NONLINEAR STABILITY ANALYSIS OF AN OSCILLATOR WITH DISTRIBUTED ELEMENT RESONATOR

H. Vahdati and A. Abdipour

Microwave/mm-Wave and Wireless Communication Research Lab
Radiocommunication Center of Excellence
Electrical Engineering Department
Amirkabir University of Technology (Tehran Polytechnic)
Hafez Ave., Tehran 15914, Iran

Abstract—In this paper a complete analysis to the stability of a microwave oscillator with distributed element resonator is presented. In this type of oscillators, the circuit description changes form ordinary differential equations to partial deferential equations. In this paper a Gunn diode oscillator with distributed elements resonator is analyzed. The instability condition of the startup phase and the stability condition of the steady state oscillation is investigated.

1. INTRODUCTION

Using of the distributed model is ubiquitous for microwave and milliliter wave circuits [1]. While the lumped circuits have been described using the ordinary differential equations "*ODE*", the distributed element circuits have been expressed using the partial differential equations "*PDE*". Theoretically, the stability issue of these two types of equations are different [2].

In this paper, the PDE of a Gunn oscillator has been investigated for its stability. Usually these types of oscillators have been analyzed using the lumped elements equivalent of the distributed element resonator [3]. But the distributed circuits could have multiple resonance frequencies and the lumped circuits could have only one, and hence the lumped elements equivalent is not adequate. On the other hand these types of oscillators are more prone to instability and even chaos [4–6]. While there are several tools for stability analysis of ODE s [7–13], there is a few for PDE s. And the most important tool for PDE is separation of variables method [14]. In the following
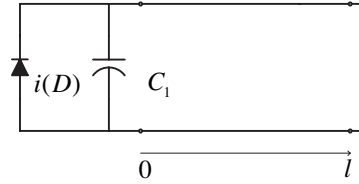
Figure 1. A model of Gunn oscillator with transmission line resonator.

sections, the stability of a Gunn oscillator at different phases from the start up to steady state has been studied.

2. FORMULATION OF A GUNN DIODE OSCILLATOR WITH DISTRIBUTED RESONATOR

A configuration of a Gunn oscillator has been shown in Fig. 1. Assume that the transmission line has a length of l with constructive parameters of $R(\frac{\Omega}{m})$, $L(\frac{H}{m})$, $C(\frac{F}{m})$ and very small resistance i.e.,

$$\mu = \frac{Rl}{\sqrt{\frac{L}{C}}} \ll 1 \tag{1}$$

Using the telegraphic equations and the boundary conditions, the mathematical model of the oscillator is:

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + RC\frac{\partial v}{\partial t} \tag{2}$$

$$\frac{\partial v}{\partial x} = LC_1\frac{\partial^2 v}{\partial t^2} - L\frac{\partial i_D}{\partial v}\cdot\frac{\partial v}{\partial t} + RC_1\frac{\partial v}{\partial t} - Ri_D(v)$$

$$v(l, t) = 0$$

The I-V characteristics of the Gunn diode which is the active element of the oscillator is

$$i_D(v) = S_0 v\left(1 - \frac{v^2}{3.v_0^2}\right) \tag{3}$$

Where S_0 denotes the negative small signal resistance of the Gunn diode. By normalizing the variables as in (4)

$$y = \frac{x}{l}, \quad \tau = \frac{t}{l\sqrt{LC}}, \quad u = \frac{v}{v_0}, \quad \gamma = \frac{C_1}{C.l}, \quad S = \frac{S_0 L}{RCl} \tag{4}$$

the (2) could be simplified as

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial \tau^2} = \mu\frac{\partial u}{\partial \tau}$$

$$\frac{\partial u}{\partial y} - \gamma \frac{\partial^2 u}{\partial \tau^2} = \mu \left[\gamma - S \left(1 - u^2 \right) \right] \frac{\partial u}{\partial \tau} \tag{5}$$
$$u(1, \tau) = 0$$

