \quad \text{for } t \geq 0 \text{ and } \forall\vec{r} \in V \tag{28}$$

$$\overset{\frown}{\vec{B}}(\vec{r}, t) = \vec{B}_1(\vec{r}, t) - \vec{B}_2(\vec{r}, t) \quad \text{for } t \geq 0 \text{ and } \forall\vec{r} \in V \tag{29}$$

$$\overset{\frown}{\vec{H}}(\vec{r}, t) = \vec{H}_1(\vec{r}, t) - \vec{H}_2(\vec{r}, t) \quad \text{for } t \geq 0 \text{ and } \forall\vec{r} \in V \tag{30}$$

satisfy the homogeneous Maxwell-Minkowski equations for which the *current density* and the *charge density* is zero, i.e., $\vec{J}(\vec{r}, t) = \vec{0}$ and

$\rho(\vec{r},\, t) = 0$.

the *Faraday's law* :

$$\nabla \times \overset{\frown}{\vec{E}}(\vec{r},\, t) \;=\; -\frac{\partial}{\partial t}\overset{\frown}{\vec{B}}(\vec{r},\, t) \quad t \geq 0 \text{ and } \vec{r} \in V \tag{31}$$

the *Ampere's law* :

$$\nabla \times \overset{\frown}{\vec{H}}(\vec{r},\, t) \;=\; \frac{\partial}{\partial t}\overset{\frown}{\vec{D}}(\vec{r},\, t) \quad t \geq 0 \text{ and } \vec{r} \in V \tag{32}$$

the *Gauss's law* :

$$\nabla \cdot \overset{\frown}{\vec{D}}(\vec{r},\, t) \;=\; \vec{0} t \geq 0 \text{ and } \vec{r} \in V \tag{33}$$

the *magnetic Gauss L=law* :

$$\nabla \cdot \overset{\frown}{\vec{B}}(\vec{r},\, t) \;=\; 0 \quad t \geq 0 \text{ and } \vec{r} \in V \tag{34}$$

the constitutive relations

$$\overset{\frown}{\vec{D}}(\vec{r},\, t) = \varepsilon \overset{\frown}{\vec{E}}(\vec{r},\, t) + \xi \overset{\frown}{\vec{H}}(\vec{r},\, t) \tag{35}$$

$$\overset{\frown}{\vec{B}}(\vec{r},\, t) = \xi \overset{\frown}{\vec{E}}(\vec{r},\, t) + \mu \overset{\frown}{\vec{H}}(\vec{r},\, t) \tag{36}$$

homogeneous boundary conditions:

$$\vec{n} \times \overset{\frown}{\vec{H}}(\vec{r},\, t) \;=\; \vec{0} \quad t \geq 0 \text{ and } \vec{r} \in S_H, \text{ and} \tag{37}$$

$$\vec{n} \times \overset{\frown}{\vec{E}}(\vec{r},\, t) \;=\; \vec{0} \quad t \geq 0 \text{ and } \vec{r} \in S_E, \tag{38}$$

homogeneous initial conditions

$$\begin{aligned} \overset{\frown}{\vec{E}}_o(\vec{r}\,) &= \overset{\frown}{\vec{E}}(\vec{r},\, 0) = \vec{0} \quad \text{for } \forall \vec{r} \in V, \\ \overset{\frown}{\vec{B}}_o(\vec{r}\,) &= \overset{\frown}{\vec{B}}(\vec{r},\, 0) = \vec{0} \quad \text{for } \forall \vec{r} \in V \end{aligned} \tag{39}$$

Let us multiply the Ampere's law for difference field (32) with the difference electric field (27) (scalar multiplication), and multiply the Faraday's law for difference field (31) with the difference magnetic field (30) (scalar multiplication), and subtract from each other:

$$\overset{\frown}{\vec{E}} \cdot \left(\nabla \times \overset{\frown}{\vec{H}}\right) - \overset{\frown}{\vec{H}} \cdot \left(\nabla \times \overset{\frown}{\vec{E}}\right) = \overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{B}}}{\partial t} + \overset{\frown}{\vec{E}} \frac{\partial \overset{\frown}{\vec{D}}}{\partial t} \tag{40}$$

