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THE PHASE CENTRE OF A PURE MODE, SMOOTH WALL, CONICAL HORN

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Abstract—In this paper the phase centre of a pure mode, smooth wall, conical horn is studied. The method employed uses phase information obtained by aperture integration of the phase corrected mode field but differs from earlier presentations in first reducing the radiation integrals in terms of an identified family of functions prior to integration. The result is formulas for the principal plane radiation patterns that better exemplify operation of the antenna. The procedure uses these formulas together with least squares minimization to determine a best fit phase centre for each principal plane. It is shown that contrary to published results, even in its principal planes, there is no unique phase centre that is a property of the horn alone. The use of such a horn as the feed element of a radio-optical antenna system is then considered. It is shown that in situations where cost is paramount and the well known drawbacks of its unequal principal plane beamwidths and the diminished gain that results from lack of a uniquely defined phase centre can be tolerated, perhaps contrary to conventional wisdom, the pure mode, smooth wall, conical horn does have a useful role to play. This is illustrated by a design example in which such a horn was used to feed a 6.4 m. dish working in X-band with a Gregorian feed system. The result is confirmation of the usefulness of the method presented here for location of a best fit phase centre.

1. INTRODUCTION

As the feed element in a radio-optical antenna, either prime focus or intermediated through a sub-reflector in a Cassegrain or Gregorian configuration, the pure mode, smooth wall, conical horn (PMSWCH) has had a bad press. It is generally rejected out of hand because of the inequality of its principal plane radiation patterns, possible mode conversion at its throat and lack of a unique phase centre, all of which for the system as a whole result in reduced aperture efficiency, diminished gain and poor cross-polar performance.

In applications like radio astronomy or in receiving telemetry from deep space probes, where performance is more important than cost, these are valid objections and such horns have rightly been set aside. However the one great feature that the PMSWCH does have in its favour is cost. By comparison with its much preferred corrugated cousins, it is extremely simple to make and so has a clear margin in situations where there can be greater tolerance of the defects listed above and only one or a few horns are needed, so that there are no manufacturing economies of scale to be had. This suggests that a revisit may be timely.

As a step in this direction, recently new formulas for radiation from a PMSWCH have been derived [1]. Like many preceding analyses, they are based on aperture integration of a TE_{11} mode launched from the throat of the horn and containing a quadratic phase term to correct for the spreading over a plane aperture of what is essentially a spherical wave. Also like their predecessors, obtaining results from them ultimately requires numerical integration.

However by first partly reducing the raw integral through analysis prior to any numerical integration, the E and H plane patterns are able to be cast into the form of the sum and difference of the first two members of a generic family of functions, in this way better showing the reasons for the observed differences in the two patterns and the displacement of their phase centres. Radiation patterns calculated from these formulas have been shown to be in excellent agreement with those obtained from an electromagnetic simulator for any flare angle likely to be of practical importance [1]. Using the phase information to determine phase centre is therefore something that can be done with a high degree of confidence in the result.

2. PRINCIPAL PLANE RADIATION FIELDS

The geometry of a PMSWCH driven from a circular waveguide propagating only the fundamental TE_{11} mode is shown in Fig. 1. Referred to a coordinate system having the centre of the aperture as origin, the following have been shown to be the spherical components of the principal plane radiation fields:

In the E-plane

$$E_\theta = \frac{jkAV_{11}e^{-jkr}}{2r} \sin\phi C_{11} k_{11} \left(W_0 - W_2\right) \tag{1}$$

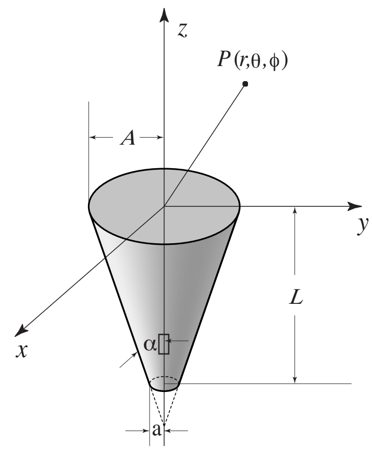

Figure 1. Horn geometry.

