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METHOD OF LARGEST EXTENDED CIRCLE FOR THE CAPACITANCE OF ARBITRARILY SHAPED CONDUCTING PLATES

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Abstract—The most difficult step in the analysis of the capacitance of arbitrarily shaped conducting plates is the determination of the electric center, or the expansion point of the charge density. This paper presents the generalized Huygens' principle, which indicates that the charge distribution on a conducting plate of convex shape has a tendency to be a circle before approaching the fringe. Therefore, the center of the largest extended circle can be taken as the electric center. The agreement with numerical methods is demonstrated.

1. INTRODUCTION

With the quick development of microwave communication, the line between the technique of wave and the technique of circuit becomes more and more obscure in engineering. We can not only add some lumped elements to a distributed parameter filter, but also consider distributed parameters in a lumped element system. Therefore, the capacitance of conducting plates and the inductance of conducting wires of arbitrary shape attract many scholars' attention.

Two problems arise from the determination of the closed form for the capacitance of the conducting plate. One is the charge distribution on the plate, and another is the optimal expansion center of the charge density, or what is called the electric center.

Reference [1] take a disc whose analytic results has been obtained as an example. Considering that the charge density on the edge of the disc is a singular distribution, namely

$$\sigma(r) = \frac{\sigma_0}{\sqrt{1 - (r/r_0)^2}} \tag{1}$$

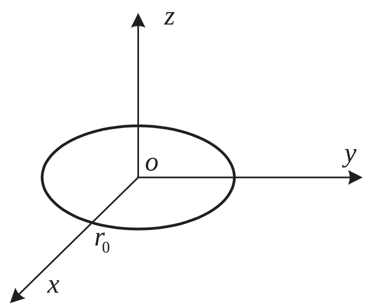


Figure 1. Infinitely thin conducting disc.

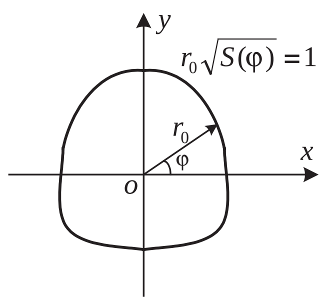


Figure 2. Symmetrical convex conducting plate.

We can obtain the accurate capacitance.

$$C_0 = \frac{Q}{V} = 8\varepsilon_0 r_0 = \frac{2r_0}{9\pi} \cdot 10^{-9} F \tag{2}$$

For a general symmetric conducting plate, the boundary curve equation can be introduced. It is

$$r_0\sqrt{S(\varphi)} = 1 \tag{3}$$

And then the charge density is expanded at the symmetric center, so we have

$$\sigma(r, \varphi) = \frac{\sigma_0}{\sqrt{1 - r^2 S(\varphi)}} \tag{4}$$

Finally, the closed form of the capacitance C is

$$C = \frac{Q}{V} = 8\varepsilon_0 \int_0^{\frac{\pi}{2}} \frac{d\varphi}{S(\varphi)} \Big/ \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{S(\varphi)}} \tag{5}$$

Applying formula (5) to an elliptical disc, in which

$$S(\varphi) = \frac{\cos^2 \varphi}{a^2} + \frac{\sin^2 \varphi}{b^2} \tag{6}$$

and $b > a$ is supposed. So the capacitance of an elliptical disc is

$$C = \frac{4\pi\varepsilon_0 a}{K(k)} \cdot \left(\frac{b}{a}\right) F \tag{7}$$

in which $K(k)$ represents the complete elliptical functions of the first kind with modulus k, and

$$k = \frac{\sqrt{b^2 - a^2}}{b} \quad (b > a) \tag{8}$$

Formula (7) is also an accurate result by [2].

2. SELECTION OF THE ELECTRIC CENTER

The success of a closed form lies mainly in the fact that the assumed singular distribution of the charge density is in accordance with the practical distribution. However, the determination of the electric center remains to be a problem. Different electric center may lead to different results.

