Progress In Electromagnetics Research, PIER 75, 383–396, 2007

FRACTIONAL RECTANGULAR WAVEGUIDE

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Abstract—Fractional curl operator is utilized to construct the solutions corresponding to fractional dual rectangular waveguides. Fractional dual rectangular waveguides may be considered as intermediate of two given waveguides, where both waveguides are related through principle of duality. Characteristic impedance of fractional waveguide is determined. Behavior of field lines in transverse plane is also investigated.

1. INTRODUCTION

Fractional Calculus is a branch of Mathematics which deals with the operators of general orders, including integer order, real non integer order and even complex order [1]. Concept of fractional curl operator was introduced by Engheta [2] and work was extended to operator having higher and complex order [3]. Fractional curl operator may be utilized to find the intermediate solutions between a given solution and electromagnetic dual to the given solution. This operator had been applied to many problems to study the intermediate or fractional dual solutions [4–8]. Waveguide problems had been analyzed in different respects [9–11]. In this paper, we have extended work on fractional dual parallel plates waveguide [5] to rectangular waveguide. Field expressions corresponding to fractional dual rectangular waveguide are

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determined. Impedance of fractional dual rectangular waveguide has been derived.

2. FRACTIONAL RECTANGULAR WAVEGUIDE

Consider a rectangular waveguide, filled with a dielectric medium with constitutive parameters ϵ and μ. Walls of the guide are PEC and are located at $x = 0$, $x = a$ and $y = 0$, $y = b$. Let us suppose that a TM wave ($H_z = 0$) is propagating in z-direction. The axial component of electric field is given by [9]

$$\hat{z}E_z(x,y,z) = \hat{z}A_{mn}\sin(k_x x)\sin(k_y y)\exp(-j\beta z) \tag{1a}$$

where $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b}$, $\beta^2 = k^2 - k_c^2$ and $k_c^2 = k_x^2 + k_y^2$. m and n are arbitrary integers. The transverse components of the fields propagating inside the waveguide are [9]

$$\hat{x}E_x(x,y,z) = -\hat{x}\frac{j\beta k_x}{k_c^2}A_{mn}\cos(k_x x)\sin(k_y y)\exp(-j\beta z) \tag{1b}$$

$$\hat{y}E_y(x,y,z) = -\hat{y}\frac{j\beta k_y}{k_c^2}A_{mn}\sin(k_x x)\cos(k_y y)\exp(-j\beta z) \tag{1c}$$

$$\hat{x}\eta H_x(x,y,z) = -\hat{x}\frac{k}{\beta}E_y(x,y,z) \tag{1d}$$

$$\hat{y}\eta H_y(x,y,z) = \hat{y}\frac{k}{\beta}E_x(x,y,z) \tag{1e}$$

where $k = \sqrt{k_c^2 + \beta^2} = \omega\sqrt{\mu\epsilon}$.

We may express the field in the waveguide as superposition of 4 TEM plane waves. That is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4 \tag{2a}$$

$$\eta\mathbf{H} = \eta\mathbf{H}_1 + \eta\mathbf{H}_2 + \eta\mathbf{H}_3 + \eta\mathbf{H}_4 \tag{2b}$$

where $(\mathbf{E}_i, \eta\mathbf{H}_i)$ and $i = 1, 2, 3, 4$ are the electric and magnetic fields associated with ith plane wave and are given below

$$\mathbf{E}_1 = \frac{A_{mn}}{4}\left(-\hat{x}\frac{\beta k_x}{k_c^2} - \hat{y}\frac{\beta k_y}{k_c^2} - \hat{z}\right)e^{j(k_x x + k_y y - \beta z)} \tag{3a}$$

$$\mathbf{E}_2 = \frac{A_{mn}}{4}\left(\hat{x}\frac{\beta k_x}{k_c^2} - \hat{y}\frac{\beta k_y}{k_c^2} + \hat{z}\right)e^{j(k_x x - k_y y - \beta z)} \tag{3b}$$

$$\mathbf{E}_3 = \frac{A_{mn}}{4}\left(-\hat{x}\frac{\beta k_x}{k_c^2} + \hat{y}\frac{\beta k_y}{k_c^2} + \hat{z}\right)e^{j(-k_x x + k_y y - \beta z)} \tag{3c}$$

$$\mathbf{E}_4 = \frac{A_{mn}}{4}\left(\hat{x}\frac{\beta k_x}{k_c^2} + \hat{y}\frac{\beta k_y}{k_c^2} - \hat{z}\right)e^{j(-k_x x - k_y y - \beta z)} \tag{3d}$$

$$\eta \mathbf{H}_1 = \frac{kA_{mn}}{4k_c^2}\left(\hat{x}k_y - \hat{y}k_x\right) e^{j(k_x x + k_y y - \beta z)} \tag{3e}$$

$$\eta \mathbf{H}_2 = \frac{kA_{mn}}{4k_c^2}\left(\hat{x}k_y + \hat{y}k_x\right) e^{j(k_x x - k_y y - \beta z)} \tag{3f}$$

$$\eta \mathbf{H}_3 = \frac{kA_{mn}}{4k_c^2}\left(-\hat{x}k_y - \hat{y}k_x\right) e^{j(-k_x x + k_y y - \beta z)} \tag{3g}$$

$$\eta \mathbf{H}_4 = \frac{kA_{mn}}{4k_c^2}\left(-\hat{x}k_y + \hat{y}k_x\right) e^{j(-k_x x - k_y y - \beta z)} \tag{3h}$$

