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CLOSED FORM SOLUTIONS FOR NONUNIFORM TRANSMISSION LINES

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Abstract—In this paper, three analytic closed form solutions are introduced for arbitrary Nonuniform Transmission Lines (NTLs). The differential equations of NTLs are written in three suitable matrix equation forms, first. Then the matrix equations are solved to obtain the chain parameter matrix of NTLs. The obtained solutions are applicable to arbitrary lossy and dispersive NTLs. The validation of the proposed solutions is verified using some comprehensive examples.

1. INTRODUCTION

Nonuniform Transmission Lines (NTLs) are widely used in microwave circuits [1–5]. The differential equations describing NTLs have non-constant coefficients, so except for a few special cases, no analytical solution exists for them. There are some methods to analyze NTLs such as cascading many short sections [6, 7], finite difference [8], Taylor's series expansion [9], Fourier series expansion [10], the equivalent sources method [11] and the method of Moments [12]. These methods are numerical and do not yield an analytic closed form solutions. Only in [13], a closed form solution has been yielded, whose accuracy is not so acceptable at all frequencies. In this paper, three analytic closed form solutions are given, which have acceptable accuracy for most of NTLs at all frequencies. First, the differential equations of NTLs are written in three suitable matrix equation forms. Then the matrix equations are solved to obtain the chain parameter matrix of NTLs. The obtained solutions are applicable to arbitrary NTLs. The validation of the proposed solutions is studied using some comprehensive examples.

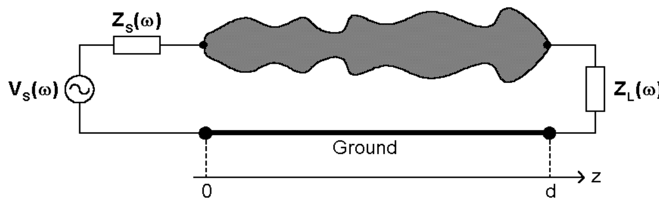

Figure 1. A typical NTL of length d terminated by arbitrary loads.

2. THE EQUATIONS OF NTLS

The equations related to NTLs are reviewed in this section. Figure 1 shows a typical NTL of length d with arbitrary terminal loads Z_S and Z_L. The differential equations describing NTLs in the frequency domain are given by

$$\frac{dV(z)}{dz} = -Z(z)I(z) \tag{1}$$

$$\frac{dI(z)}{dz} = -Y(z)V(z) \tag{2}$$

where

$$Z(z) = R(z) + j\omega L(z) \tag{3}$$

$$Y(z) = G(z) + j\omega C(z) \tag{4}$$

In (3)–(4), R, L, G and C are the distributed primary parameters of NTLs. The secondary parameters of NTLs (the characteristic impedance and the propagation coefficient) will be as follows

$$Z_c(z) = \sqrt{\frac{Z(z)}{Y(z)}} \tag{5}$$

$$\gamma(z) = \sqrt{Z(z)Y(z)} \tag{6}$$

Furthermore, the terminal conditions for loaded NTLs are as follows

$$V(0) + Z_S I(0) = V_S \tag{7}$$

$$V(d) - Z_L I(d) = 0 \tag{8}$$

Finally, the voltages and currents of terminals can be related to each other by the chain parameter or $ABCD$ matrices as follows.

$$\begin{bmatrix} V(d) \\ I(d) \end{bmatrix} = \Phi \begin{bmatrix} V(0) \\ I(0) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix} \tag{9}$$

3. SOLVING MATRIX DIFFERENTIAL EQUATION

In this section an approach is proposed to solve a particular matrix differential equation. This approach is identical to that is usually used to solve scalar differential equations. Consider the following matrix differential equation.

$$\frac{d}{dz}\mathbf{X}(z) = -\mathbf{A}(z)\mathbf{X}(z) \tag{10}$$

where

$$\mathbf{X}(z) = \begin{bmatrix} X_1(z) & X_3(z) \\ X_2(z) & X_4(z) \end{bmatrix} \tag{11}$$

and

$$\mathbf{A}(z) = \begin{bmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{bmatrix} \tag{12}$$

are the unknown and known matrices, respectively. From the matrix algebra and using (10), we can write the following

$$\begin{aligned} &\mathbf{X}^{-0.5}(z)(d\mathbf{X}(z))\mathbf{X}^{-0.5}(z) = d\left(\ln(\mathbf{X}(z))\right) \\ &= -\left(\mathbf{X}^{-0.5}(z)\mathbf{A}(z)\mathbf{X}^{0.5}(z)\right)dz \cong -\mathbf{A}(z)dz \end{aligned} \tag{13}$$

Again, from the matrix algebra and using (13), we can write the following

$$\ln(\mathbf{X}(z)) - \ln(\mathbf{X}(0)) = -\int_0^z \mathbf{A}(z')dz' \tag{14}$$

Therefore, one can determine the following solution for the unknown matrix in (10) from (14).

$$\mathbf{X}(z) \cong \mathbf{X}^{0.5}(0)\exp\left(-\int_0^z \mathbf{A}(z')dz'\right)\mathbf{X}^{0.5}(0) \cong \exp\left(-\int_0^z \mathbf{A}(z')dz'\right)\mathbf{X}(0) \tag{15}$$

The approximations used in (13) and (15) are dependent to the matrix $\mathbf{A}(z)$ and its integral.