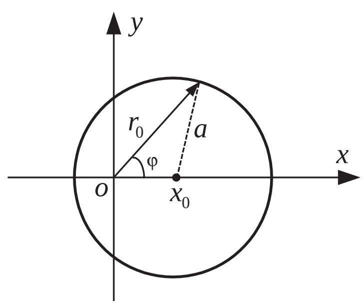


Figure 3. Disc with eccentric center.

Figure 3 shows a conducting plate expanded at an eccentric center. And the equation of which is

$$(x - x_0)^2 + y^2 = a^2 \tag{9}$$

Another equation can be obtained by normalized polar coordinate. It is

$$(\bar{r}_0 - \bar{x}_0 \cos\varphi)^2 + \bar{x}_0^2 \sin^2\varphi = 1 \tag{10}$$

in which $\bar{r}_0 = r_0/a$ and $\bar{x}_0 = x_0/a$ are normalized values. Note that $\bar{r}_0$ and r_0 are both on the fringe of the disc. So the function of the charge density of the disc is

$$\begin{aligned}\sigma(r,\varphi) &= \frac{\sigma_0}{\sqrt{1-(\bar{x}_0\sin\varphi)^2-(r-\bar{x}_0\cos\varphi)^2}} \\ &= \frac{\sigma_0}{\sqrt{1-(\bar{x}_0\sin\varphi)^2}\sqrt{1-\left[\frac{\bar{r}-\bar{x}_0\cos\varphi}{1-(\bar{x}_0\sin\varphi)^2}\right]^2}}\end{aligned} \tag{11}$$

in which r is a variable and $r \leq r_0$.

In this case, the original quarter symmetry of the disc has turned to be a half even symmetry, and the potential V at the eccentric center O is

$$V = \frac{2\sigma_0 a}{4\pi\varepsilon_0}\int_0^\pi d\varphi\int_0^{\frac{(\bar{r}_0-\bar{x}_0\cos\varphi)}{\sqrt{1-(\bar{x}_0\sin\varphi)^2}}}\frac{du}{\sqrt{1-u^2}} = \frac{\pi a\sigma_0}{4\varepsilon_0} \tag{12}$$

Note that $(\bar{r}_0 - \bar{x}_0\cos\varphi) = \sqrt{1-(\bar{x}_0\sin\varphi)^2}$ has been taken into account in the derivation process of the formula above.

Besides, the total charge Q is

$$\begin{aligned}Q &= 2a^2\sigma_0\int_0^\pi d\varphi\int_0^{\bar{r}_0}\frac{\bar{r}d\bar{r}}{\sqrt{1-(\bar{x}_0\sin\varphi)^2}\sqrt{1-\left[\frac{(\bar{r}-\bar{x}_0\cos\varphi)}{1-(\bar{x}_0\sin\varphi)^2}\right]^2}} \\ &= 2a^2\sigma_0\left[\int_0^\pi\sqrt{1-(\bar{x}_0\sin\varphi)^2}d\varphi+\int_0^\pi\bar{x}_0\cos\varphi d\varphi\int_0^1\frac{du}{\sqrt{1-u^2}}\right] \\ &= 4a^2\sigma_0 E(\bar{x}_0)\end{aligned} \tag{13}$$

in which $E(\bar{x}_0)$ is the complete elliptical integral of the second kind with modulus $\bar{x}_0$.

Finally, the capacitance of the disc expanded at the eccentric center is

$$C = \frac{Q}{V} = 8\varepsilon_0 a\left[\frac{E(\bar{x}_0)}{\pi/2}\right]F \tag{14}$$

Figure 4 shows the functional relationship of the normalized capacitance $\bar{C} = C/C_0$ with the modulus $\bar{x}_0$, in which $C_0 = 8\varepsilon_0 a$ is the capacitance obtained by expanding the charge at the center of the disc.