In the H-plane

$$E_\phi = \frac{jkAV_{11}e^{-jkr}}{2r} \cos\phi \cos\theta C_{11} k_{11} (W_0 + W_2) \quad (2)$$

where V_{11} is the mode voltage,

$C_{11} = J_1(k_{11}) \sqrt{\frac{\pi}{2}(k_{11}^2 - 1)}$,

$k_{11} = 1.841\ldots$ is the first zero of $J_1'(x)$,

$W_n(kA, \tan\alpha, \theta) = \int_0^1 w J_n(k_{11}w) J_n(kAw\sin\theta)\, e^{-j\frac{1}{2}kAw^2\tan\alpha} dw$

is a family of integrals that must be evaluated numerically,

A is the horn aperture radius,

α is the semi-angle of the horn, and

θ is the angle measured from the axis of symmetry (boresight axis) of the horn.

It is from the phase information contained in these formulas that the corresponding phase centres are to be found.

3. DEFINING PHASE CENTRE

There is a certain amount of elasticity in the literature in relation to defining phase centre [2]. The concept of there being a phase centre is rooted in the idealised notion that the radiation field of an antenna is emitted as a spherical wave. Traced back to its apparent origin, the phase centre is then the point from which the wave appears to have come. If the antenna is to act as the primary feed in some kind of radio-optical system, then the aim will be to have the phase centre so defined coincide with a focal point of the optical system, either that of the primary reflector in the case of a prime focus system or of a sub-reflector in the case of a multi-element system.

Frequently in practice a problem arises because no unique phase centre exists, leaving open the question of where to go from there. Sometimes the problem solves itself in that the optical element that the antenna is illuminating necessarily intercepts only a portion of its field and if a phase centre can be arrived at for only that portion, what may pertain elsewhere holds little interest. However very often, the PMSWCH being a case in point, even this more limited objective is not met and further retreat to a next most satisfactory alternative becomes necessary.

One is more or less to abandon the chase altogether and adopt a more application oriented approach in which, with the axis of the horn, along which any phase centre must be presumed to lie, and the radio-optical axis having first been made coincident, the horn is then positioned to maximise the aperture efficiency of the reflector that produces the final diffraction pattern [3, 4]. The equivalent phase centre of the horn is then taken as coincident with the optical focus point and its position in the horn can be fixed as a distance from its aperture plane. Despite its obvious overall practical merit, doing this produces an application specific, after the fact result that is not solely a property of the horn and so may not be of much interest in itself, knowing the position of *any* reference mark on the horn serving equally well to fix its position within the associated radio-optical system.

The root to any defining of phase centre as a property unique to the horn lies in first determining it in any plane that contains its axis of symmetry. In any such plane, either component of the radiation field can be written in the form

$$E = |V(\theta)|\, e^{j\psi(\theta)} \frac{e^{-jkr}}{r} \tag{3}$$

The phase of the field at any radius r in a direction θ is then

$$\Psi(\theta) = \psi(\theta) - kr \tag{4}$$

Hence to remain on the same wavefront as contains a point on the axis of symmetry at radius r, the ray emitted at an angle θ will need to travel an additional distance $\delta r(\theta)$ chosen so that

$$\psi(0) - kr = \psi(\theta) - kr - k\delta r(\theta) \tag{5}$$

i.e.,

$$\delta r(\theta) = \frac{\lambda}{2\pi}\{\psi(\theta) - \psi(0)\} \tag{6}$$

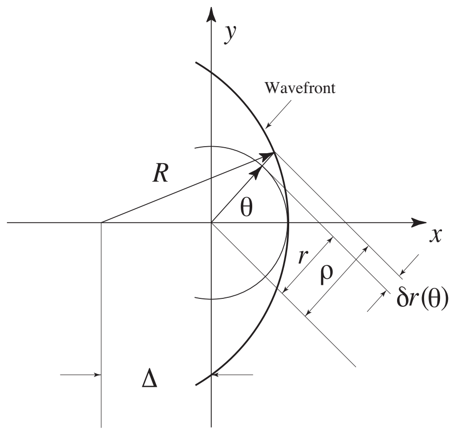

Figure 2. Phase centre not coincident with origin of coordinates.

