Progress In Electromagnetics Research B, Vol. 4, 79–99, 2008

UNIFIED DERIVATION OF THE TRANSLATIONAL ADDITION THEOREMS FOR THE SPHERICAL SCALAR AND VECTOR WAVE FUNCTIONS

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Abstract—The translational addition theorems for the spherical scalar and vector wave functions are derived in a novel, unified way based on the simple and well-known concepts of the radiation and incoming wave patterns. This approach makes the derivation simpler and more transparent compared to the previous approaches. As a result, we also obtain alternative and partly simpler expressions for the translation coefficients in the vector case.

1. INTRODUCTION

Spherical wave functions are basic solutions to the Maxwell's equations and the Helmholtz equation in the spherical system of co-ordinates and they form the basis for the expansion of any field satisfying these equations. Addition theorems are needed when it is necessary to expand fields in more than one system of co-ordinates. This happens, for example, in the case of multiple scattering from a collection of spheres [1], in evaluation of a field in spheres due to external sources [2] and in antenna measurements [3]. More recently, there has been a renewed interest in the addition theorems because of their important role in fast multipole methods [4–7]. There are two types of addition theorems for the spherical wave functions: rotational and translational. This paper considers the latter for both the spherical scalar and vector wave functions.

The translational addition theorems for the spherical scalar wave functions were derived first by Friedman and Russek [8] and later by Danos and Maximon [9]. The first derivations of the translational addition theorems for the spherical vector wave functions were due to Stein [10] and Cruzan [11]. They derived the theorems starting from the scalar theorems and through tedious algebraic manipulations obtained the vector theorems. Later, Borghese et al. [12] and Felderhof and Jones [13] presented more compact derivations making use of a so-called irreducible spherical tensor familiar in quantum mechanics. Wittmann [14] presented an interesting approach based on differential operator representations of the wave functions. Chew [15] derived the theorems quite similarly as in [10] and [11]; Chew and Wang [16] also presented recurrence formulas. Kim [17] derived the theorems as well as recurrence formulas by applying similar spherical tensor technique as in [12] and [13]. He also discussed the symmetry relations of the coefficients of both the scalar and vector theorems in [18] and presented more efficient recurrence procedure for the coefficients of the scalar theorem in [19]. Quite recently, Chew presented a new derivation of the vector theorems [20], which is similar to his derivation of the scalar theorems [15], but still relies on some peculiar integral results.

In this paper we present a unified derivation of the translational addition theorems for the spherical scalar and vector wave functions based on the simple and well-known concepts of radiation pattern and incoming wave pattern. These concepts play a crucial role also in the field of multilevel fast multipole algorithms [5–7, 21–24], where a lot of new techniques with these wave patterns has been developed recently. The motivation for this paper is to present a different and simpler derivation of the theorems than has been given in the above references. We also present alternative and partly simpler expressions for the translation coefficients that have not, to the best knowledge of the authors, been published elsewhere.

At the end of the paper, a note is made on an efficient calculation of the translation coefficient applying the following idea used in the fast multipole methods [4, 7]: an arbitrary translation is performed more efficiently by combining the rotation of the co-ordinates and the translation along the z-axis, which is considerably simpler than the general translation. It turns out that the computational cost is reduced from $\sim N^4$ to $\sim N^3$, if N is the truncation point of the expansion.

2. DEFINITIONS AND NOTATIONS

Since the definitions and notations of the functions associated to the topic are quite diverse in the literature, we begin by defining the functions as they are used in this paper. We assume the time dependence $e^{-i\omega t}$ where ω denotes the angular frequency. Moreover, $k = \omega\sqrt{\mu\epsilon}$ is the wave number in a medium with the magnetic permeability μ and the electric permittivity ϵ.

The spherical wave functions $\psi_{l,m}$ satisfying the scalar Helmholtz equation $(\nabla^2 + k^2)F = 0$ are defined as

$$\psi_{l,m}(\mathbf{r}) = z_l(kr)Y_{l,m}(\hat{r}) \tag{1}$$

where $\mathbf{r}$ is the position vector, $r = |\mathbf{r}|$, $\hat{r} = \mathbf{r}/r$, $l = 0, 1, \ldots$ and $m = 0, \pm 1, \ldots, \pm l$. The radial function z_l is the spherical Bessel function of the first kind j_l in the case of an incoming wave and the spherical Hankel function of the first kind $h_l^{(1)}$ when the wave is outgoing; when necessary, we emphasize the type of the wave function by writing $\psi_{l,m}^{\text{in}}$ or $\psi_{l,m}^{\text{out}}$, respectively. In defining the angular function, the spherical harmonic $Y_{l,m}$, we follow Jackson [25, Sec. 3.5]:

$$Y_{l,m}(\hat{r}) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi} \tag{2}$$

where θ and ϕ are the spherical angles of $\hat{r}$ and P_l^m is the associated Legendre function defined by

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l. \tag{3}$$

Defined as above, the spherical harmonics are orthonormal over a unit sphere:

$$\int_B Y_{l,m}Y_{n,p}^*\,d\Omega = \delta_{l,n}\delta_{m,p} \tag{4}$$

where B denotes the unit sphere, the asterisk the complex conjugation and $\delta_{l,n}$ the Kronecker delta being equal to 1 when $l = n$ and 0 otherwise.