The example forcefully demonstrates that the determination of the electric center is of great importance. It is proved that the capacitance is variational standing for the charge test function. Proper electric center could lead to

$$\max C \quad (\bar{x}_0 = 0) \tag{15}$$

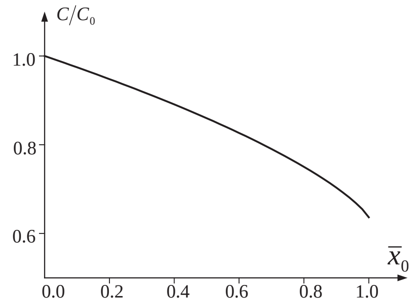


Figure 4. Relationship of capacitance $\bar{C} = C/C_0$ with $\bar{x}_0$.

For symmetric objects like the disc, the geometrical center coincides with the electric center. However, new methods for choosing electric center must be given for objects of arbitrary shape.

3. LARGEST EXTENDED CIRCLE METHOD

Huygens' principle is of special importance in electromagnetism. There are two wonderful ideas. One is that every point at the wavefront has wavelets. Another is that every wavelet transmits spherical waves outwards. Huygens' principle implies the most important idea, that is, waves have a tendency to be a sphere, cylinder or circle without the restriction of the boundary conditions.

In [3], Huygens' principle is advanced further and generalized Huygens' principle is introduced. It means that the charge distribution on a conducting plate tends to be concentric circles before reaching the edge.

Whether be static field or dynamic field, wave or non-wave, this property can be termed as electromagnetic inertia, just as Newton's mechanic inertia.

Therefore, we can put forward the selecting principle of the electric center of a convex edged conducting plate. The electric center of a convex edged conducting plate should be the center of its largest extended circle, as shown in Figure 5.

This method can satisfy the free state of the charge distribution, or being concentric circles, as much as possible.

4. EXAMPLES

The triangle plate of arbitrary shape is the most important and basic example, whose largest extended circle is its encircle, as shown in Figure 6.

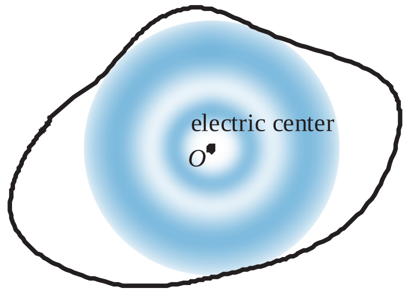


Figure 5. Center of largest extended circle.

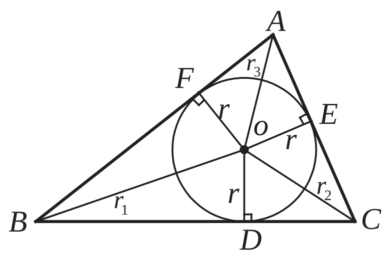


Figure 6. Arbitrary triangle plate.

r is the radius of the encircle. Let

$$\begin{cases} x = BD = BF \\ y = CD = CE \\ z = AE = AF \end{cases} \tag{16}$$

Δ represents the area of the triangle, and the basic geometrical relations is shown in Table 1.

If the center of the encircle is taken as the expansion center, the triangle ΔABC can be divided into three basic triangles, ΔOAB, ΔOBC and ΔOCA.

In which AB, BC and AC are the edges of the plate. The capacitance of ΔABC is

$$C = \sum_{i=1}^{3} Q_i \Big/ \sum_{i=1}^{3} V_i \tag{17}$$

in which Q_i and V_i represent the charge and potential of the ith triangle. In general, we establish the basic triangle OMN with MN as the edge of the plate and ON and OM as the common edge between different triangles, as shown in Figure 7.

Table 1. Basic geometrical relations in a triangle.

$S = \frac{1}{2}(a+b+c)$
$r = \frac{\Delta}{S}$
$r_1 = \sqrt{x^2 + r^2}$
$r_2 = \sqrt{y^2 + r^2}$
$r_3 = \sqrt{z^2 + r^2}$

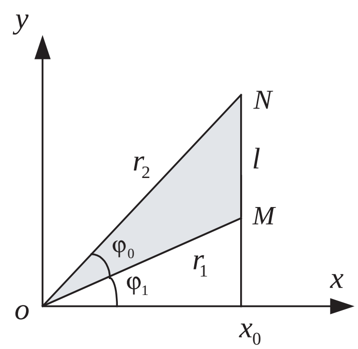


Figure 7. Basic triangle MON.