Fields $\mathbf{E}_i$ and $\eta\mathbf{H}_i$ given by Equations (3) are related through the Maxwell equations as

$$\begin{aligned} \nabla \times \mathbf{E}_i &= -j\omega\mu \mathbf{H}_i \\ \mathbf{k}_i \times \mathbf{E}_i &= \eta \mathbf{H}_i \end{aligned} \tag{4a}$$

Similarly

$$\mathbf{k}_i \times \eta \mathbf{H}_i = -\mathbf{E}_i \tag{4b}$$

where

$$\mathbf{k}_1 = \frac{1}{jk}(-jk_x\hat{x} - jk_y\hat{y} + j\beta\hat{z}) \tag{5a}$$

$$\mathbf{k}_2 = \frac{1}{jk}(-jk_x\hat{x} + jk_y\hat{y} + j\beta\hat{z}) \tag{5b}$$

$$\mathbf{k}_3 = \frac{1}{jk}(jk_x\hat{x} - jk_y\hat{y} + j\beta\hat{z}) \tag{5c}$$

$$\mathbf{k}_4 = \frac{1}{jk}(jk_x\hat{x} + jk_y\hat{y} + j\beta\hat{z}) \tag{5d}$$

It may be deduced from above expressions that for set of fields $(\mathbf{E}_i, \eta\mathbf{H}_i)$, the operator $\left(\frac{1}{jk}\nabla\times\right)$ is equivalent to cross product operator $(\mathbf{k}_i\times)$. It is also obvious that if $(\mathbf{E}_i, \eta\mathbf{H}_i)$ is one set of solutions to Maxwell's equation then other set of solutions to Maxwell's equations is $(\eta\mathbf{H}_i, -\mathbf{E}_i)$. Our interest is to determine the fields which may be regarded as intermediate step of the field $(\mathbf{E}, \eta\mathbf{H})$ and $(\eta\mathbf{H}, -\mathbf{E})$, that is, new set of solutions $(\mathbf{E}_{fd}, \eta\mathbf{H}_{fd})$. For this purpose solutions sets $(\mathbf{E}_{ifd}, \eta\mathbf{H}_{ifd})$ with $i = 1, 2, 3, 4$ are required. $(\mathbf{E}_{ifd}, \eta\mathbf{H}_{ifd})$ may be obtained by using the following relations [2]

$$\begin{aligned} \mathbf{E}_{ifd} &= \frac{1}{(jk)^\alpha}\left[(\nabla\times)^\alpha \mathbf{E}_i\right] \\ &= (\mathbf{k}_i\times)^\alpha\, \mathbf{E}_i \end{aligned} \tag{6a}$$

$$\eta \mathbf{H}_{ifd} = \frac{1}{(jk)^{\alpha}} \left[(\nabla \times)^{\alpha} \eta \mathbf{H}_i \right]$$
$$= (\mathbf{k}_i \times)^{\alpha} \eta \mathbf{H}_i, \qquad i = 1, 2, 3, 4 \tag{6b}$$

Solutions $(\mathbf{E}_{fd}, \eta \mathbf{H}_{fd})$ may be obtained by linear combination of $(\mathbf{E}_{ifd}, \eta \mathbf{H}_{ifd})$, that is

$$\mathbf{E}_{fd} = \mathbf{E}_{1fd} + \mathbf{E}_{2fd} + \mathbf{E}_{3fd} + \mathbf{E}_{4fd} \tag{7a}$$
$$\eta \mathbf{H}_{fd} = \eta \mathbf{H}_{1fd} + \eta \mathbf{H}_{2fd} + \eta \mathbf{H}_{3fd} + \eta \mathbf{H}_{4fd} \tag{7b}$$

In order to determine the fractional dual solutions $(\mathbf{E}_{ifd}, \eta \mathbf{H}_{ifd})$, the eigenvalues and eigenvectors of the cross product operators $\mathbf{k}_i \times$ are required. Eigenvectors and eigenvalues of the operator $(\mathbf{k}_1 \times)$ are

$$\mathbf{A}_{11} = \frac{1}{\sqrt{2} k k_c} \left[(\beta k_x + jkk_y)\hat{x} + (\beta k_z - jkk_x)\hat{y} + k_c^2 \hat{z} \right], \ a_1 = j \tag{8a}$$
$$\mathbf{A}_{12} = \frac{1}{\sqrt{2} k k_c} \left[(\beta k_x - jkk_y)\hat{x} + (\beta k_z + jkk_x)\hat{y} + k_c^2 \hat{z} \right], \ a_2 = -j \tag{8b}$$
$$\mathbf{A}_{13} = \frac{1}{k}(-k_x \hat{x} - k_y \hat{y} + k_z \hat{z}), \ a_3 = 0 \tag{8c}$$

Eigenvectors and eigenvalues of the operator $(\mathbf{k}_2 \times)$ are

$$\mathbf{A}_{21} = \frac{1}{\sqrt{2} k k_c} \left[(\beta k_x - jkk_y)\hat{x} + (-\beta k_z - jkk_x)\hat{y} + k_c^2 \hat{z} \right], \ a_1 = j \tag{8d}$$
$$\mathbf{A}_{22} = \frac{1}{\sqrt{2} k k_c} \left[(\beta k_x + jkk_y)\hat{x} + (-\beta k_z + jkk_x)\hat{y} + k_c^2 \hat{z} \right], \ a_2 = -j \tag{8e}$$
$$\mathbf{A}_{23} = \frac{1}{k}(-k_x \hat{x} + k_y \hat{y} + k_z \hat{z}), \ a_3 = 0 \tag{8f}$$