This is a hyperbolic partial differential equation with nonhomogeneous boundary conditions. The first step in analyzing the oscillator is finding its resonance frequencies [15–18]. For this purpose, it is assumed that the circuit is lossless i.e., $\mu = 0$ and hence (6) should be solved,

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial \tau^2} = 0 \tag{6}$$
$$\frac{\partial u}{\partial y} - \gamma \frac{\partial^2 u}{\partial \tau^2} = 0$$
$$u(1, \tau) = 0$$

Using the separation of variables method, the solution of (6) is,

$$u = (A \cos \Lambda y + B \sin \Lambda y) \cos(\omega \tau + \phi) \tag{7}$$

Applying the boundary condition at $y = 0$ and $y = 1$, it could be found that the resonance frequencies satisfy (8),

$$\cot \omega_k = \gamma \omega_k \tag{8}$$

This is a nonlinear equation and could be solved numerically or graphically Fig. 2. It should be noted that the solutions of (8) are not equidistant, it means that if ω is a resonance frequency then $n\omega$ is not another resonance frequency. It should be also noted that this solution is a normalized one and the actual resonant frequency is:

$$\Omega = \frac{\omega_k}{l\sqrt{LC}} \tag{9}$$

For a typical oscillator with $C_1 = 1\,\text{pf}$, $S_0 = 0.02$, $v_0 = 1$ and $l = 100\,\text{mm}$ on a RO4003 substrate with $\epsilon_r = 3.5$, the resonance frequencies are

$$\Omega = 0.37, 1.11, 1.87, 2.63, \ldots, 20.5, 21.6 \text{ GHz} \tag{10}$$

The existence of multiple resonance frequencies could be verified using the commercial microwave CAD s. The Nyquist diagram Fig. 3, turn around the point (0,1) for several times and each crossing the x-axis is correspond to a resonance frequency.

Graphical solution for resonance frequencies
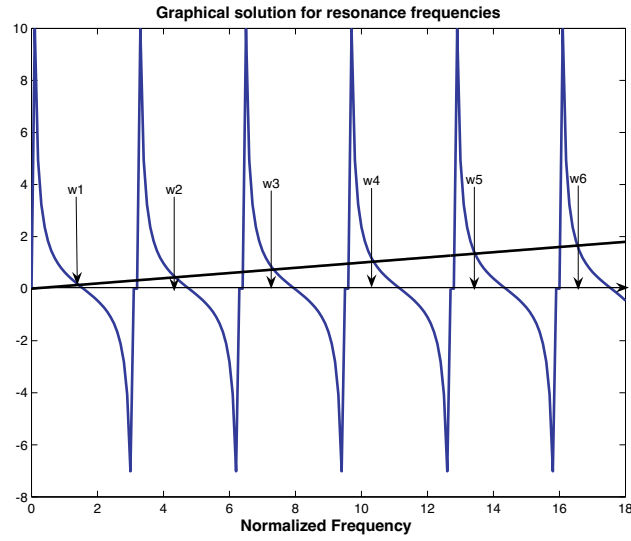
Figure 2. Multiple resonance frequency in a distributed circuit.

freq (100.0MHz to 20.00GHz)
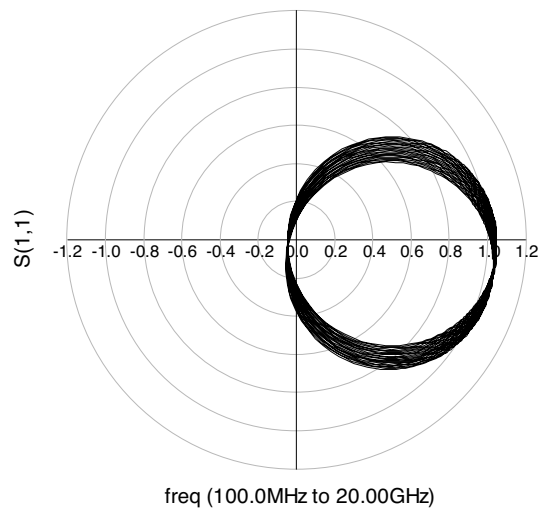
Figure 3. Nyquist diagram of the oscillator using the tools of ADS software.