The solenoidal spherical vector wave functions $\mathbf{M}_{l,m}$ and $\mathbf{N}_{l,m}$ satisfying the homogeneous vector Helmholtz equation $(\nabla^2 + k^2)\mathbf{F} = 0$ are most often defined as (see, for example, [26, Sec. 7.11])

$$\mathbf{M}_{l,m}(\mathbf{r}) = \nabla\psi_{l,m}(\mathbf{r}) \times \mathbf{r}, \tag{5}$$

$$\mathbf{N}_{l,m}(\mathbf{r}) = \frac{1}{k}\nabla \times \mathbf{M}_{l,m}(\mathbf{r}); \tag{6}$$

it follows that also

$$\mathbf{M}_{l,m}(\mathbf{r}) = \frac{1}{k}\nabla \times \mathbf{N}_{l,m}(\mathbf{r}) \tag{7}$$

since $\nabla \times \nabla \times \mathbf{M}_{l,m} = -\nabla^2\mathbf{M}_{l,m} = k^2\mathbf{M}_{l,m}$. Borghese et al. [12, Eq. (7)], Wittmann [14, Eq. (32)] and Kim [17, Eq. (3)], among others, multiply these by $i/\sqrt{l(l+1)}$ making their wave patterns conveniently orthonormal. We, however, adhere to above definitions because of their prevalence in the literature. The wave functions can be presented as

$$\mathbf{M}_{l,m}(\mathbf{r}) = z_l(kr)\mathbf{U}_{l,m}(\hat{r}), \tag{8}$$

$$\mathbf{N}_{l,m}(\mathbf{r}) = -iz_l'(kr)\mathbf{V}_{l,m}(\hat{r}) + \frac{z_l(kr)}{kr}\nabla_\Omega \times \mathbf{U}_{l,m}(\hat{r}) \tag{9}$$

if we define the vector spherical harmonics $\mathbf{U}_{l,m}$ and $\mathbf{V}_{l,m}$ as

$$\mathbf{U}_{l,m}(\hat{r}) = \nabla_\Omega Y_{l,m}(\hat{r}) \times \hat{r}, \tag{10}$$

$$\mathbf{V}_{l,m}(\hat{r}) = i\hat{r} \times \mathbf{U}_{l,m}(\hat{r}). \tag{11}$$

Above,

$$\nabla_\Omega = \hat{\theta}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \tag{12}$$

is the surface gradient on the unit sphere. We have not normalised the vector harmonics because it would make them somewhat inconsistent with the wave functions. The normalisation could be implemented by multiplying (10) by $i/\sqrt{l(l+1)}$. Hence, the vector spherical harmonics, defined as above, are orthogonal but not orthonormal:

$$\int_B \mathbf{U}_{l,m} \cdot \mathbf{U}^*_{n,p}\, d\Omega = l(l+1)\delta_{l,n}\delta_{m,p}, \tag{13}$$

$$\int_B \mathbf{V}_{l,m} \cdot \mathbf{V}^*_{n,p}\, d\Omega = l(l+1)\delta_{l,n}\delta_{m,p}, \tag{14}$$

$$\int_B \mathbf{U}_{l,m} \cdot \mathbf{V}^*_{n,p}\, d\Omega = 0. \tag{15}$$

3. RADIATION AND INCOMING WAVE PATTERNS

In the derivation of the addition theorems we make use of the concepts of radiation pattern and incoming wave pattern. Next we review these consepts and explain how they are applied.

Let F be a scalar field, i.e., a solution of the scalar Helmholtz equation $(\nabla^2 + k^2)F = 0$, with the sources in a sphere B centered at the origin. Then, outside B the field can be expanded in outgoing spherical wave functions as

$$F(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m} \psi_{l,m}^{\text{out}}(\mathbf{r}). \tag{16}$$

In addition, in the far zone, with $kr \gg 1$, F can be approximated as

$$F(\mathbf{r}) \approx \frac{e^{ikr}}{r} F_\infty(\hat{r}) \tag{17}$$

where

$$F_\infty(\hat{r}) = \lim_{r\to\infty} \frac{rF(\mathbf{r})}{e^{ikr}}, \tag{18}$$

which is called the radiation pattern of F. In particular, the radiation pattern of an outgoing spherical wave function is given by a spherical harmonic as

$$(\psi_{l,m}^{\text{out}})_\infty(\hat{r}) = \frac{1}{ik}(-i)^l Y_{l,m}(\hat{r}), \tag{19}$$

which is easy to see from (1) by examining the large argument behaviour of $h_l^{(1)}$. By applying (18) on both sides of (16) and using (19), the equation (16) reduces to

$$F_\infty(\hat{r}) = \frac{1}{ik} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l a_{l,m} Y_{l,m}(\hat{r}). \tag{20}$$

Now, the coefficients $a_{l,m}$ can be determined from (20) by applying the orthogonality of the spherical harmonics (4). Another important result, which we exploit in the derivation of the addition theorems, is the way how the radiation pattern transforms in the translation of the origin of the co-ordinate system. This transform is given by the equation

$$G_\infty(\hat{r}) = e^{ik\hat{r}\cdot\mathbf{t}} F_\infty(\hat{r}) \tag{21}$$

where $G(\mathbf{r}) = F(\mathbf{t}+\mathbf{r})$ and $\mathbf{t}$ is the translation vector from the old origin to the new one. This follows easily from (17) and (18), since $|\mathbf{t}+\mathbf{r}| \approx r + \hat{r}\cdot\mathbf{t}$ when $r \gg t$.

Consider next a scalar field F with the sources outside B. Then, inside B the field can be expanded in incoming spherical wave functions as

$$F(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m} \psi_{l,m}^{\text{in}}(\mathbf{r}). \tag{22}$$

In addition, let us assume that the field has the form

$$F(\mathbf{r}) = \int_B F_0(\hat{k}) e^{ik\hat{k}\cdot\mathbf{r}}\, d\Omega(\hat{k}) \tag{23}$$

where the integration is with respect to $\hat{k}$ over the unit sphere B. Then, we say that F is expanded in plane waves with the incoming wave pattern F_0. For instance,

$$\psi_{l,m}^{\text{in}}(\mathbf{r}) = (-i)^l \frac{1}{4\pi} \int_B Y_{l,m}(\hat{k}) e^{ik\hat{k}\cdot\mathbf{r}}\, d\Omega(\hat{k}), \tag{24}$$

which is the counterpart to the well-known expansion of a plane wave in the spherical wave functions or the spherical harmonics (see, for example [26, Sec. 7.6] or [27, Eqs. (2.29) and (2.45)]) as

$$e^{ik\hat{k}\cdot\mathbf{r}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l Y_{l,m}^*(\hat{k}) \psi_{l,m}^{\text{in}}(\mathbf{r}). \tag{25}$$