4. ANALYSIS OF NTLS

In this section three closed form solutions are obtained for NTLs utilizing the approach proposed in the previous section.

4.1. Solution No. 1

The set of differential Equations (1)–(2), can be written similar to the matrix Equation (10), assuming

$$\mathbf{X}(z)=\begin{bmatrix}V(z) & V(z)\\ I(z) & I(z)\end{bmatrix} \tag{16}$$

and

$$\mathbf{A}(z)=\begin{bmatrix}0 & Z(z)\\ Y(z) & 0\end{bmatrix} \tag{17}$$

From (15) and (9), the following closed form solution (as Solution 1) is obtained as the chain parameter matrix of NTLs.

$$\Phi=\exp\left(-\int_0^d \mathbf{A}(z')dz'\right) \tag{18}$$

4.2. Solution No. 2

One may transform the set of differential Equations (1)–(2) to the following ones

$$\frac{d\hat{V}(z)}{dz} = \frac{1}{2Y(z)}\frac{dY(z)}{dz}\hat{V}(z)-\gamma(z)\hat{I}(z) \tag{19}$$

$$\frac{d\hat{I}(z)}{dz} = \frac{1}{2Z(z)}\frac{dZ(z)}{dz}\hat{I}(z)-\gamma(z)\hat{V}(z) \tag{20}$$

where

$$\hat{V}(z) = \sqrt{Y(z)}V(z) \tag{21}$$

$$\hat{I}(z) = \sqrt{Z(z)}I(z) \tag{22}$$

are defined as the normalized voltage and current, respectively. The set of differential Equations (19)–(20), can be written similar to the matrix Equation (10), assuming

$$\mathbf{X}(z)=\begin{bmatrix}\hat{V}(z) & \hat{V}(z)\\ \hat{I}(z) & \hat{I}(z)\end{bmatrix} \tag{23}$$

and

$$\mathbf{A}(z)=\begin{bmatrix}\frac{-1}{2Y(z)}\frac{dY(z)}{dz} & \gamma(z)\\ \gamma(z) & \frac{-1}{2Z(z)}\frac{dZ(z)}{dz}\end{bmatrix} \tag{24}$$

From (15), (9) and (21)–(22), the following closed form solution (as Solution 2) is obtained as the chain parameter matrix of NTLs.

$$\Phi = \begin{bmatrix} 1/\sqrt{Y(d)} & 0 \\ 0 & 1/\sqrt{Z(d)} \end{bmatrix} \exp\left(-\int_0^d \mathbf{A}(z')dz' \right) \begin{bmatrix} \sqrt{Y(0)} & 0 \\ 0 & \sqrt{Z(0)} \end{bmatrix} \tag{25}$$

4.3. Solution No. 3

Also, one may transform the set of differential Equations (1)–(2) to the following ones

$$\frac{d\overline{V}(z)}{dz} = -\frac{1}{Z_c(z)}\frac{dZ_c(z)}{dz}\overline{V}(z) - \gamma(z)I(z) \tag{26}$$

$$\frac{dI(z)}{dz} = -\gamma(z)\overline{V}(z) \tag{27}$$

where

$$\overline{V}(z) = \frac{1}{Z_c(z)}V(z) \tag{28}$$

is defined as the normalized voltage. The set of differential Equations (26)–(27), can be written similar to the matrix Equation (10), assuming

$$\mathbf{X}(z) = \begin{bmatrix} \overline{V}(z) & \overline{V}(z) \\ I(z) & I(z) \end{bmatrix} \tag{29}$$

and

$$\mathbf{A}(z) = \begin{bmatrix} \frac{1}{Z_c(z)}\frac{dZ_c(z)}{dz} & \gamma(z) \\ \gamma(z) & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2Z(z)}\frac{dZ(z)}{dz} - \frac{1}{2Y(z)}\frac{dY(z)}{dz} & \gamma(z) \\ \gamma(z) & 0 \end{bmatrix} \tag{30}$$

From (15), (9) and (28), the following closed form solution (as Solution 3) is obtained as the chain parameter matrix of NTLs.

$$\Phi = \begin{bmatrix} Z_c(d) & 0 \\ 0 & 1 \end{bmatrix} \exp\left(-\int_0^d \mathbf{A}(z')dz' \right) \begin{bmatrix} 1/Z_c(0) & 0 \\ 0 & 1 \end{bmatrix} \tag{31}$$

The chain parameter matrix corresponding to the arbitrary chosen point z with respect to the point z=0, can be obtained using (18), (25) and (31) but replacing z instead of d. Therefore, one can obtain the voltage and current at any point of NTLs using the chain parameter matrix of terminals and the boundary conditions (7)–(8).

5. EXAMPLES AND DISCUSSION

In this section, three obtained closed form solutions are validated using three examples.