3. STABILITY ANALYSIS AT THE STARTUP PHASE

For stability analysis of the startup phase, the oscillator equations are linearize around the operation point.

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial \tau^2} = 0 \tag{11}$$

$$\frac{\partial u}{\partial y} - \gamma \frac{\partial^2 u}{\partial \tau^2} = \mu[\gamma - S]\frac{\partial u}{\partial \tau}$$

$$u(1, \tau) = 0$$

The general solution of (11) could be written as

$$u = Y(y).T(t) = \left(A.e^{\Lambda y} + Be^{-\Lambda y}\right)\left(C.e^{P_1\tau} + D.e^{P_2\tau}\right) \quad (12)$$

where $P_{1,2} = \frac{-\mu \mp \sqrt{\mu^2 + 4\Lambda^2}}{2}$. The exponents of the exponential functions determine the response stability. Applying the boundary conditions at $y = 0$ and $y = 1$, it could be found that,

$$\frac{B}{A} = \mp j\mu S - \Lambda \gamma \quad (13)$$

$$\frac{B}{A} = -\cot \Lambda$$

By (13) and using the fact that as $\mu \to 0$, $\Lambda \to \omega$, it is reasonable to expand Λ as

$$\Lambda = \omega + \mu \Lambda_1 \quad (14)$$

Substituting (14) in (9), it could be obtained

$$\Lambda_1 = \pm jS/\left(1 + \gamma + \gamma^2 * \omega^2\right) \quad (15)$$

Hence, the instability "startup" condition of the DC point is

$$S > \left(1 + \gamma + \gamma^2\omega^2\right)/2 \quad (16)$$

The two sides of this condition has been plotted in Fig. 4. It shows that, when ω increases, the possibility of the oscillation startup decreases. Hence the oscillator can oscillate at a limited number of its resonance frequencies. Using the finite difference method, the oscillator has been analyzed. Applying the finite difference method to the oscillator, needs some cautions about its numerical stability [20–23]. In Fig. 5, it has been shown that when the circuit is perturbed by a small pulse, the oscillator enters its startup phase and finally settles to a steady state response. Using the harmonic balance algorithm and with the first