Now, if we expand F_0 in (23) in spherical harmonics as

$$F_0(\hat{k}) = \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l a_{l,m} Y_{l,m}(\hat{k}), \tag{26}$$

we obtain the expansion in (22) by applying (24). The reverse of this result holds for the truncated expansion (22). Namely, if F_N is the expansion (22) truncated at $l = N$, then by (24) F_N has a representation of the form (23) with the incoming wave pattern being the expansion (26) truncated at $l = N$. Finally, we easily see that the incoming wave pattern F_0 of the field F transforms like the radiation pattern in the translation of the origin by a vector $\mathbf{t}$. Namely, for the incoming wave pattern G_0 of the field G defined by $G(\mathbf{r}) = F(\mathbf{t}+\mathbf{r})$ we have:

$$G_0(\hat{k}) = e^{ik\hat{k}\cdot\mathbf{t}} F_0(\hat{k}). \tag{27}$$

In the case of a vector field similar results can be written in terms of the spherical vector wave functions and the vector spherical harmonics. Namely, if $\mathbf{F}$ is either the electric field $\mathbf{E}$ or the magnetic field $\mathbf{H}$, due to the sources inside B, then outside B the vector field $\mathbf{F}$ can be expanded in outgoing spherical vector wave functions as [27, p. 174]

$$\mathbf{F}(\mathbf{r}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} [A_{l,m} \mathbf{M}_{l,m}^{\text{out}}(\mathbf{r}) + B_{l,m} \mathbf{N}_{l,m}^{\text{out}}(\mathbf{r})]. \tag{28}$$

The radiation pattern,

$$\mathbf{F}_{\infty}(\hat{r}) = \lim_{r\to\infty} \frac{r\mathbf{F}(\mathbf{r})}{e^{ikr}}, \tag{29}$$

can be expanded in vector spherical harmonics as

$$\mathbf{F}_{\infty}(\hat{r}) = \frac{1}{ik} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} (-i)^l [A_{l,m} \mathbf{U}_{l,m}(\hat{r}) + B_{l,m} \mathbf{V}_{l,m}(\hat{r})], \tag{30}$$

because from (8) and (9), respectively, we easily get

$$(\mathbf{M}_{l,m}^{\text{out}})_{\infty}(\hat{r}) = \frac{1}{ik}(-i)^l \mathbf{U}_{l,m}(\hat{r}), \tag{31}$$

$$(\mathbf{N}_{l,m}^{\text{out}})_{\infty}(\hat{r}) = \frac{1}{ik}(-i)^l \mathbf{V}_{l,m}(\hat{r}). \tag{32}$$

The coefficients $A_{l,m}$ and $B_{l,m}$ can now be determined from (30) by applying the orthogonality of the vector spherical harmonics (13)–(15). Also, we easily see that the radiation pattern $\mathbf{F}_{\infty}$ transforms in the translation of the origin just like in the scalar case.

Finally, consider a vector field $\mathbf{F}$, with $\mathbf{F}$ being either $\mathbf{E}$ or $\mathbf{H}$, due to the sources outside B. Then, inside B the field can be expanded in incoming spherical vector wave functions as

$$\mathbf{F}(\mathbf{r}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} [A_{l,m} \mathbf{M}_{l,m}^{\text{in}}(\mathbf{r}) + B_{l,m} \mathbf{N}_{l,m}^{\text{in}}(\mathbf{r})]. \tag{33}$$

Like in the scalar case, we next assume that the field also has the form

$$\mathbf{F}(\mathbf{r}) = \int_B \mathbf{F}_0(\hat{k}) e^{ik\hat{k}\cdot\mathbf{r}} \, d\Omega(\hat{k}). \tag{34}$$

If we then expand the incoming wave pattern $\mathbf{F}_0$ in vector spherical harmonics as

$$\mathbf{F}_0(\hat{k}) = \frac{1}{4\pi}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}(-i)^l[A_{l,m}\mathbf{U}_{l,m}(\hat{k}) + B_{l,m}\mathbf{V}_{l,m}(\hat{k})], \tag{35}$$

we obtain the expansion (33) by using the equations

$$\mathbf{M}^{\text{in}}_{l,m}(\mathbf{r}) = (-i)^l\frac{1}{4\pi}\int_B \mathbf{U}_{l,m}(\hat{k})e^{ik\hat{k}\cdot\mathbf{r}}\,d\Omega(\hat{k}), \tag{36}$$

$$\mathbf{N}^{\text{in}}_{l,m}(\mathbf{r}) = (-i)^l\frac{1}{4\pi}\int_B \mathbf{V}_{l,m}(\hat{k})e^{ik\hat{k}\cdot\mathbf{r}}\,d\Omega(\hat{k}), \tag{37}$$

which can be derived by substituting (24) into the definitions (5) and (6), respectively. They are also given, for example, in [28, Eqs. (4.17a) and (4.17b)]. In analogy to the scalar plane wave, (36) and (37) imply that a vector plane wave $\mathbf{F}_0e^{ik\hat{k}\cdot\mathbf{r}}$ with $\hat{k}\cdot\mathbf{F}_0 = 0$ can be expanded as

$$\mathbf{F}_0e^{ik\hat{k}\cdot\mathbf{r}} = 4\pi\mathbf{F}_0\cdot\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{1}{l(l+1)}i^l[\mathbf{U}^*_{l,m}(\hat{k})\mathbf{M}^{\text{in}}_{l,m}(\mathbf{r}) + \mathbf{V}^*_{l,m}(\hat{k})\mathbf{N}^{\text{in}}_{l,m}(\mathbf{r})]. \tag{38}$$

We note that here (and in several equations in Sec. 5), the factor $1/[l(l+1)]$ is a consequence of the unnormalised definitions of the wave functions and the harmonics.