Eigenvectors and eigenvalues of the operator $(\mathbf{k}_3 \times)$ are

$$\mathbf{A}_{31} = \frac{1}{\sqrt{2} k k_c} \left[(-\beta k_x + jkk_y)\hat{x} + (\beta k_z + jkk_x)\hat{y} + k_c^2 \hat{z} \right], \ a_1 = j \tag{8g}$$
$$\mathbf{A}_{32} = \frac{1}{\sqrt{2} k k_c} \left[(-\beta k_x - jkk_y)\hat{x} + (\beta k_z - jkk_x)\hat{y} + k_c^2 \hat{z} \right], \ a_2 = -j \tag{8h}$$
$$\mathbf{A}_{33} = \frac{1}{k}(k_x \hat{x} - k_y \hat{y} + k_z \hat{z}), \ a_3 = 0 \tag{8i}$$

Eigenvectors and eigenvalues of the operator $(\mathbf{k}_4 \times)$ are

$$\mathbf{A}_{41} = \frac{1}{\sqrt{2} k k_c} \left[(-\beta k_x - jkk_y)\hat{x} + (-\beta k_z + jkk_x)\hat{y} + k_c^2 \hat{z} \right], \ a_1 = j \tag{8j}$$
$$\mathbf{A}_{42} = \frac{1}{\sqrt{2} k k_c} \left[(-\beta k_x + jkk_y)\hat{x} + (-\beta k_z - jkk_x)\hat{y} + k_c^2 \hat{z} \right], \ a_2 = -j \tag{8k}$$

$$\mathbf{A}_{43} = \frac{1}{k}(k_x\hat{x} + k_y\hat{y} + k_z\hat{z}), \;\; a_3 = 0 \tag{8l}$$

Fields $(\mathbf{E}_i, \eta\mathbf{H}_i)$ may be expressed in terms of the eigenvectors of the operator (dropping the exponential terms), that is

$$\mathbf{E}_i = [P_i\mathbf{A}_{i1} + Q_i\mathbf{A}_{i2} + R_i\mathbf{A}_{i3}] \tag{9a}$$

$$\eta\mathbf{H}_i = [S_i\mathbf{A}_{i1} + T_i\mathbf{A}_{i2} + U_i\mathbf{A}_{i3}] \tag{9b}$$

The coefficients are given below

$$P_1 = Q_1 = P_4 = Q_4 = -jS_1 = -jS_4 = -jT_2 = -jT_3 = -\frac{A_{mn}}{4\sqrt{2}}\frac{k}{k_c}$$

$$P_2 = Q_2 = P_3 = Q_3 = -jT_1 = -jT_4 = -jS_2 = -jS_3 = \frac{A_{mn}}{4\sqrt{2}}\frac{k}{k_c}$$

Using (6), (8) and (9) the fractional fields are given as

$$\mathbf{E}_{1fd} = -\frac{A_{mn}}{8k_c^2}\left[\cos\left(\alpha\frac{\pi}{2}\right)\left\{2\beta k_x\hat{x} + 2\beta k_y\hat{y} + 2k_c^2\hat{z}\right\} - j\sin\left(\alpha\frac{\pi}{2}\right)\left\{2jkk_y\hat{x} - 2jkk_x\hat{y}\right\}\right]e^{jk_xx+jk_yy-j\beta z} \tag{10a}$$

$$\mathbf{E}_{2fd} = \frac{A_{mn}}{8k_c^2}\left[\cos\left(\alpha\frac{\pi}{2}\right)\left\{2\beta k_x\hat{x} - 2\beta k_y\hat{y} + 2k_c^2\hat{z}\right\} + j\sin\left(\alpha\frac{\pi}{2}\right)\left\{-2jkk_y\hat{x} - 2jkk_x\hat{y}\right\}\right]e^{jk_xx-jk_yy-j\beta z} \tag{10b}$$

$$\mathbf{E}_{3fd} = \frac{A_{mn}}{8k_c^2}\left[\cos\left(\alpha\frac{\pi}{2}\right)\left\{-2\beta k_x\hat{x} + 2\beta k_y\hat{y} + 2k_c^2\hat{z}\right\} + j\sin\left(\alpha\frac{\pi}{2}\right)\left\{2jkk_y\hat{x} + 2jkk_x\hat{y}\right\}\right]e^{-jk_xx+jk_yy-j\beta z} \tag{10c}$$

$$\mathbf{E}_{4fd} = -\frac{A_{mn}}{8k_c^2}\left[\cos\left(\alpha\frac{\pi}{2}\right)\left\{-2\beta k_x\hat{x} - 2\beta k_y\hat{y} + 2k_c^2\hat{z}\right\} - j\sin\left(\alpha\frac{\pi}{2}\right)\left\{-2jkk_y\hat{x} + 2jkk_x\hat{y}\right\}\right]e^{-jk_xx-jk_yy-j\beta z} \tag{10d}$$