Example 1: Consider a lossless and exponential NTL with the following distributed primary parameters.

$$L(z) = L_0 \exp(kz/d) \tag{32}$$
$$C(z) = C_0 \exp(-kz/d) \tag{33}$$
$$R(z) = G(z) = 0 \tag{34}$$

This type of transmission line will have the following secondary parameters.

$$Z_c(z) = \sqrt{L_0/C_0} \exp(kz/d) \tag{35}$$
$$\gamma(z) = j\beta(z) = j\omega\sqrt{L_0 C_0} \tag{36}$$

Therefore, the matrix $\mathbf{A}(z)$ will be as follows for three solutions, respectively.

$$\mathbf{A}_1(z) = \begin{bmatrix} 0 & j\omega L_0 \exp(kz/d) \\ j\omega C_0 \exp(-kz/d) & 0 \end{bmatrix} \tag{37}$$
$$\mathbf{A}_2(z) = \begin{bmatrix} \frac{k}{2d} & j\omega\sqrt{L_0 C_0} \\ j\omega\sqrt{L_0 C_0} & -\frac{k}{2d} \end{bmatrix} \tag{38}$$
$$\mathbf{A}_3(z) = \begin{bmatrix} k/d & j\omega\sqrt{L_0 C_0} \\ j\omega\sqrt{L_0 C_0} & 0 \end{bmatrix} \tag{39}$$

Assume that $d = 20\,\text{cm}$, $Z_c(0) = \sqrt{L_0/C_0} = 50\,\Omega$ and $\beta = \omega\sqrt{L_0 C_0} = \omega/c$, in which c is the velocity of the light. Figure 2, compare the parameter A versus frequency assuming $k = 1$, obtained from three introduced solutions with the exact one [14]. One sees that the accuracy of solution 1 is not acceptable unless at very low frequencies. However, the accuracies of solutions 2 and 3 are very excellent at all frequencies. Figures 3–6, compare the $ABCD$ parameters versus frequency assuming $k = 1$ and 10, obtained from the solutions 2 and 3 with the exact one. One sees that the accuracies of solutions 2 and 3 are very excellent at all frequencies. Indeed, the solutions 2 and 3 have been given exact results.

Example 2: Consider a lossless and linear NTL with the following distributed primary parameters.

$$L(z) = L_0 \left(1 + k(z/d)\right) \tag{40}$$
$$C(z) = \frac{C_0}{1 + k(z/d)} \tag{41}$$

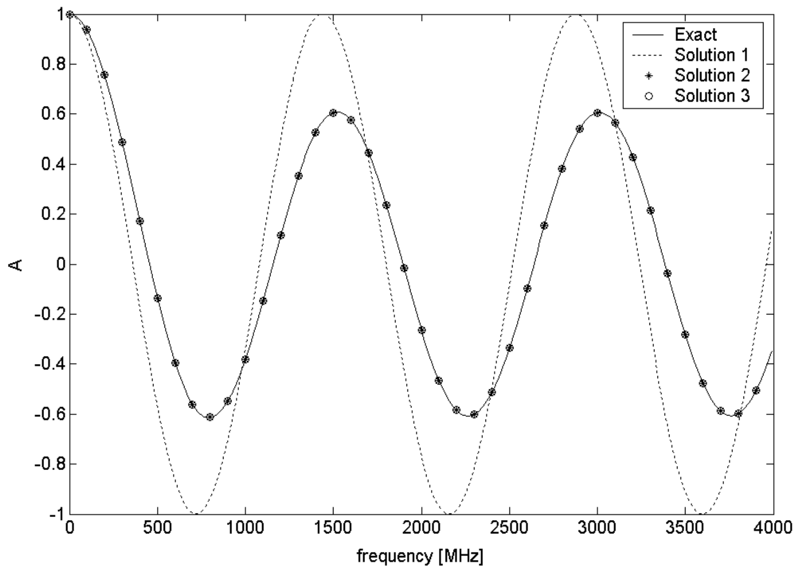

Figure 2. The parameter A for the lossless exponential NTL with $k = 1$.

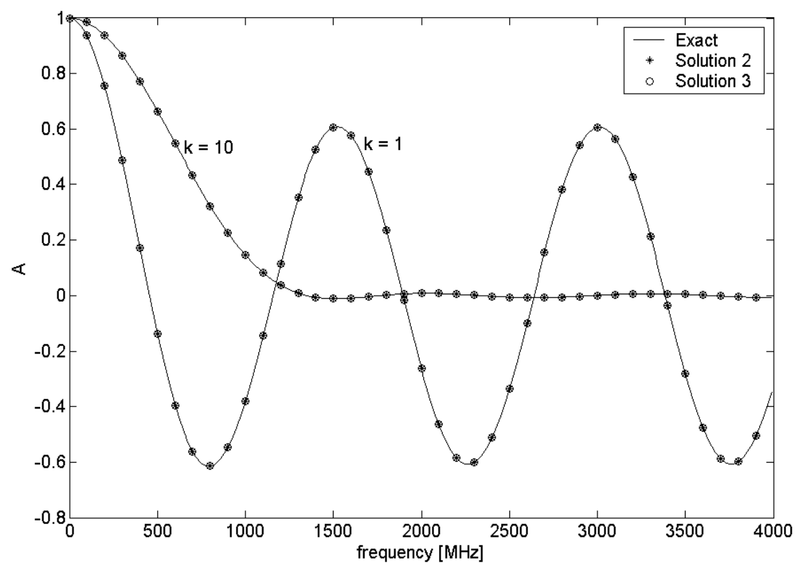

Figure 3. The parameter A for the lossless exponential NTL.