4. SCALAR ADDITION THEOREMS

Assume that we have expanded a scalar field F in spherical wave functions as

$$F(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} b_{l,m}\psi_{l,m}(\mathbf{r}) \tag{39}$$

and that we then want to translate the origin by the vector $\mathbf{t}$ and expand $F(\mathbf{t}+\mathbf{r})$ around the new origin as

$$F(\mathbf{t}+\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{p=-n}^{n} c_{n,p}\psi_{n,p}(\mathbf{r}). \tag{40}$$

To accomplish that we need to find $c_{n,p}$ in terms of $b_{l,m}$, i.e., we wish to find out how the coefficients $b_{l,m}$ transform in the translation. It is obvious that we only need to find the so-called translation coefficients $a_{l,m;n,p}$ for the expansion

$$\psi_{l,m}(\mathbf{t}+\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{p=-n}^{n} a_{l,m;n,p}(\mathbf{t})\psi_{n,p}(\mathbf{r}), \tag{41}$$

since by substituting (41) into (39), changing the order of summation and comparing the result to (40) we find the wanted coefficients as

$$c_{n,p} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{l,m;n,p}(\mathbf{t})b_{l,m}. \tag{42}$$

It is customary to call (41) the addition theorem of the spherical scalar wave functions.

There are three kinds of translations, and consequently addition theorems, depending on the types of the initial and translated wave functions. We refer them as out-to-out, in-to-in and out-to-in translations.

Consider first the out-to-out translation, in which case the wave functions on both sides of (41) are of an outgoing type and $t < r$. Then, by taking far field limits on both sides according to (18) and using (19), as explained in Sec. 3, the equation (41) reduces to

$$(-i)^l Y_{l,m}(\hat{r})e^{ik\hat{r}\cdot\mathbf{t}} = \sum_{n=0}^{\infty}\sum_{p=-n}^{n} (-i)^n a^{\text{out-out}}_{l,m;n,p}(\mathbf{t})Y_{n,p}(\hat{r}) \tag{43}$$

where the exponent factor on the left-hand-side is due to the translation of the origin by $\mathbf{t}$. Owing to the orthogonality of the spherical harmonics (4) we then obtain

$$a^{\text{out-out}}_{l,m;n,p}(\mathbf{t}) = (-i)^{l-n}\int_B Y_{l,m}(\hat{r})Y^*_{n,p}(\hat{r})e^{ik\hat{r}\cdot\mathbf{t}}\,d\Omega(\hat{r}). \tag{44}$$

The coefficients could already be calculated from the above expression by using the standard numerical integration rule over the unit sphere. An explicit expression, however, can be obtained as follows. By making use of (25) and changing the order of the integration and the summation, (44) becomes

$$a^{\text{out-out}}_{l,m;n,p}(\mathbf{t}) = 4\pi\sum_{q=0}^{\infty}\sum_{s=-q}^{q}(-i)^{l-n-q}\psi^{\text{in}}_{q,s}(\mathbf{t})\int_B Y_{l,m}Y^*_{n,p}Y^*_{q,s}\,d\Omega. \tag{45}$$

Note that in the above integral in the spherical co-ordinates over the unit sphere B, the integrand $Y_{l,m}Y_{n,p}^*Y_{q,s}^*$ contains the factor $e^{i(m-p-s)\phi}$, and so the integral vanishes unless $m-p-s=0$. This reduces the summation in s to a single value $s=m-p$. Note also that $Y_{l,m}Y_{n,p}^*$ is a spherical harmonic polynomial of order $l+n$, and similarly, $Y_{l,m}Y_{q,s}^*$ and $Y_{n,p}^*Y_{q,s}^*$ are spherical harmonic polynomials of order $l+q$ and $n+q$, respectively. So, due to the orthogonality of the spherical harmonics, the integral of $Y_{l,m}Y_{n,p}^*Y_{q,s}^*$ over B vanishes unless $q \le l+n$, $n \le l+q$ and $l \le n+q$, or equivalently, $|l-n| \le q \le l+n$. Moreover, since $m-p-s=0$, due to the form of the spherical harmonics, $Y_{l,m}Y_{n,p}^*Y_{q,s}^*$ is a polynomial of $\cos\theta$ of order $l+n+q$, and so, if this order is odd, the polynomial is odd and the integral vanishes. These facts taken together imply that (45) can be simplified to

$$a_{l,m;n,p}^{\text{out-out}}(\mathbf{t}) = 4\pi \sum_{\substack{q=|l-n| \\ l+n+q \text{ even}}}^{l+n} (-i)^{l-n-q}\psi_{q,m-p}^{\text{in}}(\mathbf{t}) \int_B Y_{l,m}Y_{n,p}^*Y_{q,m-p}^*\, d\Omega, \tag{46}$$

which is the usual form for the coefficients of the out-to-out translation. The integration of the three spherical harmonics can also be expressed in terms of Clebsch-Gordan coefficients or Wigner 3-j symbol; see, for example, [29, Sec. 27.9] or [6, Sec. 3.2.1.3].

The procedure for deriving the in-to-in translation is similar. In this case, the wave functions on both sides of (41) are of an incoming type. Then, by following the steps through (22) to (26), as explained in Sec. 3, the equation (41) can be reduced to a form identical to (43). Consequently, the coefficients of the in-to-in translation are the same as those of the out-to-out translation:

$$a_{l,m;n,p}^{\text{in-in}}(\mathbf{t}) = a_{l,m;n,p}^{\text{out-out}}(\mathbf{t}). \tag{47}$$

The out-to-in translation can be determined at once from the out-to-out translation. Namely, by inserting (45) into (41), when the wave functions on both sides are of an outgoing type and $t<r$, we have

$$\begin{aligned}\psi_{l,m}^{\text{out}}(\mathbf{t}+\mathbf{r}) \\ = 4\pi \sum_{n=0}^{\infty}\sum_{p=-n}^{n}\sum_{q=0}^{\infty}\sum_{s=-q}^{q} (-i)^{l-n-q}\psi_{q,s}^{\text{in}}(\mathbf{t})\psi_{n,p}^{\text{out}}(\mathbf{r}) \int_B Y_{l,m}Y_{n,p}^*Y_{q,s}^*\, d\Omega.\end{aligned} \tag{48}$$

Now, the expressions on both sides of the equation are symmetrical with respect to $\mathbf{t}$ and $\mathbf{r}$ apart from the types of the wave functions on the right-hand-side. Thus, we have a possibility to interchange the roles of $\mathbf{t}$ and $\mathbf{r}$ (and consequently require $t > r$) and identify

$$a^{\text{out-in}}_{l,m;n,p}(\mathbf{t}) = 4\pi \sum_{\substack{q=|l-n| \\ l+n+q \text{ even}}}^{l+n} (-i)^{l-n-q}\psi^{\text{out}}_{q,m-p}(\mathbf{t}) \int_B Y_{l,m}Y^*_{n,p}Y^*_{q,m-p}\, d\Omega, \tag{49}$$

which is the equation for the coefficients of the out-to-in translation. We note that the only difference compared to the coefficients of the out-to-out and in-to-in translations is the type of the wave function in the sum, which is now outgoing instead of incoming.

