Progress In Electromagnetics Research B, Vol. 4, 101–113, 2008

INEQUALITY CONDITION FOR GRATING LOBES OF PLANAR PHASED ARRAY

C.-H. Liang, L. Li, and X.-J. Dang

National Laboratory of Antennas and Microwave Technology
Xidian University
Xi'an 710071, China

Abstract—This paper presents a general inequality for the grating lobes of the planar phased array, whether rectangular lattice or triangular lattice. And for the planar phased array with grating lobes, the maximum scanning angle is given.

1. INTRODUCTION

Phased array is an important kind of array for its flexibility, speed and multiple beams. In the theory, it is grating lobes and blindness that mainly restrict the performance of arrays, and blindness results from the coupling between array elements. Ref. [1] shows that the study of center elements active pattern can reveal the feature of blindness. The corresponding physical concept is the coupling between elements prevents the center elements radiating power at a certain angle. But a further research shows the conclusion only applies to infinite or large arrays. Otherwise the coupling difference between the center and edge elements makes the center elements uncommon and changes the feature of blindness. Ref. [2–4] clearly shows grating lobes are a consequence of the periodicity of the array pattern and depend on the arrangement of the elements.

This paper introduces the general normalized parameter A and B for rectangular and triangular lattice, presents the inequality condition for grating lobes, and shows the maximum scanning angle under the condition of grating lobes' existence. All these conclusions are general. The typical numerical examples can be found in this paper.

2. INEQUALITY CONDITION FOR GRATING LOBES

In this paper, the planar array lies in XOY plane. And (θ, φ) shows the field or pattern at (θ, φ), (θ_0, φ_0) defines the angle to which main beam is scanned. We define $\varphi_0 = 90^\circ$ as E plane and $\varphi_0 = 0^\circ$ H plane.

The discussion of the grating lobes are based on the rectangular lattice and the triangular lattice. The rectangular lattice is shown in Fig. 1, where the element distance is dx and dy.

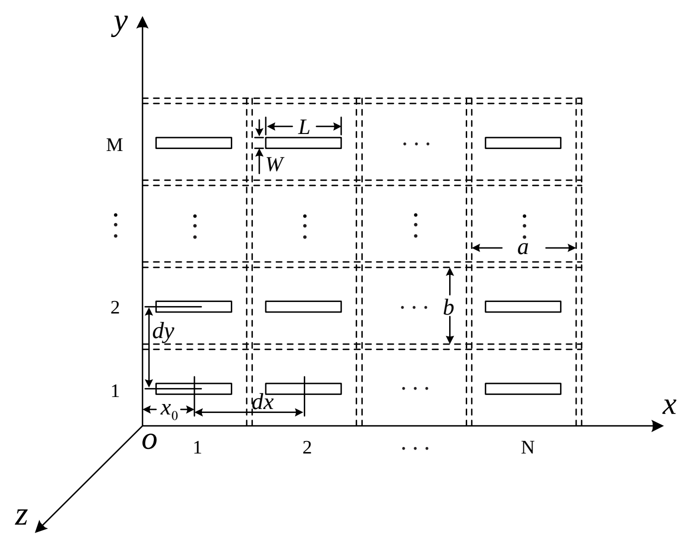

Figure 1. Phased array of rectangular lattice.

The array pattern for the array of isotropic radiators excited with equal amplitude [1] is

$$S(\theta,\varphi)=\sum_m e^{jmkdx(\sin\theta\cos\varphi-\sin\theta_0\cos\varphi_0)}\cdot\sum_n e^{jnkdy(\sin\theta\sin\varphi-\sin\theta_0\sin\varphi_0)} \tag{1}$$

Obviously, the maximum condition [1] for Eq. (1) is

$$\begin{cases} \sin\theta\cos\varphi = \sin\theta_0\cos\varphi_0 + \dfrac{p}{\overline{dx}}, & (p = 0, \pm1, \pm2, \cdots) \\ \sin\theta\sin\varphi = \sin\theta_0\sin\varphi_0 + \dfrac{q}{\overline{dy}}, & (q = 0, \pm1, \pm2, \cdots) \end{cases} \tag{2}$$

where $\overline{dx} = dx/\lambda$, $\overline{dy} = dy/\lambda$, and when

$$p = q = 0 \tag{3}$$

$\theta = \theta_0$, $\varphi = \varphi_0$, which is the main beam angular position. So p and q can not simultaneously equal 0 under the condition of grating lobes' existence.