$$\eta\mathbf{H}_{1fd} = -\frac{jA_{mn}}{8k_c^2}\left[\cos\left(\alpha\frac{\pi}{2}\right)\left\{2jkk_y\hat{x} - 2jkk_x\hat{y}\right\} - j\sin\left(\alpha\frac{\pi}{2}\right)\left\{2\beta k_x\hat{x} + 2\beta k_y\hat{y} + 2k_c^2\hat{z}\right\}\right]e^{jk_xx+jk_yy-j\beta z} \tag{10e}$$

$$\eta\mathbf{H}_{2fd} = \frac{jA_{mn}}{8k_c^2}\left[\cos\left(\alpha\frac{\pi}{2}\right)\left\{-2jkk_y\hat{x} - 2jkk_x\hat{y}\right\} + j\sin\left(\alpha\frac{\pi}{2}\right)\left\{2\beta k_x\hat{x} - 2\beta k_y\hat{y} + 2k_c^2\hat{z}\right\}\right]e^{jk_xx-jk_yy-j\beta z} \tag{10f}$$

$$\eta\mathbf{H}_{3fd} = \frac{jA_{mn}}{8k_c^2}[\cos\left(\alpha\frac{\pi}{2}\right)\{2jkk_y\hat{x}+2jkk_x\hat{y}\} +j\sin\left(\alpha\frac{\pi}{2}\right)\{-2\beta k_x\hat{x}+2\beta k_y\hat{y}+2k_c^2\hat{z}\}]e^{-jk_xx+jk_yy-j\beta z} \quad (10g)$$

$$\eta\mathbf{H}_{4fd} = -\frac{jA_{mn}}{8k_c^2}[\cos\left(\alpha\frac{\pi}{2}\right)\{-2jkk_y\hat{x}+2jkk_x\hat{y}\} -j\sin\left(\alpha\frac{\pi}{2}\right)\{-2\beta k_x\hat{x}-2\beta k_y\hat{y}+2k_c^2\hat{z}\}]e^{-jk_xx-jk_yy-j\beta z} \quad (10h)$$

Putting (10) in (7), the fractional field $(\mathbf{E}_{fd}, \eta\mathbf{H}_{fd})$ are given by

$$\mathbf{E}_{fd} = E_{xfd}\hat{x}+E_{yfd}\hat{y}+E_{zfd}\hat{z} \quad (11a)$$

$$\eta\mathbf{H}_{fd} = \eta H_{xfd}\hat{x}+\eta H_{yfd}\hat{y}+\eta H_{zfd}\hat{z} \quad (11b)$$

where

$$\mathbf{E}_{xfd} = -\frac{jA_{mn}}{k_c^2}\left\{\beta k_x\cos\left(\alpha\frac{\pi}{2}\right)+kk_y\sin\left(\alpha\frac{\pi}{2}\right)\right\}\cos\left(k_xx-\alpha\frac{\pi}{2}\right) \times\sin\left(k_yy-\alpha\frac{\pi}{2}\right)e^{-j\beta z} \quad (12a)$$

$$\mathbf{E}_{yfd} = -\frac{jA_{mn}}{k_c^2}\left\{\beta k_y\cos\left(\alpha\frac{\pi}{2}\right)-kk_x\sin\left(\alpha\frac{\pi}{2}\right)\right\}\sin\left(k_xx-\alpha\frac{\pi}{2}\right) \times\cos\left(k_yy-\alpha\frac{\pi}{2}\right)e^{-j\beta z} \quad (12b)$$

$$\mathbf{E}_{zfd} = A_{mn}\cos\left(\alpha\frac{\pi}{2}\right)\sin\left(k_xx-\alpha\frac{\pi}{2}\right)\sin\left(k_yy-\alpha\frac{\pi}{2}\right)e^{-j\beta z} \quad (12c)$$

$$\eta\mathbf{H}_{xfd} = \frac{jA_{mn}}{k_c^2}\left\{kk_y\cos\left(\alpha\frac{\pi}{2}\right)-\beta k_x\sin(\alpha\frac{\pi}{2})\right\}\sin\left(k_xx-\alpha\frac{\pi}{2}\right) \times\cos\left(k_yy-\alpha\frac{\pi}{2}\right)e^{-j\beta z} \quad (12d)$$

$$\eta\mathbf{H}_{yfd} = -\frac{jA_{mn}}{k_c^2}\left\{kk_x\cos\left(\alpha\frac{\pi}{2}\right)+\beta k_y\sin\left(\alpha\frac{\pi}{2}\right)\right\}\cos\left(k_xx-\alpha\frac{\pi}{2}\right) \times\sin\left(k_yy-\alpha\frac{\pi}{2}\right)e^{-j\beta z} \quad (12e)$$

$$\eta\mathbf{H}_{zfd} = -A_{mn}\sin\left(\alpha\frac{\pi}{2}\right)\cos\left(k_xx-\alpha\frac{\pi}{2}\right)\cos\left(k_yy-\alpha\frac{\pi}{2}\right)e^{-j\beta z} \quad (12f)$$

Equation (12) gives the fractional dual solution.