The above reasoning, however, does not work in the vector case treated in the next section. So, for the sake of completeness and as an introduction to the vector case, let us derive (49) anew starting from (41) when the wave function on the left-hand-side is outgoing and the ones on the right-hand-side incoming, and $t > r$. We assume that $a^{\text{out-in}}_{l,m;n,p}$ has a far field limit according to (18) and a radiation pattern denoted by $(a^{\text{out-in}}_{l,m;n,p})_\infty$. Then, by taking the limits on both sides, now with respect to $\mathbf{t}$ instead of $\mathbf{r}$, we obtain

$$\frac{1}{ik}(-i)^l Y_{l,m}(\hat{t})e^{ik\hat{t}\cdot\mathbf{r}} = \sum_{n=0}^{\infty}\sum_{p=-n}^{n} (a^{\text{out-in}}_{l,m;n,p})_\infty(\hat{t})\psi^{\text{in}}_{n,p}(\mathbf{r}). \tag{50}$$

By comparing (50) to (25) with $\hat{k} = \hat{t}$, it can be identified that

$$\left(a^{\text{out-in}}_{l,m;n,p}\right)_\infty(\hat{t}) = \frac{1}{ik}(-i)^{l-n}4\pi Y_{l,m}(\hat{t})Y^*_{n,p}(\hat{t}). \tag{51}$$

The right-hand-side of (51) can be expanded in spherical harmonics yielding

$$(a^{\text{out-in}}_{l,m;n,p})_\infty(\hat{t}) = \frac{1}{ik}4\pi \sum_{q=0}^{\infty}\sum_{s=-q}^{q} (-i)^{l-n}Y_{q,s}(\hat{t}) \int_B Y_{l,m}Y^*_{n,p}Y^*_{q,s}\, d\Omega. \tag{52}$$

Since the coefficients of the expansion of an outgoing field in the spherical wave functions are equal to the coefficients of the expansion of the corresponding radiation pattern in the spherical harmonics, as explained in Sec. 3, the equation (49) follows directly from (52).

5. VECTOR ADDITION THEOREMS

In this section we derive the addition theorems for the spherical vector wave functions, which are formulated as

$$\mathbf{M}_{l,m}(\mathbf{t}+\mathbf{r}) = \sum_{n=1}^{\infty}\sum_{p=-n}^{n}[A_{l,m;n,p}(\mathbf{t})\mathbf{M}_{n,p}(\mathbf{r}) + B_{l,m;n,p}(\mathbf{t})\mathbf{N}_{n,p}(\mathbf{r})], \quad (53)$$

$$\mathbf{N}_{l,m}(\mathbf{t}+\mathbf{r}) = \sum_{n=1}^{\infty}\sum_{p=-n}^{n}[B_{l,m;n,p}(\mathbf{t})\mathbf{M}_{n,p}(\mathbf{r}) + A_{l,m;n,p}(\mathbf{t})\mathbf{N}_{n,p}(\mathbf{r})] \quad (54)$$

where $A_{l,m;n,p}$ and $B_{l,m;n,p}$ are the translation coefficients to be determined. Notice that the same coefficients with an interchanged order appear in both equations (53) and (54) due to (6) and (7).

In the first subsection we present a derivation completely analogous to the derivation of the scalar addition theorem in the previous section. For comparison, in the latter subsection we briefly review a derivation similar to the traditional derivation presented, for example, in [10, 11, 15].

5.1. Derivation by Means of Wave Patterns

As in the scalar case, let us begin with the out-to-out translation when the wave functions on both sides of (53) are of an outgoing type and $t < r$. Note that the translation coefficients could be derived starting from (54) as well. Then, by taking far field limits on both sides according to (29), and using (31) and (32), as explained in Sec. 3, the equation (53) can be reduced to

$$(-i)^l\mathbf{U}_{l,m}(\hat{r})e^{ik\hat{r}\cdot\mathbf{t}} = \sum_{n=1}^{\infty}\sum_{p=-n}^{n}(-i)^n[A^{\text{out-out}}_{l,m;n,p}(\mathbf{t})\mathbf{U}_{n,p}(\hat{r}) + B^{\text{out-out}}_{l,m;n,p}(\mathbf{t})\mathbf{V}_{n,p}(\hat{r})] \quad (55)$$

where the exponent factor on the left-hand-side is due to the translation of the origin by $\mathbf{t}$. Owing to the orthogonality of the vector spherical harmonics (13)–(15) we then obtain

$$A^{\text{out-out}}_{l,m;n,p}(\mathbf{t}) = \frac{1}{n(n+1)}(-i)^{l-n}\int_B \mathbf{U}_{l,m}(\hat{r})\cdot\mathbf{U}^*_{n,p}(\hat{r})e^{ik\hat{r}\cdot\mathbf{t}}\,d\Omega(\hat{r}), \quad (56)$$

$$B^{\text{out-out}}_{l,m;n,p}(\mathbf{t}) = \frac{1}{n(n+1)}(-i)^{l-n}\int_B \mathbf{U}_{l,m}(\hat{r})\cdot\mathbf{V}^*_{n,p}(\hat{r})e^{ik\hat{r}\cdot\mathbf{t}}\,d\Omega(\hat{r}). \quad (57)$$