Now consider Fig. 2 which shows the assumed situation of a spherical wave being emitted from a point on the axis of symmetry not coincident with the origin of coordinates but offset from it by a distance Δ. Then in terms of the chosen coordinate system, the equation of the wavefront is

$$(x + \Delta)^2 + y^2 = R^2 \tag{7}$$

This can be converted to polar coordinates by the transformation $x = \rho\cos\theta$, $y = \rho\sin\theta$ to give

$$\rho = -\Delta\cos\theta + \sqrt{R^2 - \Delta^2\sin^2\theta} \tag{8}$$

when, subject to being in the radiation field where $R >> \Delta$,

$$\rho = R - \Delta\cos\theta \tag{9}$$

This is essentially to say that the radius vectors to a given point on the wavefront from the phase centre and the origin of coordinates are parallel, making the added distance that the ray must travel to reach the wavefront

$$\delta r(\theta) = \rho - r = R - \Delta\cos\theta - r = 2\Delta\sin^2\frac{1}{2}\theta \tag{10}$$

Equating this with the previously determined excess distance then gives

$$\psi(\theta) - \psi(0) = \delta\psi(\theta) = \frac{4\pi\Delta}{\lambda}\sin^2\frac{1}{2}\theta \tag{11}$$

Following Rusch and Potter [5], we apply this model to the horn in question in the plane of interest by determining a best value for Δ in a least squares sense as that which minimizes

$$\mathcal{E}^2 = \int_0^{\theta_0}\left[\delta\psi(\theta) - \frac{4\pi\Delta}{\lambda}\sin^2\frac{1}{2}\theta\right]^2 d\theta \tag{12}$$

where θ_0 bounds the semi-angular range within which phase centre is to be determined.

The value of Δ that minimizes $\mathcal{E}^2$ can be found by applying Leibnitz' Rule [6, p. 454] to determine $d\mathcal{E}^2/d\Delta$ and setting it to zero. The result is

$$\Delta = \frac{\lambda}{4\pi}\frac{\int_0^{\theta_0}\delta\psi(\theta)\sin^2\frac{1}{2}\theta d\theta}{\int_0^{\theta_0}\sin^4\frac{1}{2}\theta d\theta} \tag{13}$$

We can test the usefulness of this value of Δ by calculating $\sqrt{\mathcal{E}^2/\theta_0}$ over the chosen angular range, this being the RMS phase error that results from its use as the supposed phase centre. To be useful the error must be zero or sufficiently small.

It is to be noted that this procedure weights as equally significant the phase at all angles within the range of integration, even though the antenna pattern has a maximum at $\theta = 0$ from which, at least initially, it falls away with increasing θ. Whether or not values of θ for which the radiation intensity is small should equally influence choice of best phase centre is something to consider, even though introduction of any weighting scheme designed to avoid this would necessarily be somewhat arbitrary. Often a value of θ_0 will suggest itself. For example, if the horn is acting as source to illuminate a reflector, the semi-angle subtended by the reflector could be considered an appropriate choice. In less obvious cases setting $\theta_0 \approx \bar{\theta}_{3\,\mathrm{dB}}$, the average 3-dB beamwidth (at which angle the lobe width $2\theta_0$ is about twice the beamwidth), might be considered, (at which angle the radiation intensity will have fallen by about 12 dB from its maximum).

4. AN ILLUSTRATIVE EXAMPLE

We will now apply this procedure to the phase information contained in Eqs. (1) and (2). W_0 and W_2, and hence their sum and difference, are complex quantities which, when written in amplitude-phase form, in the radiation field principal planes lead directly to

$$\delta\psi(\theta) = \arg\left[W_0(\theta) \mp W_2(\theta)\right] - \arg\left[W_0(0)\right] \tag{14}$$

Implicit in this is the realisation that on boresight where $\theta = 0$, $W_2(0) = 0$ and so makes no contribution to the phase

Since it will soon emerge that no general result can be presented, it is most illustrative to continue the discussion with an example. For this we choose a horn with a semi-angle $\alpha = 16.63°$, an aperture radius $A = 80$ mm and is driven from a circular waveguide with a 25 mm internal diameter ($a = 12.5$ mm). The length of the horn is $L = 226$ mm and it is assumed to operate at a wavelength of $\lambda = 36.35$ mm (frequency of 8.253 GHz). The principal plane radiation diagrams of this antenna calculated from Eqs. (1) and (2) are shown in Fig. 3 wherein the interplay as between the two patterns of the addition and subtraction of the secondary W_2 term with the primary W_0 term is clearly evident.