Figure 2 shows the triangular lattice, where the element distance is dx and dy. The array pattern [2] is as follows.

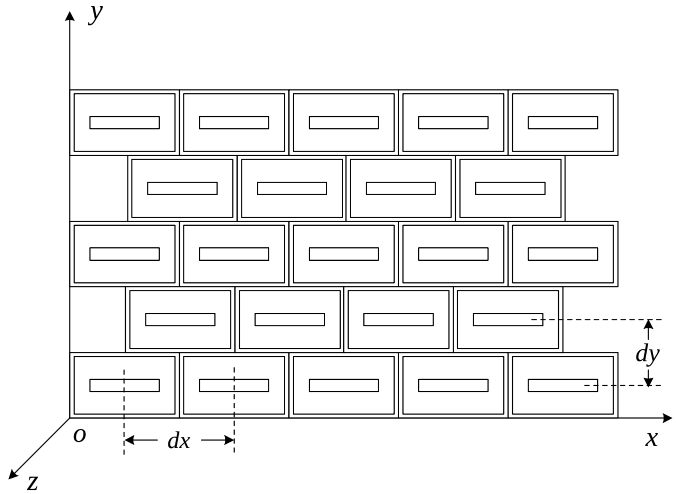

Figure 2. Phased array of triangular lattice.

$$S(\theta,\varphi)=\left[1+e^{j\frac{1}{2}kdx(\sin\theta\cos\varphi-\sin\theta_0\cos\varphi_0)}e^{jkdy(\sin\theta\sin\varphi-\sin\theta_0\sin\varphi_0)}\right] \cdot\sum_m e^{jmkdx(\sin\theta\cos\varphi-\sin\theta_0\cos\varphi_0)}\cdot\sum_n e^{jnkdy(\sin\theta\sin\varphi-\sin\theta_0\sin\varphi_0)} \quad (4)$$

The maximum condition [2] is

$$\begin{cases} \sin\theta\cos\varphi = \sin\theta_0\cos\varphi_0 + \dfrac{p}{dx}, & (p = 0, \pm 1, \pm 2, \cdots) \\ \sin\theta\sin\varphi = \sin\theta_0\sin\varphi_0 + \dfrac{q}{2dy}, & (q = 0, \pm 1, \pm 2, \cdots) \\ p + q = \text{even} \end{cases} \quad (5)$$

The last equation is the even restriction condition for triangular lattice. This paper introduces the normalized parameter A and B.

$$A = \frac{p}{dx}$$

$$B = \begin{cases} \dfrac{q}{\overline{\overline{dy}}} & \text{rectangular lattice} \\ \dfrac{q}{2\overline{\overline{dy}}} & \text{triangular lattice} \end{cases} \tag{6}$$

Thus, Eq. (2) and Eq. (5) can be united as

$$\begin{cases} \sin\theta\cos\varphi = \sin\theta_0\cos\varphi_0 + A \\ \sin\theta\sin\varphi = \sin\theta_0\sin\varphi_0 + B \end{cases} \tag{7}$$

Squaring Eq. (7) and considering that the grating lobes begin to form at $\theta = 90^\circ$ and $\sin\theta = 1$ [2], we can obtain

$$\sin^2\theta_0 + 2\sin\theta_0(A\cos\varphi_0 + B\sin\varphi_0) + (A^2 + B^2 - 1) = 0 \tag{8}$$

Equation (8) shows the position of the main beam θ_0, φ_0 when the grating lobes begin to form. We introduce

$$\begin{cases} \cos\varphi_1 = \dfrac{A}{\sqrt{A^2 + B^2}} \\ \sin\varphi_1 = \dfrac{B}{\sqrt{A^2 + B^2}} \end{cases} \tag{9}$$

Equation (8) can be simplified as

$$\sin^2\theta_0 + 2\sqrt{A^2 + B^2}\cos(\varphi_0 - \varphi_1)\sin\theta_0 + (A^2 + B^2 - 1) = 0 \tag{10}$$

Let

$$\begin{cases} \xi = \sqrt{A^2 + B^2}\cos(\varphi_0 - \varphi_1) \\ \eta = \xi + \sin\theta_0 \end{cases} \tag{11}$$