resonance frequency in a microwave CAD like ADS, the steady state solution could be found. This response has been plotted at the bottom of Fig. 5. Here there arises an important question and it is, "as the circuit has several resonance frequencies and for each resonance frequency, the harmonic balance converge to specific response, which of them, is the actual response?". This is the issue of oscillation stability analysis.
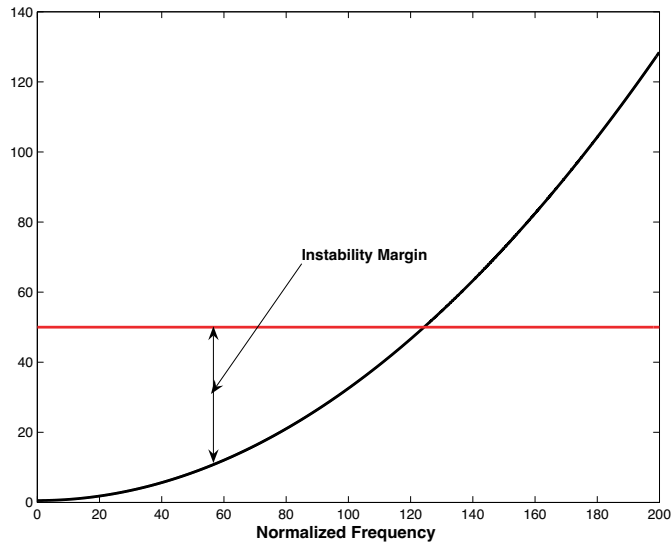
Figure 4. Instability margin for the startup phase.
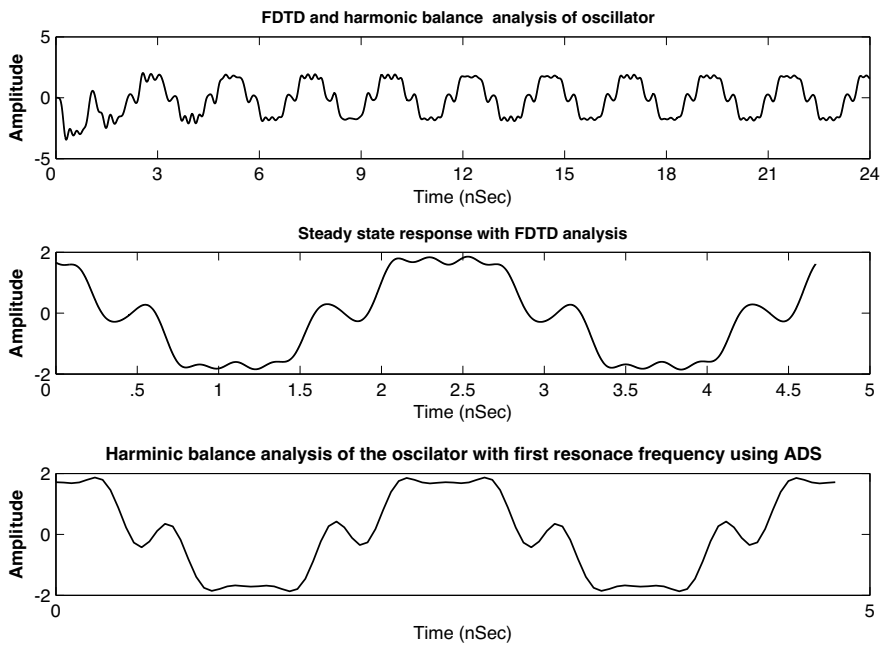
Figure 5. Oscillator analysis by FDTD with a pulse perturbation and the comparison of its steady state response with ADS solution.

4. STABILITY ANALYSIS OF THE STEADY STATE RESPONSE OF THE OSCILLATOR

To check the the oscillation stability, the reaction of steady state response to small perturbations should be analyzed. As shown in Appendix A, the steady state response of the oscillator is:

$$u = A_k(\cos(\omega_k y) - \omega_k \gamma \sin(\omega_k)) \cos(\omega_k \tau) \tag{17}$$

where

$$A_k = 4 \left[1 - \left(1 + \gamma + \omega^2 \gamma^2 \right) / 2 \right] \tag{18}$$

For stability analysis a small perturbation is exerted to the steady state solution and it is checked, whether the effects of this perturbation vanishes during the time or not [24, 25]? If $V_s(y, \tau)$ be the steady state response and $\delta(y, \tau)$ be a small perturbation, then we apply (19) to the oscillator equation.

$$v(y, \tau) = v_s(y, \tau) + \delta(y, \tau) \tag{19}$$

By substitution of (19) in (7), we could write:

$$\frac{\partial^2 \delta}{\partial y^2} - \frac{\partial^2 \delta}{\partial \tau^2} = \mu \frac{\partial \delta}{\partial \tau} \tag{20}$$

$$\frac{\partial \delta}{\partial y} - \gamma \frac{\partial^2 \delta}{\partial \tau^2} = \mu \left[\gamma - S \left(1 - (A_k \cos \omega_k)^2 \right) \right] \frac{\partial \delta}{\partial \tau}$$