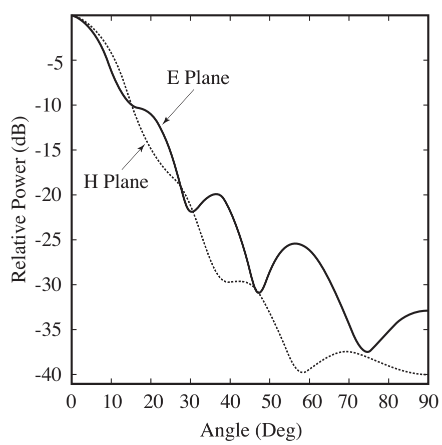

Figure 3. Sample horn radiation diagram.

For each of these principal planes, Table 1 shows the distance inside the horn, measured from its aperture plane, of the best fit, principal plane phase centres as a function of θ_0 together with the RMS phase error that results from this choice. This shows that in both cases the phase centre falls within the body of the horn but more importantly that it depends on θ_0, with the RMS error, though remaining tolerably small over a wide range of this variable, generally increasing as θ_0 increases. The clear implication of this result is that even in a specific plane, phase centre position is not a unique property of the horn but is specific to the application.

Table 1. Best fit phase centre.

	E-plane		H-plane	
θ_0 (deg.)	Position (mm)	RMS error (deg)	Position (mm)	RMS error (deg)
10	160.1	6.53	64.8	3.11
11	168.2	8.01	66.7	3.84
12	177.1	9.57	68.9	4.68
13	186.4	11.11	71.4	5.64
14	195.2	12.49	74.3	6.71
15	203.0	13.56	77.6	7.91
16	208.8	14.17	81.3	9.23
17	212.4	14.35	85.4	10.67
18	213.6	14.22	89.9	12.18
19	212.7	14.09	94.7	13.72
20	210.1	14.42	99.6	15.19

5. COMPARISON WITH PUBLISHED RESULTS

While so far this has been shown only in relation to a specific example, the result is general and should act to proscribe any possibility of presenting data on phase centre location as a function of the horn geometry alone. Interestingly however, such data appears in the literature. There is a table in Milligan's book [7] and a graph in the book edited by Olver et al. [2]. Data used to produce both is claimed to be based on phase information generated by numerical integration of the horn aperture field, taken to be that of the TE_{11} mode together with a quadratic phase deviation term, exactly as assumed in deriving Eqs. (1) and (2). However in neither case is anything said about the actual process of moving from radiation field phase data to phase centre.

Milligan cites an earlier paper [8] as the source of the data contained in his book but, while this contains his universal radiation patterns, it is silent on the subject of phase centre. He does, however, also cite a paper by Muehldorf [9], which, while it does not specifically consider the PMSWCH, does provide a method for translating the phase information into a phase centre that is applicable to horns in general. Presumably Milligan has applied this to his raw data and the result is the table shown in the book.

Muehldorf defines phase centre as the point from which the radiation field appears to have come. He then assumes that in any plane containing the horn axis, it can be found as the centre of curvature of a far field, constant phase surface at its point of intersection with the axis. This removes any dependence on θ_0 but only at the expense of the additional assumption that the constant phase surface is perfectly spherical, in the process discarding most of the phase information save for that in the immediate neighbourhood of the axis.

In both references [2, 7] the data is given in terms of a normalised phase centre defined as the distance of the phase centre from the aperture plane divided by a normalisation parameter R. For this parameter, Ref. [2] uses the axial length of the cone, the distance measured along the axis of symmetry from the aperture plane to the point of intersection of its generators, when in terms of Fig. 1 $R = A/\tan\alpha$, while in [7] the choice is to use the slant length, measuring along a generator, when $R = A/\sin\alpha$. Obviously for $\alpha \ll 1$ there will not be a great difference.

The presentation is in terms of normalised phase centre as a function of a dimensionless parameter $s = A^2/2\lambda R$ which captures both the horn geometry and the operating condition, in [2] as a graph and in [7] as a table. Both presentations include within their range the illustrative example used above. It is therefore interesting to carry out a critical examination of the consistency of the two sets of published data and the degree to which they agree with the example results obtained in this paper.