For $\alpha=0$,

$$\mathbf{E}_{xfd} = -\frac{jA_{mn}}{k_c^2}\beta k_x\cos(k_xx)\sin(k_yy)e^{-j\beta z} = E_x \quad (13a)$$

$$\mathbf{E}_{yfd} = -\frac{jA_{mn}}{k_c^2}\beta k_y\sin(k_xx)\cos(k_yy)e^{-j\beta z} = E_y \quad (13b)$$

$$\mathbf{E}_{zfd} = A_{mn} \sin(k_x x) \sin(k_y y) e^{-j\beta z} = E_z \tag{13c}$$

$$\eta \mathbf{H}_{xfd} = \frac{jA_{mn}}{k_c^2} k k_y \sin(k_x x) \cos(k_y y) e^{-j\beta z} = \eta H_x \tag{13d}$$

$$\eta \mathbf{H}_{yfd} = -\frac{jA_{mn}}{k_c^2} k k_x \cos(k_x x) \sin(k_y y) e^{-j\beta z} = \eta H_y \tag{13e}$$

$$\eta \mathbf{H}_{zfd} = 0 = \eta H_z \tag{13f}$$

yields original TM solution in PEC waveguide.

For $\alpha = 1$, we get

$$\mathbf{E}_{xfd} = \frac{jA_{mn}}{k_c^2} k k_y \sin(k_x x) \cos(k_y y) e^{-j\beta z} = \eta H_x \tag{14a}$$

$$\mathbf{E}_{yfd} = -\frac{jA_{mn}}{k_c^2} k k_x \cos(k_x x) \sin(k_y y) e^{-j\beta z} = \eta H_y \tag{14b}$$

$$\mathbf{E}_{zfd} = 0 = \eta H_z \tag{14c}$$

$$\eta \mathbf{H}_{xfd} = \frac{jA_{mn}}{k_c^2} \beta k_x \cos(k_x x) \sin(k_y y) e^{-j\beta z} = -E_x \tag{14d}$$

$$\eta \mathbf{H}_{yfd} = \frac{jA_{mn}}{k_c^2} \beta k_y \sin(k_x x) \cos(k_y y) e^{-j\beta z} = -E_y \tag{14e}$$

$$\eta \mathbf{H}_{zfd} = -A_{mn} \sin(k_x x) \sin(k_y y) e^{-j\beta z} = -E_z \tag{14f}$$

The field for $\alpha = 1$ is dual to the field for $\alpha = 0$. For $0 < \alpha < 1$, the field given by (12) may be regarded as intermediate between (13) and (14), and may be called fractional dual field or fractional field. Field expressions (14) are TE fields in a rectangular waveguide with perfect magnetic conductor (PMC) walls. From (12) it may be noted, for $0 < \alpha < 1$, behavior is changing in a same way as discussed in [7]. PEC guide is changing to PMC and TM mode is changing to TE mode.

For $\alpha = 2$,

$$\mathbf{E}_{xfd} = \frac{jA_{mn}}{k_c^2} \beta k_x \cos(k_x x) \sin(k_y y) e^{-j\beta z} = -E_x \tag{15a}$$

$$\mathbf{E}_{yfd} = \frac{jA_{mn}}{k_c^2} \beta k_y \sin(k_x x) \cos(k_y y) e^{-j\beta z} = -E_y \tag{15b}$$

$$\mathbf{E}_{zfd} = -A_{mn} \sin(k_x x) \sin(k_y y) e^{-j\beta z} = -E_z \tag{15c}$$

$$\eta \mathbf{H}_{xfd} = -\frac{jA_{mn}}{k_c^2} k k_y \sin(k_x x) \cos(k_y y) e^{-j\beta z} = -\eta H_x \tag{15d}$$

$$\eta \mathbf{H}_{yfd} = \frac{jA_{mn}}{k_c^2} k k_x \cos(k_x x) \sin(k_y y) e^{-j\beta z} = -\eta H_y \tag{15e}$$

$$\eta \mathbf{H}_{zfd} = 0 = -\eta H_z \tag{15f}$$

and we have TM mode in PEC waveguide and these solution are dual to solution for $\alpha = 1$.

For $\alpha = 3$,

$$\begin{aligned}
\mathbf{E}_{xfd} &= -\frac{jA_{mn}}{k_c^2} k k_y \sin(k_x x)\cos(k_y y) e^{-j\beta z} = -\eta H_x && (16a)\\
\mathbf{E}_{yfd} &= \frac{jA_{mn}}{k_c^2} k k_x \cos(k_x x)\sin(k_y y) e^{-j\beta z} = -\eta H_y && (16b)\\
\mathbf{E}_{zfd} &= 0 = -\eta H_z && (16c)\\
\eta\mathbf{H}_{xfd} &= -\frac{jA_{mn}}{k_c^2} \beta k_x \cos(k_x x)\sin(k_y y) e^{-j\beta z} = E_x && (16d)\\
\eta\mathbf{H}_{yfd} &= -\frac{jA_{mn}}{k_c^2} \beta k_y \sin(k_x x)\cos(k_y y) e^{-j\beta z} = E_y && (16e)\\
\eta\mathbf{H}_{zfd} &= A_{mn} \sin(k_x x)\sin(k_y y) e^{-j\beta z} = E_z && (16f)
\end{aligned}$$

we have TE solution in PMC waveguide and these solution are dual to solution for $\alpha = 2$.