Under this condition the charge distribution function expanded at point O is

$$\sigma(r,\varphi) = \frac{\sigma_0}{\sqrt{1-\left(\frac{x}{x_0}\right)^2}} = \frac{\sigma_0}{\sqrt{1-\left(\frac{r\cos\varphi}{x_0}\right)^2}} \tag{18}$$

and the potential V_i of the ΔOMN at the center is

$$\begin{aligned} V_i &= \frac{1}{4\pi\varepsilon_0}\int_{\varphi_1}^{\varphi_1+\varphi_2} d\varphi \int_0^{r_0} \frac{\sigma_0 dr}{\sqrt{1-\left(\frac{r\cos\varphi}{x_0}\right)^2}} \\ &= \frac{\sigma_0 x_0}{8\varepsilon_0}\left\{\frac{1}{2}\ln\left[\frac{1-\sin\varphi_1}{1+\sin\varphi_1}\cdot\frac{1+\sin(\varphi_1+\varphi_0)}{1-\sin(\varphi_1+\varphi_0)}\right]\right\} \end{aligned} \tag{19}$$

in which,

$$\begin{cases} \sin\varphi_1 = \dfrac{r_2\cos\varphi_0 - r_1}{l} \\ \sin(\varphi_1+\varphi_0) = \dfrac{r_2 - r_1\cos\varphi_0}{l} \\ \tan\varphi_1 = \dfrac{r_2\cos\varphi_0 - r_1}{r_2\sin\varphi_0} \\ \tan(\varphi_1+\varphi_0) = \dfrac{r_2 - r_1\cos\varphi_0}{r_1\sin\varphi_0} \end{cases} \tag{20}$$

On account of

$$\cos\varphi_0 = \frac{r_1^2 + r_2^2 - l^2}{2r_1r_2} \tag{21}$$

$$\frac{1}{2}\ln\left[\frac{1-\sin\varphi_1}{1+\sin\varphi_1}\cdot\frac{1+\sin(\varphi_1+\varphi_0)}{1-\sin(\varphi_1+\varphi_0)}\right] = \ln\left[\frac{r_1+r_2+l}{r_1+r_2-l}\right] \tag{22}$$

that is

$$V_i = \frac{\sigma_0 x_0}{8\varepsilon_0}\ln\left(\frac{r_1+r_2+l}{r_1+r_2-l}\right) \tag{23}$$

Similarly, the charge Q_i of the ΔMON is

$$\begin{aligned} Q_i &= \int_{\varphi_1}^{\varphi_1+\varphi_0} d\varphi \int_0^{r_0} \sigma r \Big/ \sqrt{1-\left(\frac{r\cos\varphi}{x_0}\right)^2}\, dr \\ &= \sigma_0 r_1 r_2 \sin\varphi_0 \end{aligned} \tag{24}$$

in which

$$x_0 = \frac{r_1 r_2 \sin\varphi_0}{l} \tag{25}$$

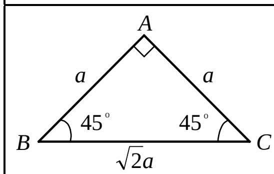
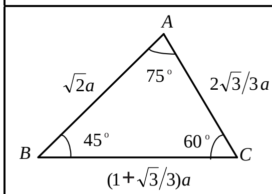
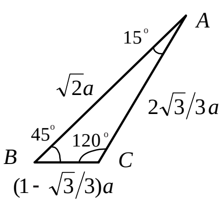
In the original triangle ABC, considering x_0 equals the radius of the encircle and formula (17), the capacitance of the triangle ΔABC is

$$C = \frac{8\varepsilon_0(a+b+c)}{\ln\left(\frac{r_1+r_2+a}{r_1+r_2-a}\right) + \ln\left(\frac{r_2+r_3+b}{r_2+r_3-b}\right) + \ln\left(\frac{r_3+r_1+c}{r_3+r_1-c}\right)} \tag{26}$$

Formula (26) represents the capacitance of the conducting plate with the center of the encircle of the triangle as the electric center. Table 2 shows the comparison of the capacitances of the triangles obtained by formula (26) and by the moment method [3–6]. It demonstrates that the relative error is less than 3%.