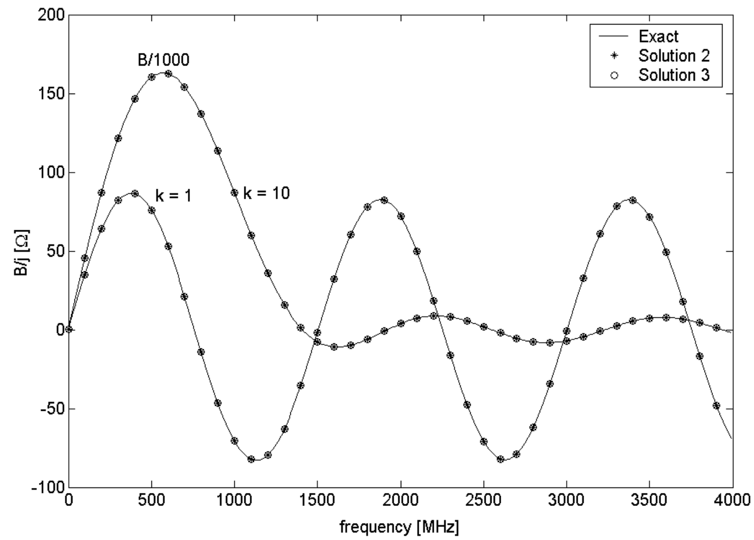

Figure 4. The parameter B for the lossless exponential NTL.

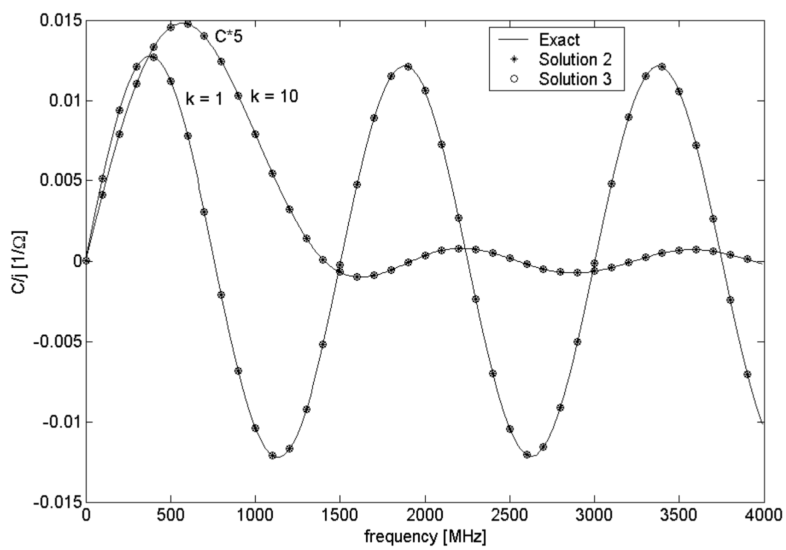

Figure 5. The parameter C for the lossless exponential NTL.

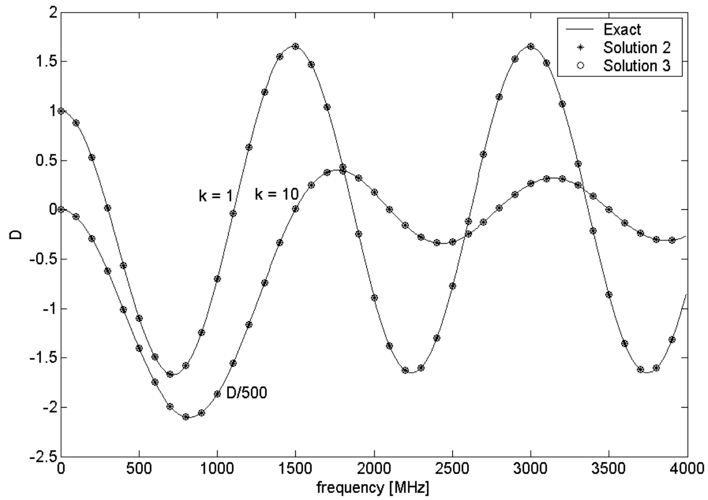

Figure 6. The parameter D for the lossless exponential NTL.

$$R(z) = G(z) = 0 \tag{42}$$

This type of transmission line will have the following secondary parameters.

$$Z_c(z) = \sqrt{L_0/C_0}\,(1 + k(z/d)) \tag{43}$$

$$\gamma(z) = j\beta(z) = j\omega\sqrt{L_0C_0} \tag{44}$$

Therefore, the matrix $\mathbf{A}(z)$ will be as follows for three solutions, respectively.