For $\alpha = 4$,

$$\begin{aligned}
\mathbf{E}_{xfd} &= -\frac{jA_{mn}}{k_c^2} \beta k_x \cos(k_x x)\sin(k_y y) e^{-j\beta z} = E_x && (17a)\\
\mathbf{E}_{yfd} &= -\frac{jA_{mn}}{k_c^2} \beta k_y \sin(k_x x)\cos(k_y y) e^{-j\beta z} = E_y && (17b)\\
\mathbf{E}_{zfd} &= A_{mn} \sin(k_x x)\sin(k_y y) e^{-j\beta z} = E_z && (17c)\\
\eta\mathbf{H}_{xfd} &= \frac{jA_{mn}}{k_c^2} k k_y \sin(k_x x)\cos(k_y y) e^{-j\beta z} = \eta H_x && (17d)\\
\eta\mathbf{H}_{yfd} &= -\frac{jA_{mn}}{k_c^2} k k_x \cos(k_x x)\sin(k_y y) e^{-j\beta z} = \eta H_y && (17e)\\
\eta\mathbf{H}_{zfd} &= 0 = \eta H_z && (17f)
\end{aligned}$$

These are original solution so, fractional dual field are periodic with period $\alpha = 4$.

The characteristic impedance of the fractional guide is

$$\overline{\overline{\mathbf{Z}}} = \begin{bmatrix} \frac{E_{xfd}}{H_{xfd}} & \frac{E_{xfd}}{H_{yfd}} \\ -\frac{E_{yfd}}{H_{xfd}} & \frac{E_{yfd}}{H_{yfd}} \end{bmatrix} \tag{18}$$

$$\overline{\overline{\mathbf{Z}}} = \eta \begin{bmatrix} -\frac{\beta k_x \cos(\alpha\frac{\pi}{2})+kk_y \sin(\alpha\frac{\pi}{2})}{kk_y \cos(\alpha\frac{\pi}{2})-\beta k_x \sin(\alpha\frac{\pi}{2})} \frac{\cot(k_x x-\alpha\frac{\pi}{2})}{\cot(k_y y-\alpha\frac{\pi}{2})} & \frac{\beta k_x \cos(\alpha\frac{\pi}{2})+kk_y \sin(\alpha\frac{\pi}{2})}{kk_x \cos(\alpha\frac{\pi}{2})+\beta k_y \sin(\alpha\frac{\pi}{2})} \\ \frac{\beta k_y \cos(\alpha\frac{\pi}{2})-kk_x \sin(\alpha\frac{\pi}{2})}{kk_y \cos(\alpha\frac{\pi}{2})-\beta k_x \sin(\alpha\frac{\pi}{2})} & \frac{\beta k_y \cos(\alpha\frac{\pi}{2})-kk_x \sin(\alpha\frac{\pi}{2})}{kk_x \cos(\alpha\frac{\pi}{2})+\beta k_y \sin(\alpha\frac{\pi}{2})} \frac{\tan(k_x x-\alpha\frac{\pi}{2})}{\tan(k_y y-\alpha\frac{\pi}{2})} \end{bmatrix} \quad (19)$$

Equation (19) gives the characteristic impedance of fractional waveguide.

For $\alpha = 0$

$$\overline{\overline{\mathbf{Z}}} = \eta \begin{bmatrix} -\frac{\beta k_x}{kk_y} \frac{\cot(k_x x)}{\cot(k_y y)} & \frac{\beta}{k} \\ \frac{\beta}{k} & \frac{\beta k_y}{kk_x} \frac{\tan(k_x x)}{\tan(k_y y)} \end{bmatrix} \quad (20)$$

we have characteristic impedance of TM mode.

For $\alpha = 1$

$$\overline{\overline{\mathbf{Z}}} = \eta \begin{bmatrix} \frac{kk_y}{\beta k_x} \frac{\tan(k_x x)}{\tan(k_y y)} & \frac{k}{\beta} \\ \frac{k}{\beta} & -\frac{kk_x}{\beta k_y} \frac{\cot(k_x x)}{\cot(k_y y)} \end{bmatrix} \quad (21)$$

we have characteristic impedance of TE mode. For $\alpha = 2$, we get the same result as for $\alpha = 0$. It can be seen from (19), that characteristic impedance is periodic with period $\alpha = 2$.

3. FIELD LINES IN TRANSVERSE PLANE

To study the behavior of field lines in the waveguide, we select xy-plane which is transverse to the guide. We have plotted the field lines for square guide($a = b = 1$) for $m = n = 1$. For this mode $k_c = \sqrt{2}\pi$ so we have taken $k = 2\pi$. To plot field lines we need to write instantaneous field expression, by multiplying (12) with $\exp(j\omega t)$ and taking the real part, that is

$$\begin{aligned} \mathbf{E}_{xfd} &= \frac{A_{mn}}{k_c^2}\{\beta k_x \cos(\alpha\frac{\pi}{2}) + kk_y \sin(\alpha\frac{\pi}{2})\} \cos(k_x x - \alpha\frac{\pi}{2}) \\ &\quad \times \sin(k_y y - \alpha\frac{\pi}{2}) \sin(\beta z - \omega t) \qquad (22a) \\ \mathbf{E}_{yfd} &= \frac{A_{mn}}{k_c^2}\{\beta k_y \cos(\alpha\frac{\pi}{2}) - kk_x \sin(\alpha\frac{\pi}{2})\} \sin(k_x x - \alpha\frac{\pi}{2}) \\ &\quad \times \cos(k_y y - \alpha\frac{\pi}{2}) \sin(\beta z - \omega t) \qquad (22b) \\ \mathbf{E}_{zfd} &= A_{mn} \cos(\alpha\frac{\pi}{2}) \sin(k_x x - \alpha\frac{\pi}{2}) \sin(k_y y - \alpha\frac{\pi}{2}) \cos(\beta z - \omega t) \qquad (22c) \end{aligned}$$