Note that there are more than one largest extended circle for some plates of arbitrary shape. In this case, a more symmetric center should be adopted. Take the parallelogram conducting plate $ABCD$ in Figure 8 for example. There are many largest extended circles, but

Table 2. Comparison of capacitances of different triangles by two methods.

Different triangles	Incircle electric center $C/8\varepsilon_0 a$	Moment Method $C/8\varepsilon_0 a$	Relative error $\delta C = \frac{C - C_0}{C_0}$
	0.41524	0.42797	-0.02975
	0.51670	0.53054	-0.02609
	0.30593	0.31500	-0.02877

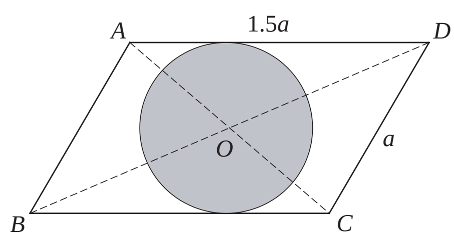


Figure 8. Largest extended circle of parallelogram.

the one whose center is in accordance with the symmetric center O should be adopted.

Similarly we lead to

$$C = 0.617075033 \cdot 8\varepsilon_0 a$$

$$C_0 = 0.62935 \cdot 8\varepsilon_0 a$$
$$\delta C = -1.950\%$$

and the two values are in good agreement with each other.

5. CONCLUSION

The difficulty to determine the closed form of the capacitance of a conducting plate of arbitrary shaped lies mainly in the selection of the electric center, or the charge expansion point.

In this paper, generalized Huygens' principle that the charge distribution has a tendency to be concentric circles as much as possible is introduced. Therefore, it is proper to take the center of the largest extended circle as the electric center. And it is fully proved by the practical examples.

REFERENCES

1. Liang, C. H., L. Li, and H. Q. Zhi, "Asymptotic closed form for the capacitance of an arbitrarily shaped conducting plate," *IEE Proc.-Microw. Antennas*, Vol. 151, No. 3, 217–220, June 2004.
2. Stratton, J. A., *Electromagnetic Theory*, McGraw-Hill, New York, 1941.
3. Harrington, R. F., *Field Computation by Moment Methods*, IEEE Press, New York, 1993.
4. Cheng, D. K. and C. H. Liang, "Thinning technique for moment method solutions," *Proc. IEEE*, Vol. 71, No. 2, 265–266, 1983.
5. Bancroft, R., "A note on the moment method solution for the capacitance of a conducting flat plate," *IEEE Trans. Antennas Propag.*, Vol. 45, No. 11, 1704, 1997.
6. Kuo, J. T. and K. Y. Su, "Analytical evaluation of the mom matrix elements for the capacitance of a charged plate," *IEEE Trans. Microw. Theory. Tech.*, Vol. 50, No. 5, 1435–1436, 2002.
7. Collin, R. E., *Field Theory of Guided Waves*, McGraw-Hill, New York, 1960.
8. Shumpert, T. H. and D. J. Galloway, "Capacitance bounds and equivalent radius," *IEEE Trans. Antennas Propag.*, Vol. 25, No. 2, 284–286, 1977.
9. Wang, C. F., L. W. Li, P. S. Kooi, and M. S. Leong, "Efficient capacitance computation for three-dimensional structures based on adaptive integral method," *Progress In Electromagnetics Research*, PIER 30, 33–46, 2001.