The corresponding domain is

$$\begin{cases} \sin\theta_0 \in [0, 1] \\ \xi \in \left[-\sqrt{A^2 + B^2}, \sqrt{A^2 + B^2}\right] \\ \eta \in \left[-\sqrt{A^2 + B^2}, 1 + \sqrt{A^2 + B^2}\right] \end{cases} \tag{12}$$

And Eq. (10) can be changed into two hyperbolic equations

$$\xi^2 - \eta^2 = A^2 + B^2 - 1, \quad \text{when} \quad A^2 + B^2 - 1 > 0$$

or

$$\eta^2 - \xi^2 = 1 - A^2 - B^2, \quad \text{when} \quad A^2 + B^2 - 1 < 0 \tag{13}$$

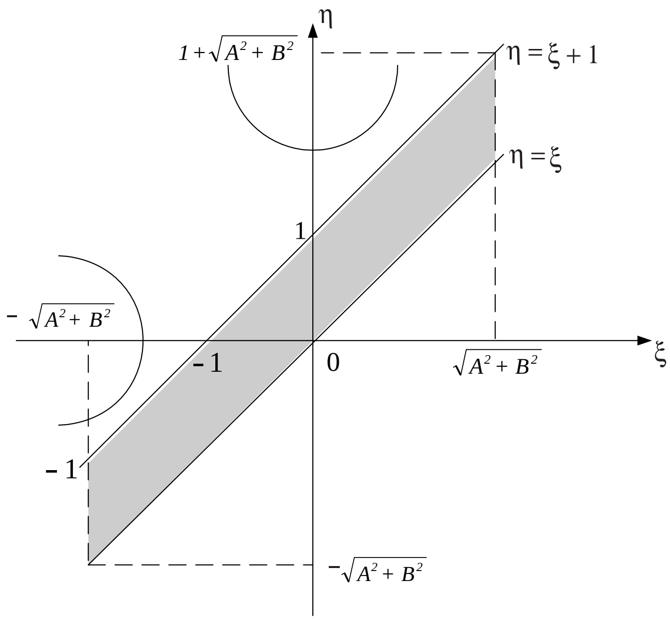

Figure 3. Hyperbolic curves and domain of grating lobes.

The hyperbolic curves and domain of grating lobes are shown in Fig. 3.

Obviously, no crossing point between hyperbolic curves and domain is the condition for no grating lobes. Let

$$\begin{cases} \eta = \xi + h \\ h > 1 \end{cases} \tag{14}$$

where $h > 1$ assures no crossing point between hyperbolic curves and domain.

The research shows only considering the first equation of Eq. (13) is sufficient. Substituting Eq. (14) into Eq. (13), we can obtain

$$\begin{cases} \xi^2 - (\xi + h)^2 = k \\ k = A^2 + B^2 - 1 > 0 \end{cases} \tag{15}$$

Only when $\xi < 0$, Eq. (15) can hold. And when Eq. (15) can not hold, the grating lobes can also not exist. So we obtain

$$h^2 - 2|\xi|h + k = 0 \tag{16}$$

Its solution is

$$h = |\xi| \pm \sqrt{|\xi|^2 - k} > 1 \tag{17}$$

When the negative root meets the inequality, the positive one automatically meets the inequality. That is $|\xi| - \sqrt{|\xi|^2 - k} > 1$ or

$$\sqrt{|\xi|^2 - k} < |\xi| - 1 \tag{18}$$

Squaring Eq. (18), we obtain

$$A^2 + B^2 > 2|\xi| \tag{19}$$

Considering $|\xi| = \sqrt{A^2 + B^2}\,|\cos(\varphi_0 - \varphi_1)|$ and the modulus of circular function less than 1, we can obtain the condition for no grating lobes of planar phased array

$$A^2 + B^2 > 4 \tag{20}$$

where A and B must be out of the circle with radius 2, as shown in Fig. 4.