Initial comparison of the two sets of published data, even at the grossest level, shows no agreement at all. According to Milligan's table [7], for example, the E-plane phase centre always lies further inside the horn than the H-plane phase centre, whereas with Shafia and Kishk's graph [2] the opposite is the case. However it is also to be observed that in the graph, the normal convention for choice of dependent and independent variable axes has been reversed. If based on this observation, one does the comparison on the basis that the axes have simply been mislabelled, the position changes completely and the result, even at the level of detail, is considerable agreement. It therefore seems safe to assume that this is what has actually occurred.

In effecting a comparison of the published data with that derived in this paper, there is a clear preference to use the table in which interpolation by differences is obviously preferable to attempting to read a small graph where obtaining even two figure accuracy is problematical. However, whatever the case, one is still left to compare a single value with a range of values.

Interpolating in the table given by Milligan for the case $s = 0.3149$, the example case, we obtain $R_E = 214.7\,\mathrm{mm}$, $R_H = 83.8\,\mathrm{mm}$, which compares fairly favourably with the data obtained in this paper where, if θ_0 is set at 15°, about the average principal plane beamwidth for the horn in question, $R_E = 203.0\,\mathrm{mm}$, $R_H = 77.6\,\mathrm{mm}$. Using the procedure outlined in this paper, it is found that the RMS phase errors that attach to this choice are 13.56° for the E-plane and 7.91° for the H-plane. This author's reading of the graph (entered at $s = 0.3149/\cos 16.63^\circ = 0.3286$) gives $R_E = 195\,\mathrm{mm}$, $R_H = 88\,\mathrm{mm}$ Since both sets of published have in common omission of any angular range of applicability, both sets of answers equally must be regarded as no more than indicative.

6. APPLICATION AS A FEED IN RADIO-OPTICAL ANTENNA SYSTEMS

Quite apart from the specifics above and however defined, there is general agreement that the phase centre is different between the two principal planes. As Milligan [7] points out, the principal plane phase centres can be expected to be at the extremes of the range that would be obtained in any plane containing the horn axis, the variation between the principal planes being elliptical. There is thus no position at which a PMSWCH can be placed along the optical axis of any radio-optical antenna system of which it may be the feed that will not result in some diminution of aperture efficiency at the final beam forming aperture with consequent loss of gain and reduced cross polar performance, although these may be tolerated where an adequate-for-application and markedly cheaper product results.

It is quite clear too that there is a wide difference in the placement of the principal plane phase centres, making the PMSWCH a highly astigmatic source. In the light of Eqs. (1) and (2), where as between the principal planes the complex W_2 term is in one case additive and the other subtractive, this is perhaps not entirely surprising. In attempting to deal with this matter, Ref. [10] offers the advice that "for horns with phase centers in the two principal planes located not too close to each other, satisfactory operation can be obtained by locating the focus at a point between the two phase centers such that the maximum phase deviations in the two planes are approximately the same".

In this instance, a route to that goal is to observe that were it not for the W_2 term, which, as already pointed out, makes no contribution on the boresight axis, the two principal plane patterns would otherwise be the same. To base a compromise on the result obtained using the W_0 term alone and putting up with the consequent partial defocusing of the part of the field associated with the W_2 term therefore suggests itself. That this compromise is possible is not evident in any solution that proceeds directly from numerical integration of the aperture field without prior reduction in terms of this set of functions. When this is done for the example horn with $\theta_0 = 15^\circ$, the result is to place the compromise phase centre at $z = -135.0\,\mathrm{mm}$ (not too different from the arithmetic mean of the principal plane phase centres, the compromise recommended in [10] for closely spaced but not exactly coincident phase centres) with an RMS phase error (based on the W_0 term alone) of $0.1932^c = 11.07^\circ$.

7. EXPERIMENTAL VALIDATION

Despite the seeming rubberyness of the whole concept of phase centre, at least as it applies to the PMSWCH, it does have practical utility. Even if one adopts the more systems oriented approach of attempting to place the horn at the position that optimises radio-optical system aperture efficiency, it is of more than passing utility to have a close idea of where to begin what is necessarily an iterative process. Kildal [3] has outlined a systematic way of going about this by correction of an adequately good starting point. This is to be supplied by the means outlined so far in this paper.

The horn that has been used as an example in the above was designed and manufactured with a specific application in mind. The project involved refurbishment of a 6.4 m dish using a nominally Gregorian radio-optical system, obtained cheaply on the second hand market and originally configured for Ka-band operation, to receive circularly polarised, telemetry signals in X-band from a constellation of earth resources satellites against a background where cost and speed of readiness for operation were paramount. Details are contained in [11] and here it suffices to say that the link budget contained sufficient slack to allow the shortcomings of the PMSWCH as a feed to be tolerated.