$$\mathbf{A}_1(z) = \begin{bmatrix} 0 & j\omega L_0(1 + kz/d) \\ j\omega C_0/(1 + kz/d) & 0 \end{bmatrix} \tag{45}$$

$$\mathbf{A}_2(z) = \begin{bmatrix} \frac{k/(2d)}{1+kz/d} & j\omega\sqrt{L_0C_0} \\ j\omega\sqrt{L_0C_0} & -\frac{k/(2d)}{1+kz/d} \end{bmatrix} \tag{46}$$

$$\mathbf{A}_3(z) = \begin{bmatrix} \frac{k/d}{1+kz/d} & j\omega\sqrt{L_0C_0} \\ j\omega\sqrt{L_0C_0} & 0 \end{bmatrix} \tag{47}$$

Assume that d=20 cm, $Z_c(0) = \sqrt{L_0/C_0} = 50\,\Omega$ and $\beta = \omega\sqrt{L_0C_0} = \omega/c$. Figure 7, compare the parameter A versus frequency assuming $k = 1$, obtained from three introduced solutions with the exact one [7]. Again, one sees that the accuracy of solution 1 is not acceptable unless

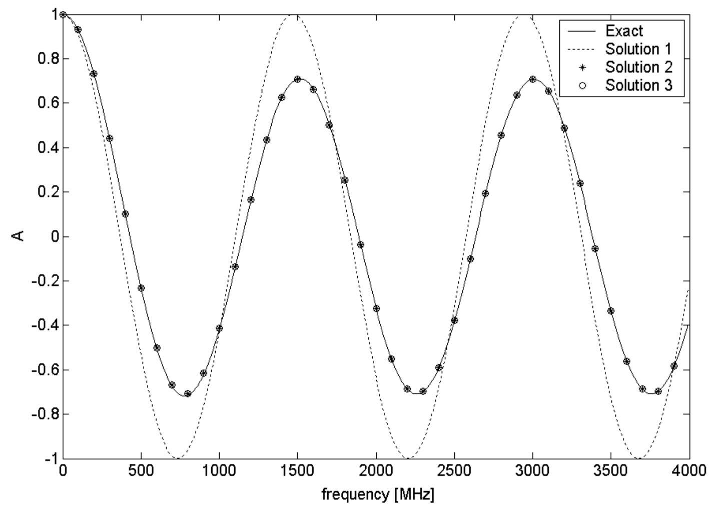

Figure 7. The parameter A for the lossless linear NTL with $k = 1$.

at very low frequencies. However, the accuracies of solutions 2 and 3 are excellent at all frequencies. Figures 8–11, compare the $ABCD$ parameters versus frequency assuming $k = 1$ and 10, obtained from the solutions 2 and 3 with the exact one. One sees that the accuracies of solutions 2 and 3 are good at all frequencies and increases as the k decreases.

Example 3: Consider a lossy and dispersive linear NTL with the following distributed primary parameters.

$$L(z) = L_0 \left(1 + k(z/d)\right) \quad (48)$$

$$C(z) = C_0 \left(1 + k(z/d)\right) \quad (49)$$

$$R(z) = R_0 \left(1 + k(z/d)\right) \quad (50)$$

$$G(z) = G_0 \left(1 + k(z/d)\right) \quad (51)$$

This type of transmission line will have the following secondary parameters.

$$Z_c(z) = \sqrt{(R_0 + j\omega L_0)/(G_0 + j\omega C_0)} \quad (52)$$

$$\gamma(z) = \gamma_0(1 + kz/d) = \sqrt{(R_0 + j\omega L_0)(G_0 + j\omega C_0)}(1 + kz/d) \quad (53)$$

Therefore, the matrix $\mathbf{A}(z)$ will be as follows for three solutions, respectively.

$$\mathbf{A}_1(z) = \begin{bmatrix} 0 & (R_0 + j\omega L_0)(1 + kz/d) \\ (G_0 + j\omega C_0)(1 + kz/d) & 0 \end{bmatrix} \quad (54)$$

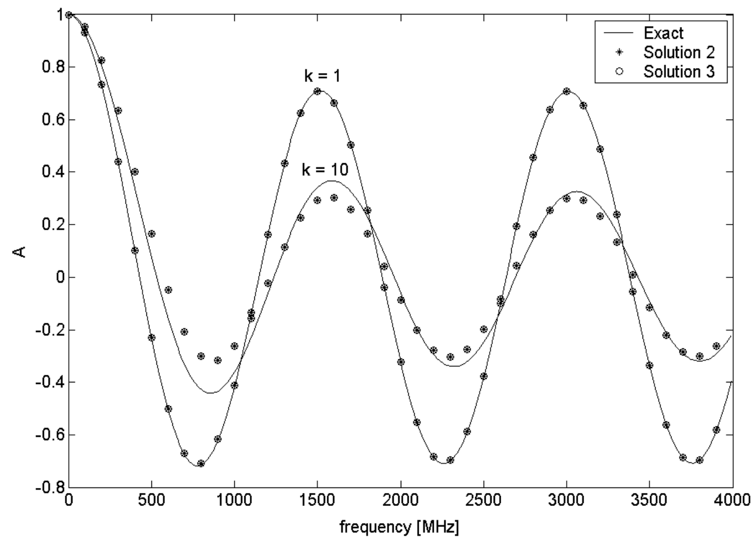

Figure 8. The parameter A for the lossless linear NTL.