The coefficients could already be calculated from the above expressions by numerical integration. An explicit expressions can be obtained by substituting (25) into (56) and (57) and changing the order of the integration and the summation:

$$\begin{aligned} & A^{\text{out-out}}_{l,m;n,p}(\mathbf{t}) \\ = \; & \frac{1}{n(n+1)} 4\pi \sum_{q=0}^{\infty} \sum_{s=-q}^{q} (-i)^{l-n-q} \psi^{\text{in}}_{q,s}(\mathbf{t}) \int_B \mathbf{U}_{l,m} \cdot \mathbf{U}^*_{n,p} Y^*_{q,s} \, d\Omega, \quad (58) \\ & B^{\text{out-out}}_{l,m;n,p}(\mathbf{t}) \\ = \; & \frac{1}{n(n+1)} 4\pi \sum_{q=0}^{\infty} \sum_{s=-q}^{q} (-i)^{l-n-q} \psi^{\text{in}}_{q,s}(\mathbf{t}) \int_B \mathbf{U}_{l,m} \cdot \mathbf{V}^*_{n,p} Y^*_{q,s} \, d\Omega. \quad (59) \end{aligned}$$

In the formulas above, most of the terms in the summation in s are zero, just like in the scalar case, but this time the non-zero terms are not that easy to identify. However, we do not care for that, because at the end of this subsection, after deriving the expressions for the coefficients for the remaining translations, we write formulas for evaluating the vector translation coefficients by using the corresponding scalar coefficients. By doing so the zero terms are automatically dropped out.

The procedure for deriving the in-to-in translation is similar. In this case, the wave functions on both sides of (53) are of an incoming type. Then, by following the steps through (33) to (35), as explained in Sec. 3, the equation (53) can be reduced to a form identical to (55). Consequently, the coefficients of the in-to-in translation are the same as the coefficients of the out-to-out translation:

$$A^{\text{in-in}}_{l,m;n,p}(\mathbf{t}) = A^{\text{out-out}}_{l,m;n,p}(\mathbf{t}), \quad (60)$$

$$B^{\text{in-in}}_{l,m;n,p}(\mathbf{t}) = B^{\text{out-out}}_{l,m;n,p}(\mathbf{t}). \quad (61)$$

Consider next the out-to-in translation when the wave function on the left-hand-side of (53) is outgoing and those on the right-hand-side are incoming, and $t > r$. We assume that $A^{\text{out-in}}_{l,m;n,p}$ and $B^{\text{out-in}}_{l,m;n,p}$ have far field limits according to (29) and radiation patterns denoted by $(A^{\text{out-in}}_{l,m;n,p})_\infty$ and $(B^{\text{out-in}}_{l,m;n,p})_\infty$. Then, by taking the limits on both sides according to (29), now with respect to $\mathbf{t}$ instead of $\mathbf{r}$, we obtain

$$\begin{aligned} & \frac{1}{ik}(-i)^l \mathbf{U}_{l,m}(\hat{t}) e^{ik\hat{t}\cdot\mathbf{r}} \\ = & \sum_{n=1}^{\infty} \sum_{p=-n}^{n} [(A^{\text{out-in}}_{l,m;n,p})_\infty(\hat{t}) \mathbf{M}^{\text{in}}_{n,p}(\mathbf{r}) + (B^{\text{out-in}}_{l,m;n,p})_\infty(\hat{t}) \mathbf{N}^{\text{in}}_{n,p}(\mathbf{r})]. \quad (62) \end{aligned}$$

By comparing (62) to (38) it can be identified that

$$(A^{\text{out-in}}_{l,m;n,p})_\infty(\hat{t}) = \frac{1}{n(n+1)}\frac{1}{ik}(-i)^{l-n}4\pi\mathbf{U}_{l,m}(\hat{t})\cdot\mathbf{U}^*_{n,p}(\hat{t}). \tag{63}$$

This can be expanded in spherical harmonics as

$$(A^{\text{out-in}}_{l,m;n,p})_\infty(\hat{t}) = \frac{1}{n(n+1)}\frac{1}{ik}4\pi\sum_{q=0}^{\infty}\sum_{s=-q}^{q}(-i)^{l-n}Y_{q,s}(\hat{t})\int_B \mathbf{U}_{l,m}\cdot\mathbf{U}^*_{n,p}Y^*_{q,s}\,d\Omega, \tag{64}$$

and, since the coefficients of the expansions of an outgoing field in the spherical wave functions and the radiation pattern in the spherical harmonics are related, as explained in Sec. 3, this readily yields

$$A^{\text{out-in}}_{l,m;n,p}(\mathbf{t}) = \frac{1}{n(n+1)}4\pi\sum_{q=0}^{\infty}\sum_{s=-q}^{q}(-i)^{l-n-q}\psi^{\text{out}}_{q,s}(\mathbf{t})\int_B \mathbf{U}_{l,m}\cdot\mathbf{U}^*_{n,p}Y^*_{q,s}\,d\Omega. \tag{65}$$

The procedure for obtaining $B^{\text{out-in}}_{l,m;n,p}$ is similar, giving as a result

$$B^{\text{out-in}}_{l,m;n,p}(\mathbf{t}) = \frac{1}{n(n+1)}4\pi\sum_{q=0}^{\infty}\sum_{s=-q}^{q}(-i)^{l-n-q}\psi^{\text{out}}_{q,s}(\mathbf{t})\int_B \mathbf{U}_{l,m}\cdot\mathbf{V}^*_{n,p}Y^*_{q,s}\,d\Omega. \tag{66}$$

To conclude this subsection, we establish formulas by which the vector translation coefficients can be evaluated by taking advantage of the corresponding scalar coefficients. By comparing the above expressions for the coefficients of the different translations, we find that, just like in the scalar case, the only difference between them is the type of the wave functions in the summations. Hence, the following formulas can be written for the three translations once and for all. Therefore, in the following, we only consider the out-to-out translation. By expanding the dot product in (58) with aid of (A1) and (A2) in Appendix A and identifying the terms similar to the scalar translation (45), we find that

$$A_{l,m;n,p} = \frac{1}{n(n+1)}\left[\frac{1}{2}\sqrt{(l-m)(l+m+1)(n-p)(n+p+1)}a_{l,m+1;n,p+1}\right.$$

$$+\frac{1}{2}\sqrt{(l+m)(l-m+1)(n+p)(n-p+1)}a_{l,m-1;n,p-1}+mpa_{l,m;n,p}\Bigg]. \tag{67}$$