$$\delta(1, \tau) = 0$$

Using the relation $\cos^2 \omega_k \tau = (1 + \cos 2\omega_k \tau)/2$, and the fact that $2\omega_k$ is not a resonance frequency, the (20) could be simplified as (21)

$$\frac{\partial^2 \delta}{\partial y^2} - \frac{\partial^2 \delta}{\partial \tau^2} = \mu \frac{\partial \delta}{\partial \tau} \tag{21}$$

$$\frac{\partial \delta}{\partial y} - \gamma \frac{\partial^2 \delta}{\partial \tau^2} = \mu \left[\gamma - S \left(1 - \left(A_k^2 \right) / 2 \right) \right] \frac{\partial \delta}{\partial \tau}$$

$$\delta(1, \tau) = 0$$

The Equation (21) has the same form of (9) which we encountered in the stability analysis of the startup phase. Here $S(1 - (A_k^2)/2)$ has been replaced S in (9). Hence the oscillator stability condition can be derived as (22).

$$A_k^2 > 2 \left[1 - \left(1 + \gamma + \gamma^2 \omega_i^2 \right) / 2S \right] \tag{22}$$

Then the oscillation with the amplitude of A_k is stable if (22) is satisfied for all resonance frequencies. In applying this condition to the above Gunn oscillator, it could be seen that (22) is violated for some resonance frequency e.g., for resonance frequency of 20.5 GHz. Then the oscillator is unstable. The oscillation instability can be verified by time domain analysis. As shown in Fig. 6, when the oscillator is perturbed with a small random signal, its steady state response is different from 5 and this shows the instability of this oscillator.
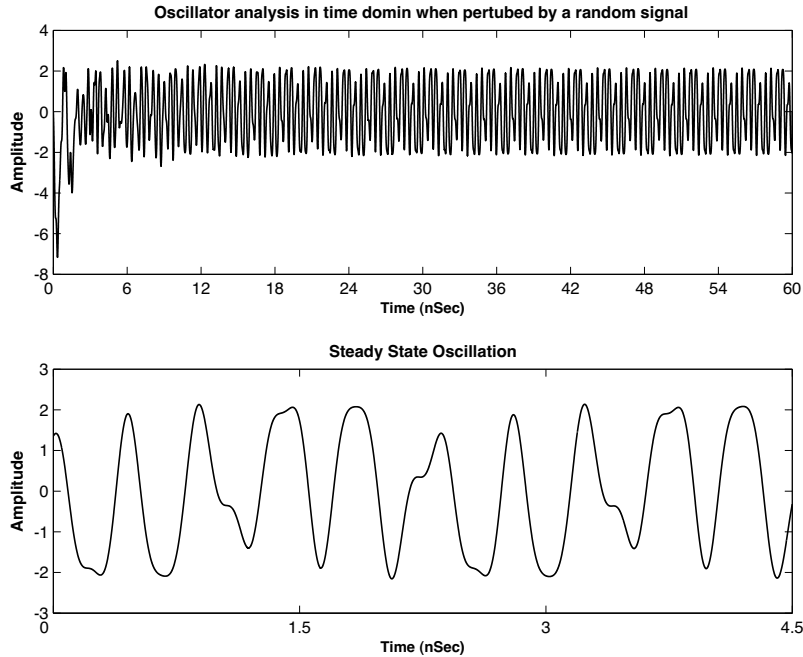
Figure 6. The oscillator response when perturbed by a noise like perturbation.

5. CONCLUSION

In this paper the stability of a microwave oscillator with distributed elements resonator has been studied. This oscillator has been described by its partial differential equation with inhomogeneous boundary conditions. Here an analysis of the startup phase, steady state behavior and oscillation stability have been presented. The instability condition for the startup phase and the stability condition for the oscillation have been derived theoretically. The oscillator response

has been also simulated numerically. Increasing applications of high frequency circuits with transmission line elements which convert the circuit description from the ODE to PDE, clarify the need for such investigation. This type of theoretical analysis is also necessary as a criterium for the correctness of the simulation and numerical methods.