$$\eta\mathbf{H}_{xfd} = -\frac{A_{mn}}{k_c^2}\{kk_y\cos(\alpha\frac{\pi}{2}) - \beta k_x\sin(\alpha\frac{\pi}{2})\}\sin(k_x x - \alpha\frac{\pi}{2})$$
$$\times\cos(k_y y - \alpha\frac{\pi}{2})\sin(\beta z - \omega t) \quad (22\text{d})$$

$$\eta\mathbf{H}_{yfd} = \frac{A_{mn}}{k_c^2}\{kk_x\cos(\alpha\frac{\pi}{2}) + \beta k_y\sin(\alpha\frac{\pi}{2})\}\cos(k_x x - \alpha\frac{\pi}{2})$$
$$\times\sin(k_y y - \alpha\frac{\pi}{2})\sin(\beta z - \omega t) \quad (22\text{e})$$

$$\eta\mathbf{H}_{zfd} = -A_{mn}\sin(\alpha\frac{\pi}{2})\cos(k_x x-\alpha\frac{\pi}{2})\cos(k_y y-\alpha\frac{\pi}{2})\cos(\beta z-\omega t) \quad (22\text{f})$$

Slope of Electric field lines in xy-plane is given by

$$\frac{dy}{dx} = \frac{E_{yfd}}{E_{xfd}} \quad (23)$$

putting values in (23) from (22) and integrating we get following equation which describes electric field lines in the guide for different values of constant

$$\frac{\ln\cos(k_y y - \alpha\frac{\pi}{2})}{k_y\{\beta k_y\cos(\alpha\frac{\pi}{2}) - kk_x\sin(\alpha\frac{\pi}{2})\}} - \frac{\ln\cos(k_x x - \alpha\frac{\pi}{2})}{k_x\{\beta k_x\cos(\alpha\frac{\pi}{2}) + kk_y\sin(\alpha\frac{\pi}{2})\}} = C_1 \quad (24)$$

where C_1 is a constant. Plots of 30 such lines are given in Figure 1 for different values of α. Similarly for magnetic field

$$\frac{dy}{dx} = \frac{H_{yfd}}{H_{xfd}} \quad (25)$$

and its solution is

$$\frac{\ln\sin(k_y y - \alpha\frac{\pi}{2})}{k_y\{kk_x\cos(\alpha\frac{\pi}{2}) + \beta k_y\sin(\alpha\frac{\pi}{2})\}} - \frac{\ln\sin(k_x x - \alpha\frac{\pi}{2})}{k_x\{kk_y\cos(\alpha\frac{\pi}{2}) - \beta k_x\sin(\alpha\frac{\pi}{2})\}} = C_2 \quad (26)$$

and C_2 is another constant. Plots of 30 field lines is given in Figure 2 for different values of α. For $\alpha = 0$, plot (1a) gives electric field and plot (2a) gives magnetic field for TM_{11} mode in PEC waveguide. These plots are in agreement with [10] and [11]. It can be seen from these plots that electric field lines are perpendicular to guide walls and magnetic field lines are parallel. Plot (1e) gives electric field lines and plot (2e) magnetic field lines for $\alpha = 1$. It is evident from these plots that electric field lines are parallel and magnetic field lines are perpendicular to guide walls, so, the guide can be considered as made