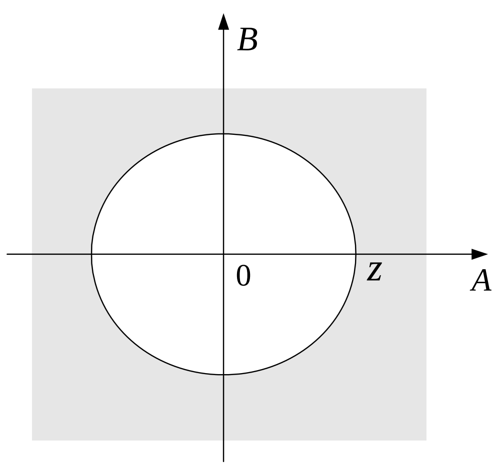

Figure 4. Circular domain inequality for A and B under the condition without grating lobes.

The distance condition for elements under different lattice will be given.

Case 1 Rectangular lattice

Inequality condition for no grating lobes is

$$\left(\frac{p}{\overline{dx}}\right)^2 + \left(\frac{q}{\overline{dy}}\right)^2 > 4 \tag{21}$$

We must consider all the possibility of $p = 0, \pm 1, \pm 2, \cdots$ and $q = 0, \pm 1, \pm 2, \cdots$. In fact, we only need consider two worst cases:

$$\text{when } (p, q) = (1, 0), \left(\frac{1}{\overline{dx}}\right) > 2, \ \overline{dy} \text{ unconfined}$$

and

$$\text{when } (p, q) = (0, 1), \left(\frac{1}{\overline{dy}}\right) > 2, \ \overline{dx} \text{ unconfined}$$

So, the distance condition for no grating lobes of rectangular lattice is

$$\overline{dx} < 0.5, \quad \overline{dy} < 0.5 \tag{22}$$

Case 2 Triangular lattice

Inequality condition for no grating lobes is

$$\left(\frac{p}{\overline{dx}}\right)^2 + \left(\frac{q}{2\overline{dy}}\right)^2 > 4 \tag{23}$$

We must consider all the possibility of $p = 0, \pm 1, \pm 2, \cdots$ and $q = 0, \pm 1, \pm 2, \cdots$. In fact, we only need consider three worst cases:

$$\text{when } (p, q) = (2, 0), \left(\frac{2}{\overline{dx}}\right) > 2, \ \overline{dy} \text{ unconfined}$$

and

$$\text{when } (p, q) = (0, 2), \left(\frac{1}{\overline{dy}}\right) > 2, \ \overline{dx} \text{ unconfined}$$

When $(p, q) = (1, 1)$, we notice that inequality condition for no grating lobes have many distribution. The physical concept shows the distance dx and dy is contradictory.

$$\left(\frac{1}{\overline{dx}}\right)^2 + \left(\frac{1}{2\overline{dy}}\right)^2 > 4 \tag{24}$$

Considering the restriction $(2, 0)$ and $(0, 2)$, we obtain the optimum restriction

$$\left(\frac{1}{\overline{dx}}\right)^2 > 3 \quad \left(\frac{1}{2\overline{dy}}\right)^2 > 1 \tag{25}$$

So $A = \sqrt{3}$, $B = 1$. Thus the optimum distance inequality for the triangular lattice is

$$\overline{dx} < \sqrt{3}/3 = 0.57735, \quad \overline{dy} < 0.5 \tag{26}$$

Obviously, the element distance for no grating lobes of triangular lattice is larger than the rectangular lattice, which mainly lies in $\overline{dx}$. Let $u = \cos\varphi_0$ and $v = \sin\theta_0$, we obtain the optimum element distance lies out of grating lobes, as shown in Fig. 5.

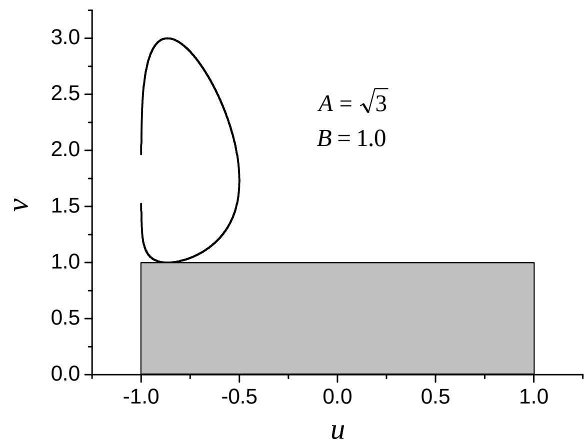

Figure 5. Optimum element distance lies out of grating lobes.