As a way of optimising it for its original design application, the optical elements of the antenna, while remaining figures of revolution, had been slightly distorted from true conics to best accommodate the characteristics of the original corrugated horn feed. The result is some blurring of the point focus that would otherwise have been the case. However a ray optical study based on geometric optics principles had shown that there was an identifiable point of best focus which, with a point source located there, would produce an acceptable result. The object then became to match this with a PMSWCH source.

The preceding analysis was used to set a starting point for experimental optimisation, this consisting of having the system produce a radiation pattern that most closely approximates that calculated using a point source of the same average beamwidth as the horn and located at the point of best focus. The degree to which this has been achieved can be gauged from Fig. 4 which compares the theoretically determined best result with that obtained from the dish with the horn located at its experimentally determined optimum position. Because no test range was available that could contain so large an antenna for measurement purposes, a satellite source was used to do this, details being given in [11].

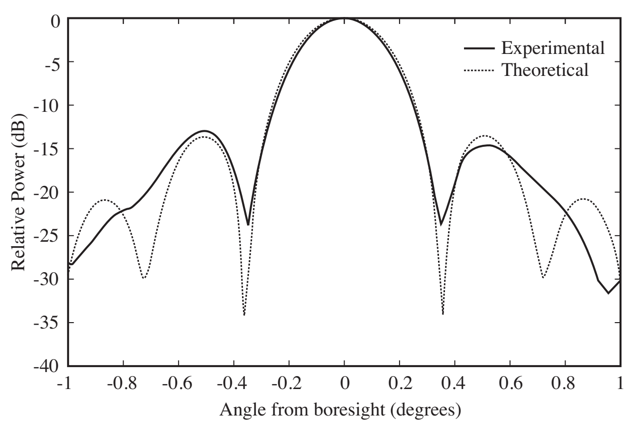

Figure 4. Theoretically and experimentally determined radiation diagrams for 6.4 m dish.

The optimum result was obtained with the horn set at a position that measured from its aperture plane would place its best fit phase centre on the axis of symmetry at $z = -139$ mm. Because the elliptical sub-reflector that the horn directly illuminated intercepted most of its main lobe, it was decided to determine the starting phase centre by setting $\theta_0 = \bar{\theta}_{3\,\mathrm{dB}}$, in this case 15°. Results for this are given above. A comparison of these with the experimentally determined result shows a discrepancy of 4 mm for the case where the W_0 derived result is used and 1.3 mm for the case where the mean of the best fit principal plane results is used, in both cases small fractions of a wavelength

($\lambda = 36.35\,\mathrm{mm}$).

A radio-optical system with a true set of conic surfaces and in consequence an unambiguous point focus would hold the potential for debate free validation, a property that, as noted, can only be approximated by a system that comes already compromised to suit its original application. However it is contended that despite the less than ideal circumstances in which they were determined, these results, a generally good reproduction of a pre-computed polar diagram, lend strong weight to the validity of the method that forms the basis of the paper.

8. CONCLUSION

In this paper we have studied the question of the phase centre of a pure mode, smooth wall, conical horn. Whichever of the several possible definitions of the term to be found in the literature is used, it has been shown that in planes containing the axis of symmetry different results are obtained in each plane, with the principal plane values constituting the extremes, and that under most definitions of the term, in any plane the PMSWCH does not have a phase centre that is a property of the horn alone, but is application dependent. Even so, it has been shown that a theoretically determined value is useful in setting a starting point for optimisation by maximisation of secondary aperture efficiency. A comparison of the results determined here with others published previously in the literature has highlighted a number of shortcomings, including that some most likely contain labelling errors.

The drawbacks that have led to the virtual discarding of the PMSWCH as a primary source element in radio-optical systems are not removed by anything contained in this paper. However, it must be remarked that the worth of any engineering solution is to be judged in terms of its economy in relation to the need that it is intended to satisfy, sometimes forgotten in a technically driven search for the very best. The paper illustrates by way of practical example that, when there are occasions where these shortcomings can be lived with, the cheapness and ease of construction of the plain conical horn makes it an attractive solution worth keeping in mind.

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