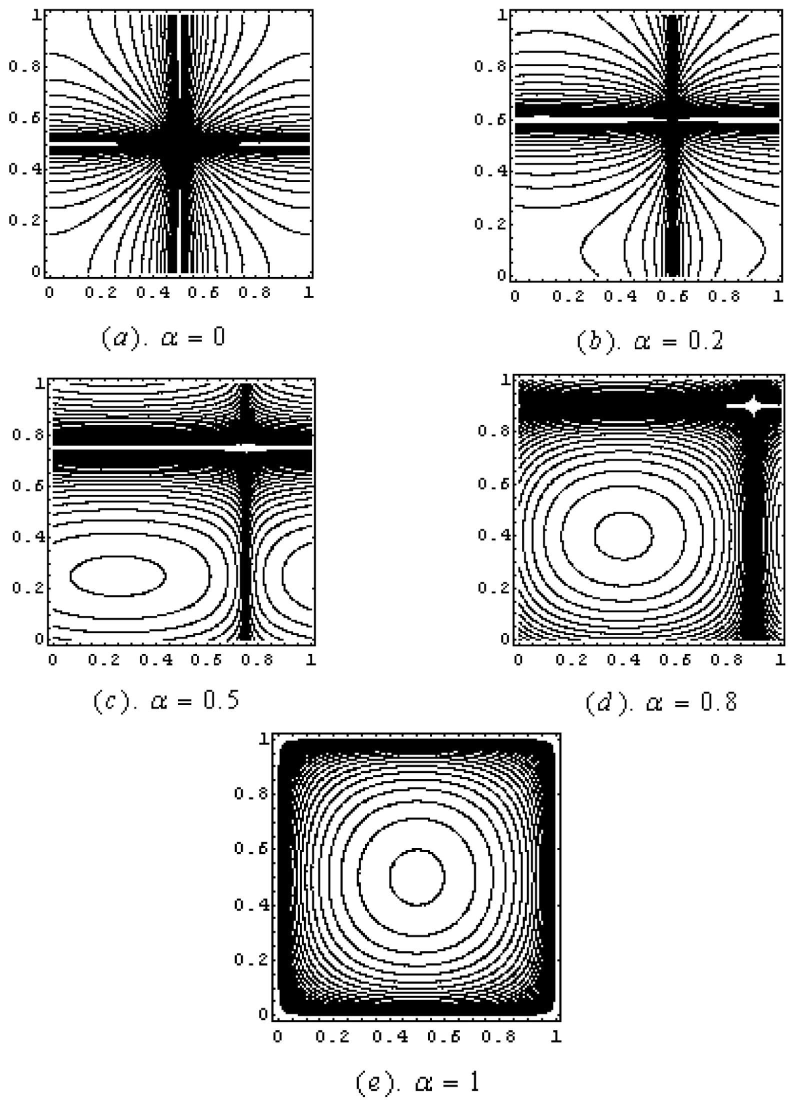

Figure 1. Plots of electric field lines in transverse plane.

up of perfect magnetic conductor (PMC). Plots (1b–1d) shows the electric field lines and plots (2b–2d) magnetic field lines for $\alpha = 0.2$, 0.5 and 0.8 respectively. The electric and magnetic fields are neither perpendicular nor parallel to walls so the walls can be considered as intermediate of PEC and PMC.

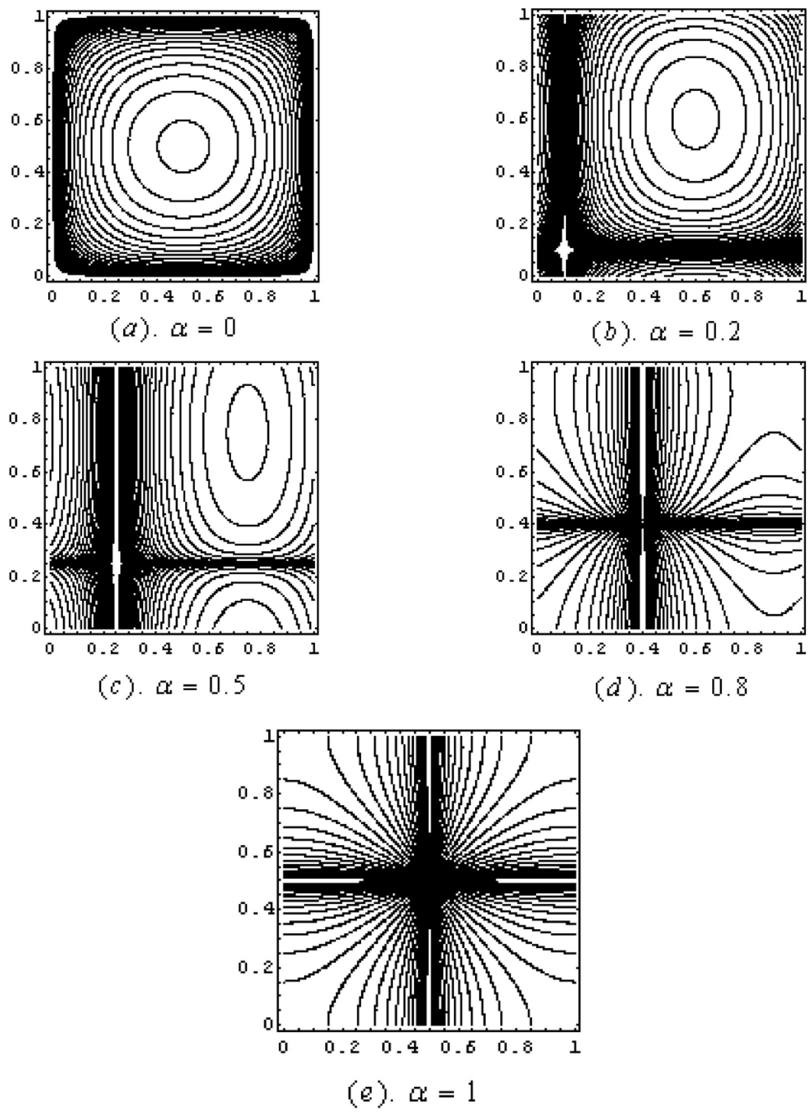

Figure 2. Plots of magnetic field lines in transverse plane.

4. CONCLUSION

In this paper we have studied the fractional dual field in rectangular waveguide. It is noted that for $\alpha = 0$, we have original solution i.e., TM solution in PEC waveguide. For $\alpha = 1$, we have solution dual to original i.e., TE solution in PMC waveguide. For $0 < \alpha < 1$, we have solution which may be regarded as intermediate of these two solutions. It is noted that these solution are periodic with period $\alpha = 4$. Also we calculated the characteristic impedance of fractional or intermediate waveguide. It is seen that for $\alpha = 0$, we have impedance for TM mode and for $\alpha = 1$, we have impedance for TE mode. For $0 < \alpha < 1$, we have fractional characteristic impedance (characteristic impedance of fractional waveguide). At the end we have shown and discussed the behavior of electric and magnetic field lines inside fractional waveguide. It is noted that for $\alpha = 0$, electric field lines are perpendicular and magnetic field lines parallel, so guide walls may be considered as made of walls PEC. For $\alpha = 1$, magnetic field lines are perpendicular and electric field lines are parallel, so, guide walls may be considered as of PMC walls. For $0 < \alpha < 1$, both electric and magnetic field lines are neither completely perpendicular nor parallel to waveguide walls, so, guide walls can be considered intermediate between PEC and PMC.

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