*Progress In Electromagnetics Research, PIER 74, 273–289, 2007*

# TRANSIENT SOLUTIONS OF MAXWELL'S EQUATIONS BASED ON SUMUDU TRANSFORM

**M. G. M. Hussain**

Electrical Engineering Department
Kuwait University
P.O. Box 5969, Safat 13060, Kuwait

**F. B. M. Belgacem**

Faculty of Information Technology
Arab Open University
P.O. Box 3322, Safat 13033, Kuwait

**Abstract**—The Sumudu transform is derived from the classical Fourier integral. Based on the mathematical simplicity of the Sumudu transform and its fundamental properties, Maxwell's equations are solved for transient electromagnetic waves propagating in lossy conducting media. The Sumudu transform of Maxwell's differential equations yields a solution directly in the time domain, which neutralizes the need to perform inverse Sumudu transform. Two sets of computer plots are generated for the solution of Maxwell's equations for transient electric field strength in lossy medium. A set of plots presents the Sumudu transform of the transient solution and another one presents inverse Sumudu transform. Both sets of plots reveal similar characteristics and convey equal information. Such property is referred to as the *Sumudu reciprocity*.

## 1. INTRODUCTION

An integral transform, referred to as Sumudu transform, was introduced by Watugala [1, 2] to facilitate the process of solving differential and integral equations in the time domain, and for the use in various applications of system engineering and applied physics. Although the mathematical properties of the Sumudu transform have been explored in some details [3–11], to the best of our knowledge, no systematic derivation of the Sumudu transform is available in the open literature.

The phenomenon of electromagnetic-wave propagation in different media is well described by solutions of Maxwell's differential equations that satisfy specified set of initial and boundary conditions [12–18]. Typically, Fourier and Laplace transforms are the convenient mathematical tools for solving differential equations [19]. In this paper, we first present a systematic derivation of the Sumudu transform, and based on its fundamental properties we use it to solve Maxwell's equations for transient excitation functions propagating in a lossy conducting medium. The Sumudu transform of Maxwell's equations yields a solution directly in the time domain. The obtained solution and its inverse Sumudu transform are of similar characteristics and provide equal information regarding the phenomenon of transient-wave propagation in lossy conducting media. This property of the Sumudu transform is referred to as the *Sumudu reciprocity*, which is an attractive one for solving differential equations and investigating many practical problems in the time domain without the need for performing inverse Sumudu transform. The Fourier transform as well as the Laplace transform methods of solving differential equations require an inverse transform operation in order to obtain a time-domain solution.

In Section 2, starting with the classical Fourier integral, a detailed derivation of the Sumudu transform is presented, and its relationship to the Laplace transform is established. The Sumudu transform of the unit-step function, the ramp function, the exponential-ramp function, and the Gaussian pulse are presented and plotted too. These functions are often used in many engineering problems. In Section 3, the Sumudu transform and some of its fundamental properties are used to solve Maxwell's equations for a transient electric field excitation function propagating in a lossy conducting medium. Two sets of computer plots are generated for the solution of Maxwell's equations for transient electric field strengths with the time variation of an exponential-ramp function and that of a Gaussian pulse. One set of plots presents the Sumudu transform of the transient solution and another set presents the inverse Sumudu transform of the same solution. Both sets of plots reveal similar characteristics and provide equal information. Conclusions are given in Section 4.

## 2. THE ORIGIN OF SUMUDU TRANSFORM

For a function $f(t)$ which is of exponential order,

$$|f(t)| < \begin{cases} Me^{-t/\tau_1} & \text{for } t \leq 0, \\ Me^{t/\tau_2} & \text{for } t \geq 0, \end{cases} \tag{1}$$

the Sumudu transform, henceforth designated by the operator $\mathbb{S}[.]$, is defined by the integral equation

$$\mathbb{S}[f(t)] = G(u) = \int_0^{\infty} f(ut)e^{-t}\,dt, \qquad -\tau_1 \leq u \leq \tau_2, \tag{2}$$

where $M$ is a real finite number and $\tau_1$ and $\tau_2$ can be finite or infinite [2]. Analogous to the Laplace transform, which was derived from the Fourier transform to account for transient functions, the Sumudu transform given in (2) can also be derived directly from the Fourier integral. Furthermore, the Sumudu transform can be related to the Laplace transform, as will be demonstrated shortly.