3. MAXIMUM SCANNING ANGLE $\max\theta_0$

In many practical engineering, many restriction makes Eq. (2) unhold. Then grating lobes are unavoidable. We defines scanning angle $\varphi_0 \in [0, 2\pi]$ and $\theta_0 \in [0, \max\theta_0]$ for planar phased array in which the domain without grating lobes are called maximum scanning domain and $\max\theta_0$ is maximum scanning angle, as shown in Fig. 6.

We define $u = \cos\varphi_0$ and $v = \sin\theta_0$. Fig. 7 shows the classical case with grating lobes, where $u \in [-1, 1]$, $v \in [0, 1]$.

Eq. (8) can be written as

$$v^2 + 2v\left(Au \pm B\sqrt{1-u^2}\right) + (A^2 + B^2 - 1) = 0 \tag{27}$$

Left-top curve in Fig. 7 shows Eq. (27). When grating lobes exist, the curves lie in u, v domain. Obviously, when $dv/du = 0$, $\max v$ holds.

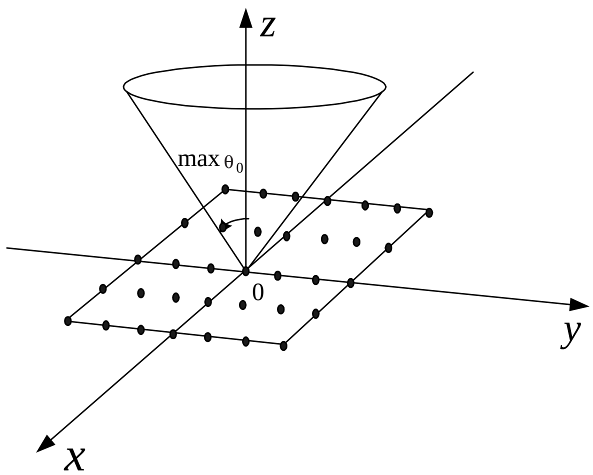

Figure 6. Maximum scanning domain and maximum scanning angle.

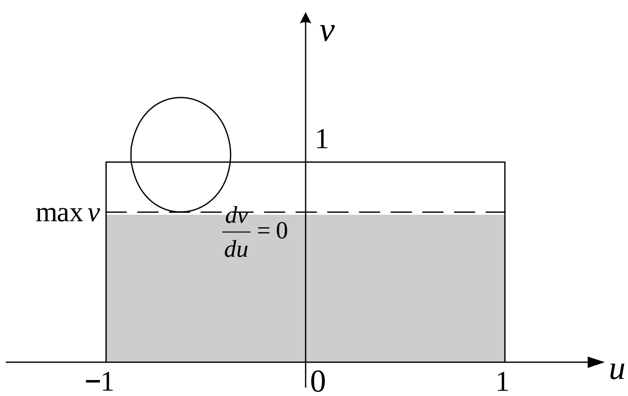

Figure 7. Maximum scanning domain with grating lobes $\max v$ corresponding to $\frac{dv}{du} = 0$.

Differentiating Eq. (27)

$$2v\frac{dv}{du} + 2\frac{dv}{du}\left(Au \pm B\sqrt{1-u^2}\right) + 2v\left(A \mp B\frac{u}{\sqrt{1-u^2}}\right) = 0 \quad (28)$$

Considering the condition that derivative equals 0. The condition for extreme is

$$A = \pm B \frac{u}{\sqrt{1-u^2}} \tag{29}$$

Squaring Eq. (29), we can obtain

$$u = \pm \frac{A}{\sqrt{A^2+B^2}} \tag{30}$$

Select the negative root, then

$$u = -\cos\varphi_1 \tag{31}$$

The equation for extreme is

$$v^2 - 2\sqrt{A^2+B^2}v + (A^2+B^2-1) = 0 \tag{32}$$

Finally, we can obtain

$$\max v = \sqrt{A^2+B^2} - 1 \tag{33}$$

that is

$$\max\theta_0 = \sin^{-1}\left(\sqrt{A^2+B^2} - 1\right) \tag{34}$$

No crossing points exist between another root of Eq. (32) and grating lobes, so we ignore it. Eq. (14) shows the maximum scanning angle $\max\theta_0$. From Eq. (33), we can obtain when

$$\sqrt{A^2+B^2} - 1 > 1 \tag{35}$$

the grating lobes do not exist. We find Eq. (3) equal to Eq. (20).