Since

$$\overset{\frown}{\vec{E}} \cdot \left(\nabla \times \overset{\frown}{\vec{H}}\right) - \overset{\frown}{\vec{H}} \cdot \left(\nabla \times \overset{\frown}{\vec{E}}\right) = -\nabla \cdot \left(\overset{\frown}{\vec{E}} \times \overset{\frown}{\vec{H}}\right)$$

For any pair of continuous vector fields, we have

$$-\nabla \cdot \left(\overset{\frown}{\vec{E}} \times \overset{\frown}{\vec{H}} \right) = \overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{B}}}{\partial t} + \overset{\frown}{\vec{E}} \frac{\partial \overset{\frown}{\vec{D}}}{\partial t} \tag{41}$$

Let us integrate this equation over the domain V:

$$-\int_V \nabla \cdot \left(\overset{\frown}{\vec{E}} \times \overset{\frown}{\vec{H}} \right) dv = \int_V \left(\overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{B}}}{\partial t} + \overset{\frown}{\vec{E}} \frac{\partial \overset{\frown}{\vec{D}}}{\partial t} \right) dv \tag{42}$$

Since the Divergence Theorem yields

$$\int_V \nabla \cdot \left(\overset{\frown}{\vec{E}} \times \overset{\frown}{\vec{H}} \right) dv = \int_S \left(\overset{\frown}{\vec{E}} \times \overset{\frown}{\vec{H}} \right) \cdot \vec{n} da$$

From (42) we have,

$$-\int_S \left(\overset{\frown}{\vec{E}} \times \overset{\frown}{\vec{H}} \right) \cdot \vec{n} da = \int_V \left(\overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{B}}}{\partial t} + \overset{\frown}{\vec{E}} \frac{\partial \overset{\frown}{\vec{D}}}{\partial t} \right) dv \tag{43}$$

Taking into account the vector identities,

$$-\vec{n} \cdot \left(\overset{\frown}{\vec{E}} \times \overset{\frown}{\vec{H}} \right) = -\left(\overset{\frown}{\vec{E}} \times \overset{\frown}{\vec{H}} \right) \cdot \vec{n} = \overset{\frown}{\vec{E}} \cdot \left(\vec{n} \times \overset{\frown}{\vec{H}} \right) = -\overset{\frown}{\vec{H}} \cdot \left(\vec{n} \times \overset{\frown}{\vec{E}} \right) \tag{44}$$

we can arrange (43) taking into account various boundary conditions.

Case 1) If the tangential component of the magnetic field is given all over the boundary surface, i.e., $S_H = S$ and $S_E = \varnothing$, then from (43), (37), and (44) we have

$$\int_S \left[\overset{\frown}{\vec{E}} \cdot \left(\vec{n} \times \overset{\frown}{\vec{H}} \right) \right] \cdot \vec{n} da = \int_V \left(\overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{B}}}{\partial t} + \overset{\frown}{\vec{E}} \frac{\partial \overset{\frown}{\vec{D}}}{\partial t} \right) dv = 0 \tag{45}$$

Case 2) If the tangential component of the electric field is given all over the boundary surface, i.e., $S_E = S$ and $S_H = \varnothing$, then from (43), (38), and (44) we have

$$-\int_S \left[\overset{\frown}{\vec{H}} \cdot \left(\vec{n} \times \overset{\frown}{\vec{E}} \right) \right] \cdot \vec{n} da = \int_V \left(\overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{B}}}{\partial t} + \overset{\frown}{\vec{E}} \frac{\partial \overset{\frown}{\vec{D}}}{\partial t} \right) dv = 0 \tag{46}$$

Case 3) If the tangential component of the electric field is given on a certain part of the boundary S_E, the magnetic field is given on a certain part of the boundary. S_H, where S_E and S_H are the non-overlapping complementary parts of the boundary (i.e., $S_E \cup S_H = S$ and $S_E \cap S_H = \varnothing$) then from (43), (37), (38) and (44) we have

$$\int_{S_H} \left[\overset{\frown}{\vec{E}} \cdot \left(\vec{n} \times \overset{\frown}{\vec{H}} \right) \right] da - \int_{S_E} \left[\overset{\frown}{\vec{H}} \cdot \left(\vec{n} \times \overset{\frown}{\vec{E}} \right) \right] da = \int_V \left(\overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{B}}}{\partial t} + \overset{\frown}{\vec{E}} \frac{\partial \overset{\frown}{\vec{D}}}{\partial t} \right) dv = 0 \tag{47}$$