Similarly, by expanding the dot product in (59) with aid of (A1)–(A4) and identifying the terms similar to the scalar translation, we get

$$\begin{aligned}B_{l,m;n,p} = &-\frac{1}{2}\sqrt{(l-m)(l+m+1)}\\ &\cdot\left[\frac{1}{n+1}\sqrt{\frac{(n+p+1)(n+p+2)}{(2n+1)(2n+3)}}a_{l,m+1;n+1,p+1}\right.\\ &\left.+\frac{1}{n}\sqrt{\frac{(n-p-1)(n-p)}{(2n-1)(2n+1)}}a_{l,m+1;n-1,p+1}\right]\\ &+\frac{1}{2}\sqrt{(l+m)(l-m+1)}\\ &\cdot\left[\frac{1}{n+1}\sqrt{\frac{(n-p+1)(n-p+2)}{(2n+1)(2n+3)}}a_{l,m-1;n+1,p-1}\right.\\ &\left.+\frac{1}{n}\sqrt{\frac{(n+p-1)(n+p)}{(2n-1)(2n+1)}}a_{l,m-1;n-1,p-1}\right]\\ &+m\left[\frac{1}{n+1}\sqrt{\frac{(n+p+1)(n-p+1)}{(2n+1)(2n+3)}}a_{l,m;n+1,p}\right.\\ &\left.-\frac{1}{n}\sqrt{\frac{(n+p)(n-p)}{(2n-1)(2n+1)}}a_{l,m;n-1,p}\right].\end{aligned} \tag{68}$$

The above formulas, to the best knowledge of the authors, are new. In the next subsection we derive the formulas used traditionally. By comparing them to above ones, we notice that our formula for $A_{l,m;n,p}$ is simpler while the one for $B_{l,m;n,p}$ is more complicated.

5.2. Traditional Derivation

The traditional procedure for deriving the vector addition theorems starts by invoking the scalar addition theorem (41) in the definition (5) and noting that the gradient operator is invariant under translation; see, for example, [10, Eq. (12)], [11, Eq. (8)], [15, Eq. (D.19)], [16, Eq. (5)] or [17, Eq. (10)]:

$$\begin{aligned}\mathbf{M}_{l,m}(\mathbf{t}+\mathbf{r}) &= \nabla\psi_{l,m}(\mathbf{t}+\mathbf{r}) \times (\mathbf{t}+\mathbf{r}) \\ &= \sum_{n=1}^{\infty}\sum_{p=-n}^{n} a_{l,m;n,p}(\mathbf{t})[\nabla\psi_{n,p}(\mathbf{r}) \times \mathbf{t} + \mathbf{M}_{n,p}(\mathbf{r})].\end{aligned} \tag{69}$$

It then remains to find the coefficients of the expansion

$$\nabla\psi_{n,p}(\mathbf{r}) \times \mathbf{t} = \sum_{q=0}^{\infty}\sum_{s=-q}^{q} [c_{n,p;q,s}(\mathbf{t})\mathbf{M}_{q,s}(\mathbf{r}) + d_{n,p;q,s}(\mathbf{t})\mathbf{N}_{q,s}(\mathbf{r})]. \tag{70}$$

These coefficients too can be found by reducing the wave functions on both sides of the equation into the corresponding harmonics as explained in Sec. 3. By doing so we get

$$\begin{aligned}&ik\hat{r}(-i)^n Y_{n,p}(\hat{r}) \times \mathbf{t} \\ &= \sum_{q=0}^{\infty}\sum_{s=-q}^{q} (-i)^q [c_{n,p;q,s}(\mathbf{t})\mathbf{U}_{q,s}(\hat{r}) + d_{n,p;q,s}(\mathbf{t})\mathbf{V}_{q,s}(\hat{r})].\end{aligned} \tag{71}$$

The coefficients are then obtained by the orthogonality:

$$c_{n,p;q,s}(\mathbf{t}) = \frac{1}{q(q+1)}(-i)^{n-q}\int_B Y_{n,p} k\mathbf{t} \cdot \mathbf{V}^*_{q,s}\, d\Omega, \tag{72}$$

$$d_{n,p;q,s}(\mathbf{t}) = \frac{1}{q(q+1)}(-i)^{n-q}\int_B Y_{n,p} k\mathbf{t} \cdot \mathbf{U}^*_{q,s}\, d\Omega. \tag{73}$$

The above integrals can be evaluated by applying (A1)–(A4) in Appendix A and the orthogonality once more. Then, insertion of (70) into (69) and re-arrangement of terms yields

$$\begin{aligned}A_{l,m;n,p} &= a_{l,m;n,p} \\ &\quad -\frac{1}{2}k(t_x + it_y)\left[\frac{1}{n+1}\sqrt{\frac{(n+p+1)(n+p+2)}{(2n+1)(2n+3)}}a_{l,m;n+1,p+1}\right. \\ &\quad \left. -\frac{1}{n}\sqrt{\frac{(n-p-1)(n-p)}{(2n-1)(2n+1)}}a_{l,m;n-1,p+1}\right] \\ &\quad +\frac{1}{2}k(t_x - it_y)\left[\frac{1}{n+1}\sqrt{\frac{(n-p+1)(n-p+2)}{(2n+1)(2n+3)}}a_{l,m;n+1,p-1}\right.\end{aligned}$$

$$-\frac{1}{n}\sqrt{\frac{(n+p-1)(n+p)}{(2n-1)(2n+1)}}a_{l,m;n-1,p-1}\Bigg]$$
$$+kt_z\left[\frac{1}{n+1}\sqrt{\frac{(n+p+1)(n-p+1)}{(2n+1)(2n+3)}}a_{l,m;n+1,p}\right.$$
$$\left.+\frac{1}{n}\sqrt{\frac{(n+p)(n-p)}{(2n-1)(2n+1)}}a_{l,m;n-1,p}\right], \tag{74}$$

$$B_{l,m;n,p} = \frac{i}{n(n+1)}\left[\frac{1}{2}k(t_x+it_y)\sqrt{(n-p)(n+p+1)}a_{l,m;n,p+1}\right.$$
$$\left.+\frac{1}{2}k(t_x-it_y)\sqrt{(n+p)(n-p+1)}a_{l,m;n,p-1}+kt_zpa_{l,m;n,p}\right], \tag{75}$$

where t_x, t_y and t_z are the Cartesian components of $\mathbf{t}$. The formulas above are the same as, for example, in [16, Eqs. (12a) and (12b)].