When $\sqrt{A^2+B^2} - 1 < 0$, no scanning domain exists, which accounts for the ignorance of the second hyperbolic curve of Eq. (13). Fig. 8 shows when $A = 1.16686$, the variation of the maximum scanning angle with B.

Figure 9 shows the pattern of 20×20 triangular lattice with $dx = 1.008\lambda$, $dy = 0.504\lambda$. Fig. 9(a) shows the pattern without phase shift. Fig. 9(b) shows the pattern of $\varphi = 0°$ when the main beam is $\theta_0 = 60°$, $\varphi_0 = 0°$. According to Eq. (23), there is grating lobes under this condition. But it is evidence that no grating lobe exists on this plane. From Fig. 9(c) ($\varphi = 82.7°$), one can see the grating lobe. It is concluded that the main beam and the grating lobe can lie on two different planes. The maximum scanning angle $\theta_0 = 23.765°$ can be obtained from Eq. (34). Fig. 9(d) shows the plane where the grating lobe exists.

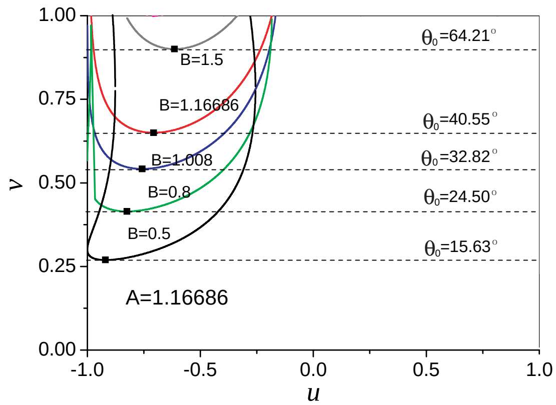

Figure 8. Variation of the maximum scanning angle with B, $A = 1.16686$.

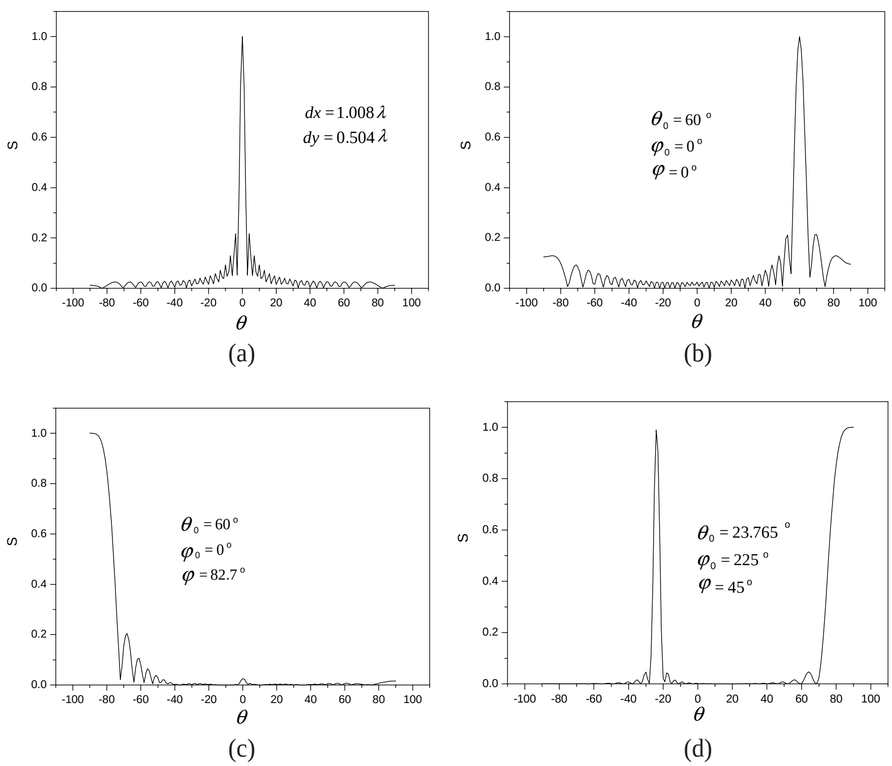

Figure 9. Pattern of phased array of triangular lattice. (a) $\theta_0 = 0°$, $\varphi_0 = 0°$, $\varphi = 0°$; (b) $\theta_0 = 60°$, $\varphi_0 = 0°$, $\varphi = 0°$; (c) $\theta_0 = 60°$, $\varphi_0 = 0°$, $\varphi = 82.7°$; (d) $\theta_0 = 23.765°$, $\varphi_0 = 22.5°$, $\varphi = 45°$.