APPENDIX A. STEADY STATE RESPONSE

Clearly an oscillator oscillates at it its resonance frequencies and we should only find its amplitude. For this purpose the PDE (5) should be solve. We assume the solution of the form,

$$u = u_0 + \mu u_1 \tag{A1}$$

where u_0 is the solution of the lossless circuit "$\mu = 0$" which is derived in the first section of this paper. In this case

$$u_0 = A(\cos(\omega y) - \omega\gamma\sin(\omega y))\cos(\omega\tau) \tag{A2}$$

and ω is the resonance frequency (8) and A is the oscillation amplitude. Substituting the (A2) in (5) and ignoring $\mu\frac{\partial v_1}{\partial\tau}$ we have

$$\frac{\partial^2 u_1}{\partial y^2} - \frac{\partial^2 u_1}{\partial\tau^2} = -A\omega(\cos(\omega y) - \omega\gamma\sin(\omega y))\sin(\omega\tau) \tag{A3}$$

$$\frac{\partial u_1}{\partial y} - \gamma\frac{\partial^2 u_1}{\partial\tau^2} = -(\gamma - S]A\omega\sin(\omega\tau) - SA^3\omega\cos^2(\omega\tau)\sin(\omega\tau)$$

$$u_1(1,\tau) = 0$$

To solve this equation, it should be noted that the resonance frequencies of this oscillator are not equidistant and hence if ω_1 be the resonance frequency, then $2\omega_1$ and $3\omega_1$ are not and these terms "like the describing function method" could be ignored. Based on this, to find the steady state response of the oscillator, (A4) should be solved.

$$\frac{\partial^2 u_1}{\partial y^2} - \frac{\partial^2 u_1}{\partial\tau^2} = -A\omega(\cos(\omega y) - \omega\gamma\sin(\omega y))\sin(\omega\tau) \tag{A4}$$

$$\frac{\partial u_1}{\partial y} - \gamma\frac{\partial^2 u_1}{\partial\tau^2} = -A\omega\left[\gamma - S\left(1 - \frac{A^2}{4}\right)\right]\sin(\omega\tau)$$

$$u_1(1,\tau) = 0$$

Assuming a solution of the form $v_1 = V(y)\sin(\omega\tau)$ and substituting in (A2),

$$\frac{d^2 V}{y^2} + \omega^2 V = -A\omega(\cos(\omega y) - \omega\gamma\sin(\omega y)) \tag{A5}$$

$$\frac{dV}{dy} + \gamma\omega^2 V = -A\omega\left[\gamma S\left(1 - \frac{A^2}{4}\right)\right]$$
$$V(1,\tau) = 0$$

This is an ODE and its solution is

$$V(y) = v_h + v_p = B\cos(\omega y) + \sin(\omega y) - 1/2 A\omega\gamma y\cos(\omega y) - 1/2 Ay\sin(\omega y) \tag{A6}$$

Using the boundary conditions, we have

$$C + \gamma\omega B = -A\gamma/2 + AS\left(1 - \frac{A^2}{4}\right) \tag{A7}$$

$$C + \gamma\omega B = A\left(1 + \omega^2\gamma^2\right)$$

Solving (A7), the amplitude of the kth resonance frequency is,

$$A_k = 4\left[1 - \left(1 + \gamma + \omega^2\gamma^2\right)/2\right] \tag{A8}$$

As (A8) shows, it is possible that the oscillator has steady state oscillation at different frequencies and it should be also noted that the amplitude of the oscillation decrease with increasing its frequency. Hence the steady state response of this oscillator could be expressed as:

$$u = A_k(\cos(\omega_k y) - \omega_k\gamma\sin(\omega_k)\cos(\omega_k\tau) \tag{A9}$$

where ω_k is the kth resonance frequency and A_k is its amplitude.

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