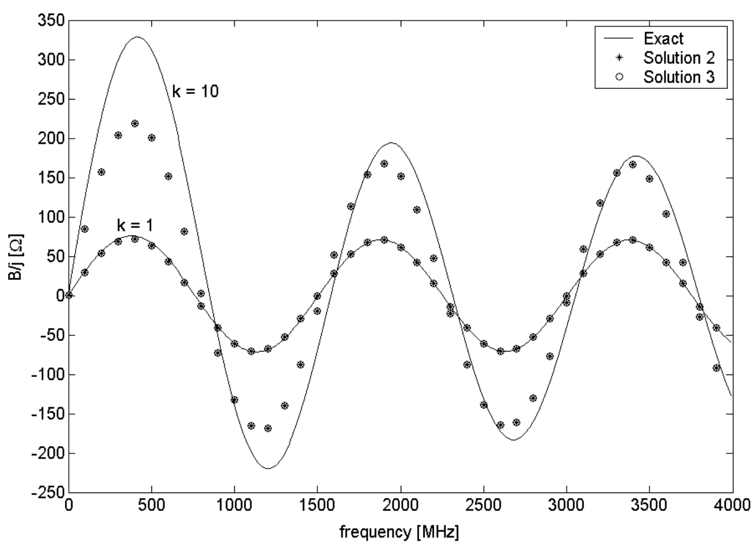

Figure 9. The parameter B for the lossless linear NTL.

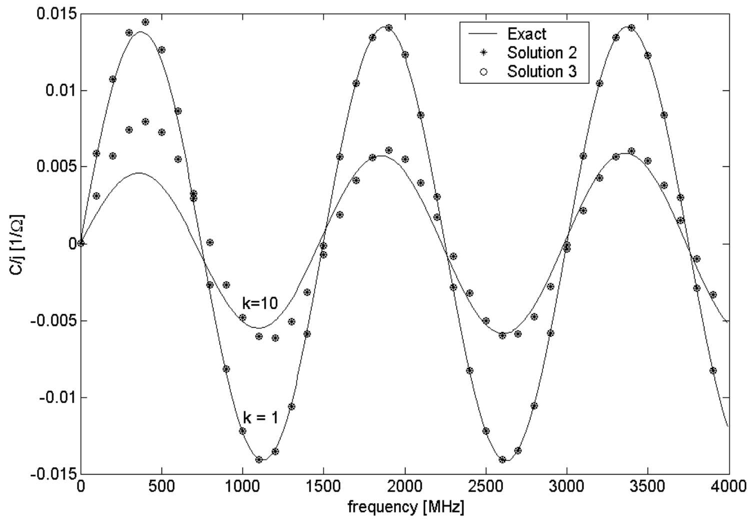

Figure 10. The parameter C for the lossless linear NTL.

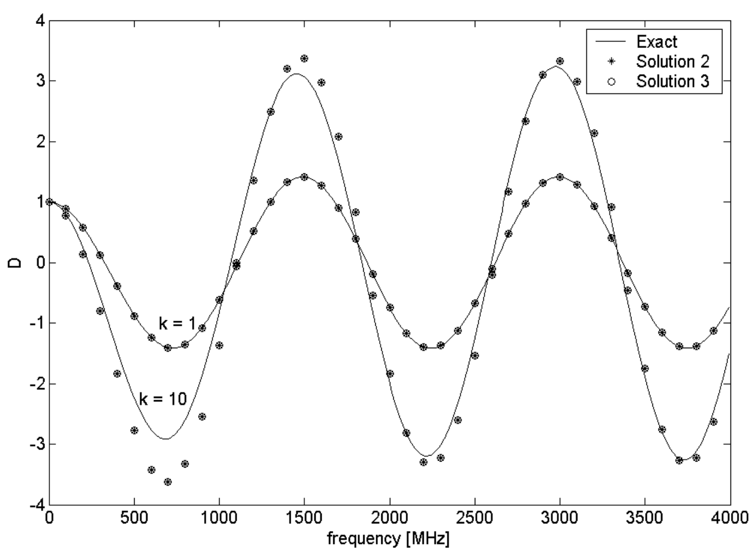

Figure 11. The parameter D for the lossless linear NTL.