Therefore we conclude that we have

$$\int_V \left(\overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{B}}}{\partial t} + \overset{\frown}{\vec{E}} \frac{\partial \overset{\frown}{\vec{D}}}{\partial t} \right) dv = 0 \tag{48}$$

for three types of boundary value problems.

Let us now employ the constitutive Equations (15), (16), in (48):

$$\int_V \left(\mu \overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{H}}}{\partial t} + \varepsilon \overset{\frown}{\vec{E}} \cdot \frac{\partial \overset{\frown}{\vec{E}}}{\partial t} + \xi \left(\overset{\frown}{\vec{E}} \cdot \frac{\partial \overset{\frown}{\vec{H}}}{\partial t} + \overset{\frown}{\vec{H}} \cdot \frac{\partial \overset{\frown}{\vec{E}}}{\partial t} \right) \right) dv = 0 \tag{49}$$

which can be further arranged that

$$\frac{\partial}{\partial t} \int_V \frac{1}{2} \left(\mu \overset{\frown}{\vec{H}} \cdot \overset{\frown}{\vec{H}} + \varepsilon \overset{\frown}{\vec{E}} \cdot \overset{\frown}{\vec{E}} + 2\xi \overset{\frown}{\vec{E}} \cdot \overset{\frown}{\vec{H}} \right) dv = 0 \tag{50}$$

Since this relation should be valid any region, we arrive at

$$\frac{\partial}{\partial t} \frac{1}{2} \left(\mu \overset{\frown}{\vec{H}} \cdot \overset{\frown}{\vec{H}} + \varepsilon \overset{\frown}{\vec{E}} \cdot \overset{\frown}{\vec{E}} + 2\xi \overset{\frown}{\vec{E}} \cdot \overset{\frown}{\vec{H}} \right) = 0 \quad \text{for } t \geq 0 \text{ and } \vec{r} \in V \tag{51}$$

If this equation is integrated with respect to time taking into account that the difference field satisfies homogeneous initial conditions, we obtain

$$\frac{1}{2} \left(\mu \overset{\frown}{\vec{H}} \cdot \overset{\frown}{\vec{H}} + \varepsilon \overset{\frown}{\vec{E}} \cdot \overset{\frown}{\vec{E}} + 2\xi \overset{\frown}{\vec{E}} \cdot \overset{\frown}{\vec{H}} \right) = 0 \quad \text{for } t \geq 0 \text{ and } \vec{r} \in V \tag{52}$$

It was shown before that the expression in (52) is positive definite for $\mu > 0$, $\varepsilon > 0$, and $\xi \leq \sqrt{\mu\varepsilon}$, the condition in (52) implies

$$\overset{\frown}{\vec{H}} = 0 \text{ and } \overset{\frown}{\vec{E}} = 0 \tag{53}$$

which means that a boundary value problem can have one solution. Q.E.D.

5. CONCLUSIONS

It is shown that the energy stored in a Tellegen medium due to the charges and the currents is positive definite if

$$\mu > 0,\ \varepsilon > 0,\ \text{and}\ \xi \leq \sqrt{\mu\varepsilon} \tag{54}$$

The first two are also necessary for an isotropic regular electromagnetic media, whereas the last one is specific for Tellegen media. It is also shown that if the energy stored in a Tellegen media is positive definite, then the initial-boundary value problems possess one solution.

It should be noted first that the results obtained in this study is based on the assumption that the there exists a solution to initial-boundary value problems defined for Tellegen media. The presented study does not have any implication for the validity of this assumption.

It is also interesting to note that even though the stored energy in a Tellegen media is always positive under the conditions given in (54), the contribution of the Tellegen parameter to the energy stored in the media due to the fields can be positive or negative.

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