6. TRANSLATION THROUGH ROTATION

Besides using the translation formulas derived above, the translation of a wave function can be performed through consecutive rotation, translation along the z'-axis of the new system of co-ordinates and inverse rotation. The advantage of this approach is that the number of non-zero coefficients in the above three sequential operations is one degree lower comparing to the straightforward translation. This approach is discussed, for example, in [4, 5, Sec. 5.3.6, 30] and for completeness it is briefly reviewed in the following. For the sake of clarity, this is done only in the scalar case; the vector case is essentially the same.

The rotational translation theorem for the spherical harmonics can be formulated as (see, for example, [3, App. A2] or [10, App. A1])

$$Y_{l,m}(\hat{r}') = \sum_{p=-l}^{l} D_{l,m;p}(\alpha,\beta,\gamma)Y_{l,p}(\hat{r}) \tag{76}$$

where α, β and γ are the three Euler angles of the rotation from $\hat{r}$ to $\hat{r}'$. The coefficient can be written as

$$D_{l,m;p}(\alpha,\beta,\gamma) = e^{im\alpha}d_{l,m;p}(\beta)e^{ip\gamma} \tag{77}$$

where

$$d_{l,m;p}(\beta) = \sqrt{\frac{(l+p)!(l-p)!}{(l+m)!(l-m)!}} \sum_u \binom{l+m}{l-p-u} \binom{l-m}{u} \cdot (-1)^{l-p-u} \left(\cos\frac{\beta}{2}\right)^{m+p+2u} \left(\sin\frac{\beta}{2}\right)^{2l-m-p-2u} \quad (78)$$

where the summation is carried out over all u for which the arguments of the binomials are positive. Since the order of the spherical harmonics is preserved in the rotation, the rotation for the spherical wave function is obtained simply by multiplying both sides of (76) by $z_l(r)$; and further, since the gradient operator is invariant under rotation, the rotation is found to be similar for the spherical vector wave functions as well.

Now, the translation of the spherical wave function $\psi_{l,m}$ by $\mathbf{t}$, in general formulated as in (41), and with $\mathbf{t}$ having the spherical angles ϑ and φ, can be separated into the following three steps:

$$\psi_{l,m}(\mathbf{t}+\mathbf{r}) = \sum_{s=-l}^{l} D_{l,m;s}(-\varphi,-\vartheta,0)\psi_{l,s}(\hat{z}'t+\mathbf{r}'), \quad (79)$$

$$\psi_{l,s}(\hat{z}'t+\mathbf{r}') = \sum_{n=0}^{\infty} a_{l,s;n,s}(\hat{z}'t)\psi_{n,s}(\mathbf{r}'), \quad (80)$$

$$\psi_{n,s}(\mathbf{r}') = \sum_{p=-n}^{n} D_{n,s;p}(\varphi,\vartheta,0)\psi_{n,p}(\mathbf{r}). \quad (81)$$

Going backwards from (81) to (79), we first rotate the system of co-ordinates so that the z'-axis points in the direction of $\mathbf{t}$; then we perform the translation along the z'-axis; and finally we rotate the system of co-ordinates into the original alignment.

Comparing to (41), we note that at every step from (81) to (79) the coefficient has three different indices instead of four and one summation instead of two. This suggests that the computational cost of the translation by (79)–(81) is one degree lower ($\sim N^3$) compared to that of the translation by (41) ($\sim N^4$).

7. CONCLUSION

In this paper we derived the translational addition theorems for the spherical scalar and vector wave functions in a simple and unified manner by applying the fundamental concepts of radiation pattern and incoming wave pattern. In addition, we obtained alternative and partly simpler expressions for the translation coefficients in the vector case, which, to the best knowledge of the authors, have not been published elsewhere.

APPENDIX A. COMPONENTS OF THE VECTOR SPHERICAL HARMONICS

The vector spherical harmonics defined as (10) and (11) can be decomposed by making use of the recurrence relations of the associated Legendre functions; see, for example, [26, Sec. 7.3, Eqs. (13) and (14)]. It turns out that expressions consisting purely of spherical harmonics can be found only for the rectangular components; in particular, the $(\hat{x} \pm i\hat{y})$- and $\hat{z}$-component are the simplest ones. They are

$$(\hat{x} \pm i\hat{y}) \cdot \mathbf{U}_{l,m} = -i\sqrt{(l \mp m)(l \pm m + 1)}Y_{l,m\pm1}, \tag{A1}$$

$$\hat{z} \cdot \mathbf{U}_{l,m} = -imY_{l,m}, \tag{A2}$$

$$(\hat{x} \pm i\hat{y}) \cdot \mathbf{V}_{l,m} = \pm il\sqrt{\frac{(l \pm m + 1)(l \pm m + 2)}{(2l+1)(2l+3)}}Y_{l+1,m\pm1} \pm i(l+1)\sqrt{\frac{(l \mp m - 1)(l \mp m)}{(2l-1)(2l+1)}}Y_{l-1,m\pm1}, \tag{A3}$$

$$\hat{z} \cdot \mathbf{V}_{l,m} = -il\sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}Y_{l+1,m} + i(l+1)\sqrt{\frac{(l+m)(l-m)}{(2l-1)(2l+1)}}Y_{l-1,m}. \tag{A4}$$

The components of $\mathbf{U}_{l,m}$ can also be found in an irreducible spherical tensor [12, Eq. (2)] familiar in quantum mechanics.

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