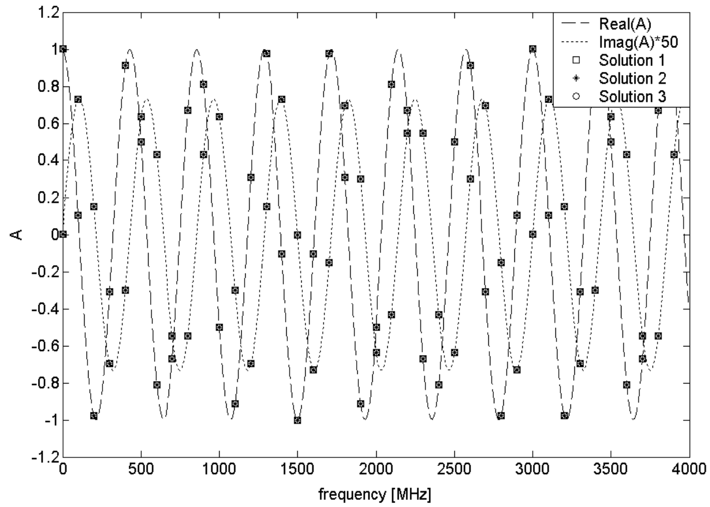

Figure 12. The parameter A for the lossy linear NTL with $k = 5$.

$$\mathbf{A}_2(z) = \begin{bmatrix} -\frac{k/(2d)}{1+kz/d} & \gamma_0(1+kz/d) \\ \gamma_0(1+kz/d) & -\frac{k/(2d)}{1+kz/d} \end{bmatrix} \tag{55}$$

$$\mathbf{A}_3(z) = \begin{bmatrix} 0 & \gamma_0(1+kz/d) \\ \gamma_0(1+kz/d) & 0 \end{bmatrix} \tag{56}$$

Assume that $d = 20\,\text{cm}$, $\sqrt{L_0/C_0} = 50\,\Omega$, $\sqrt{L_0 C_0} = 1/c$, R_0=1.0472 Ω/m and $G_0 = 4.1888*10^{-4}$ 1/Ωm. Figures 12–15, compare the $ABCD$ parameters versus frequency assuming k=5, obtained from three solutions 1, 2 and 3 with the exact one [7]. One sees that the accuracies of all three solutions are very excellent at all frequencies. Indeed, the solutions 1, 2 and 3 have been given exact results.

It is worth to mention that the relation $AD - BC = 1$ has been satisfied by all three solutions in the above examples. According to the above examples, one may conclude the following results:

1. The solutions 1, 2 and 3 give exact results at all frequencies for arbitrary NTLs, whose characteristic impedance is constant along their length.
2. The accuracies of solutions 1, 2 and 3 are excellent at low frequencies for arbitrary NTLs.
3. The accuracies of solutions 2 and 3 are the same and are acceptable at all frequencies for arbitrary NTLs.

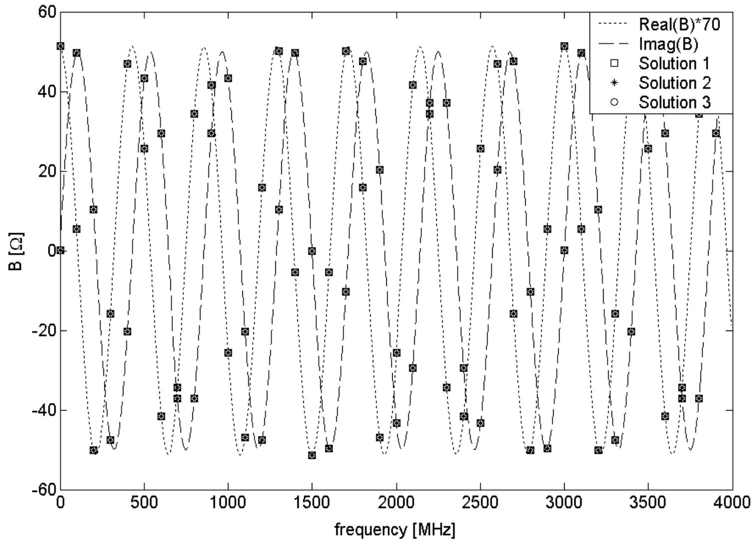

Figure 13. The parameter B for the lossy linear NTL with $k = 5$.

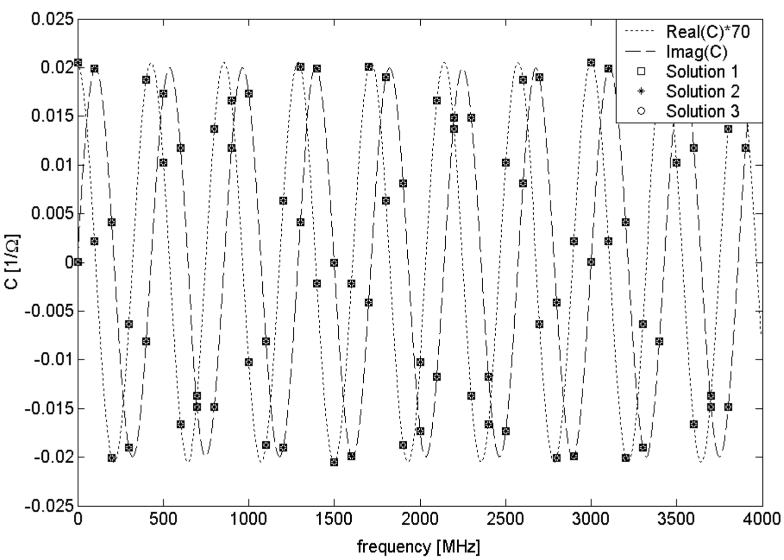

Figure 14. The parameter C for the lossy linear NTL with $k = 5$.