4. CONCLUSION

On the basis of study on all kinds of arrangement of planar phased array, this paper presents general normalized parameter A and B, and proves under the condition for no grating lobes A and B meet circle domain inequality

$$A^2 + B^2 > 4$$

Distance condition for rectangular lattice is $\overline{dx} < 0.5$ and $\overline{dy} \le 0.5$, and the optimum distance condition for triangular lattice is $\overline{dx} < \sqrt{3}/3 = 0.57735$ and $\overline{dy} < 0.5$. On the other hand, when domain inequality does not hold, the maximum scanning angle is $\max\theta_0 = \sin^{-1}\left(\sqrt{A^2+B^2} - 1\right)$. The numerical computations show that the conclusions can also applied to the arrays with unequal amplitude. The research on blindness of phased array will be reported later.

REFERENCES

1. Amitay, N., V. Galindo, and C. P. Wu, *Theory and Analysis of Phased Array Antennas*, Wiley-Interscience, 1972.
2. Sharp, E. D., "A triangular arrangement of planar-array elements that reduces the number needed," *IRE Trans. Antennas and Propagation*, 126–129, March 1961.
3. Stark, L., "Microwave theory of phased-array antennas — A review," *Proceedings of IEEE*, Vol. 62, No. 12, 1661–1701, Dec. 1974.
4. Pozar, D. M. and D. H. Schaubert, "Scan blindness in infinite phased arrays of printed dipoles," *IEEE Trans. Antennas and Propagation*, Vol. 32, No. 6, 602–610, June 1984.
5. Wheeler, H. A., "The grating-lobe series for the impedance variation in a planar phased-array antenna," *IEEE Trans. Antennas and Propagation*, 825–827, Sep. 1965.
6. Wheeler, H. A., "The grating-lobe series for the impedance variation in a planar phased-array antenna," *IEEE Trans. Antennas and Propagation*, Vol. 14, No. 6, 707–714, Nov. 1966.
7. Diamond, B. L., "A generalized approach to the analysis of infinite planar array antennas," *Proceedings of IEEE*, Vol. 56, No. 11, 1837–1851, Nov. 1968.
8. Bates, R. H. T., "Mode theory approach to arrays," *IEEE Trans. Antennas and Propagation*, 321–322, March 1965.

9. Vescovo, R., "Beam scanning with null and excitation constraints for linear arrays of antennas," *Journal of Electromagnetic Waves and Applications*, Vol. 21, No. 2, 267–277, 2007.
10. Vallecchi, A. and G. B. Gentili, "Broad band full scan coverage dual polarised spherical conformal phased array," *Journal of Electromagnetic Waves and Applications*, Vol. 16, No. 3, 385–401, 2002.
11. Prosvirnin, S. L., S. A. Tretyakov, and P. L. Mladyonov, "Electromagnetic wave diffraction by planar periodic gratings of wavy metal strips," *Journal of Electromagnetic Waves and Applications*, Vol. 16, No. 3, 421–435, 2002.
12. Abbaspour, M. and H. R. Hassani, "Wideband star-shaped microstrip patch antenna," *Progress In Electromagnetics Research Letters*, Vol. 1, 61–68, 2008.
13. Nie, X.-C., Y.-B. Gan, N. Yuan, C.-F. Wang, and L.-W. Li, "An efficient hybrid method for analysis of slot arrays enclosed by a large radome," *Journal of Electromagnetic Waves and Applications*, Vol. 20, No. 2, 249–264, 2006.
14. Lim, K.-S., V. C. Koo, and T. S. Lim, "Design, simulation and measurement of a post slot waveguide antenna," *Journal of Electromagnetic Waves and Applications*, Vol. 21, No. 12, 1589–1603, 2007.
15. Zhang, Y. and W. Cao, "Array pattern synthesis based on weighted biorthogonal modes," *Journal of Electromagnetic Waves and Applications*, Vol. 20, No. 10, 1367–1376, 2006.