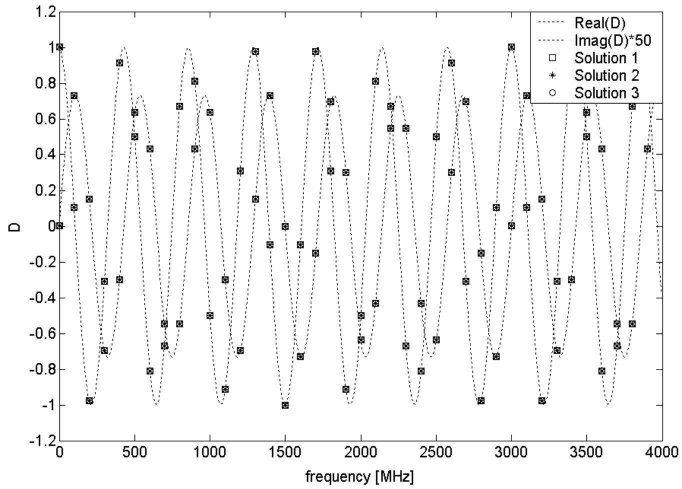

Figure 15. The parameter D for the lossly linear NTL with $k = 5$.

4. The accuracies of solutions 1, 2 and 3 are increased as the variations of the primary parameters are decreased.

6. CONCLUSION

Three analytic closed form solutions were introduced for arbitrary Nonuniform Transmission Lines (NTLs). The differential equations of NTLs are written in three suitable matrix equation forms, first. Then the matrix equations are solved to obtain the chain parameter matrix of NTLs. The validation of the proposed solutions was verified using some comprehensive examples. It was concluded that all three solutions give exact results at all frequencies for arbitrary NTLs, whose characteristic impedance is constant along their length. Also, the accuracies of all three solutions are excellent at low frequencies for arbitrary NTLs. Moreover, the accuracies of all three solutions are increased as the variations of the primary parameters are decreased. Also, the accuracies of two solutions are the same and are acceptable at all frequencies for arbitrary NTLs.

REFERENCES

1. Burkhart, S. C. and R. B. Wilcox, "Arbitrary pulse shape synthesis via nonuniform transmission lines," *IEEE Trans. Mic. Theory and Tech.*, 1514–1518, Oct. 1990.

2. Roberts, P. P. and G. E. Town, "Design of microwave filters by inverse scattering," *IEEE Trans. Mic. Theory and Tech.*, 739–743, Apr. 1995.
3. Ghose, R. N., "Exponential transmission lines as resonators and transformers," *IRE Trans. Mic. Theory and Tech.*, 213–217, July 1957.
4. Tang, C. C. H., "Delay equalization by tapered cutoff waveguides," *IEEE Trans. Mic. Theory and Tech.*, 608–615, Nov. 1964.
5. Protonotarios, E. N. and O. Wing, "Analysis and intrinsic properties of the general nonuniform transmission line," *IEEE Trans. Micro. Theory and Tech.*, 142–150, Mar. 1967.
6. Paul, C. R., *Analysis of Multiconductor Transmission Lines*, John Wiley and Sons Inc., 1994.
7. Lu, K., "An efficient method for analysis of arbitrary nonuniform transmission lines," *IEEE Trans. Micro. Theory and Tech.*, 9–14, Jan. 1997.
8. Khalaj-Amirhosseini, M., "Analysis of coupled or single nonuniform transmission lines using step-by-step numerical integration," *Progress In Electromagnetics Research*, PIER 58, 187–198, 2006.
9. Khalaj-Amirhosseini, M., "Analysis of nonuniform transmission lines using Taylor's series expansion," *Int. J. RF and Microwave Computer-Aided Engineering*, Vol. 16, No. 5, 536–544, Sep. 2006.
10. Khalaj-Amirhosseini, M., "Analysis of nonuniform transmission lines using Fourier series expansion," *Int. J. of RF and Microwave Computer-Aided Eng.*, Vol. 17, No. 3, 345–352, May 2007.
11. Khalaj-Amirhosseini, M., "Analysis of nonuniform transmission lines using the equivalent sources," *Progress In Electromagnetics Research*, PIER 71, 95–107, 2007.
12. Khalaj-Amirhosseini, M., "Analysis of nonuniform transmission lines using the method of moments," *Asia-Pacific Microwave Conf. (APMC 2007)*, Bangkok, Thailand, Dec. 12–14, 2007.
13. Cheldavi, A., M. Kamarei, and S. Safavi-Naeini, "First order approximation of the exact solution of arbitrary nonuniform transmission lines: Application in high speed integrated circuits," *IEICE Trans. Electron.*, Vol. E82-C, No. 12, 2248–2254, Dec. 1999.
14. Oufi, E. A., A. S. AlFuhaid, and M. M. Saied, "Transient analysis of lossless single-phase nonuniform transmission lines," *IEEE Trans. Power Delivery*, 1694–1700, July 1994.