The Fourier transform pair for a function $f(t)$ that satisfies the condition for absolute convergence, $\int_{-\infty}^{+\infty} |f(t)|\,dt < \infty$, is given by the integrals,

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt \tag{3}$$

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\,d\omega, \tag{4}$$

where the operators $\mathcal{F}[.]$ and $\mathcal{F}^{-1}[.]$ denote Fourier transform and inverse Fourier transform, respectively. If $f(t)$ does not vanish at $\infty$, absolute convergence can be enforced by the exponential order condition described in (1) such that,

$$\lim_{T\to\infty}\int_0^{T} |e^{-\alpha t}f(t)|\,dt < \infty, \tag{5}$$

where $\alpha$ is constant and $T$ time interval. The lower bound $\alpha_a$ of all values of $\alpha$ which satisfy (5) is called the abscissa of absolute convergence. Analogous to (3) and (4), the Fourier transform pair for the product $f(t)e^{-\alpha t}$ is written as follows,

$$F(\alpha + i\omega) = \int_{-\infty}^{+\infty} f(t)e^{-(\alpha+i\omega)t}\,dt \tag{6}$$

$$f(t)e^{-\alpha t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\alpha + i\omega)e^{i\omega t}\,d\omega \tag{7}$$

where $\alpha > \alpha_a$ and $t \geq 0$. Note that for $t < 0$ the integral given in (7) must equal to zero since $f(t)$ is said to be of exponential order. Hence, the inverse Fourier transform can be expressed in terms of the unit step function $s(t)$,

$$f(t)s(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\alpha + i\omega)e^{(\alpha+i\omega)t}\,d\omega. \tag{8}$$

$$s(t) = \begin{cases} 1, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases} \tag{9}$$

Solving first the integrals in (6) and (8) and then applying the limit $\alpha \to 0$ to the solutions, one obtains for $\alpha_a \leq 0$ the Fourier transform pair for functions that are not absolutely convergent. The introduction of the Laplace transform proved that such laborious steps are unnecessary.

With the substitution $s = \alpha + i\omega, \alpha > \alpha_a$, the Fourier transform integral given in (6) and the inverse Fourier transform integral given in (8) become the Laplace–transform pair of the function $f(t)$, respectively [19],

$$\mathcal{L}[f(t)] = F(s) = \int_{-\infty}^{+\infty} f(t)e^{-st}\,dt, \qquad Re(s) > \alpha_a, \tag{10}$$

$$\mathcal{L}^{-1}[F(s)] = f(t)s(t) = \frac{1}{i2\pi}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)e^{st}\,ds, \tag{11}$$

where the abscissa of convergence $\alpha_a$ is determined by the functional properties of $f(t)$. The operators $\mathcal{L}[.]$ and $\mathcal{L}^{-1}[.]$ denote Laplace transform and inverse Laplace transform, respectively. Since $f(t)s(t) = 0,\ \text{for } t < 0$, and $Re[s] = \alpha > \alpha_a$, the Laplace transform can be interpreted as a mapping of the points lying on the *positive real axis* of $t$ onto that portion of the *complex plane* of $s$ which lies to the right of the abscissa of convergence $\alpha_a$. Such a mapping, which is of significance in applied physics and engineering, does not exists for the Sumudu transform.

The Sumudu transform given by (2) can now be derived explicitly from the Fourier integral given in (7). Integration over time from $-\infty$ to $+\infty$ of both sides of (7) yields the following two integrals,

$$q_1(\alpha) = \int_{-\infty}^{+\infty} f(t)e^{-\alpha t}\,dt = \int_{0}^{+\infty} f(t)e^{-\alpha t}\,dt \tag{12}$$

$$q_2(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\alpha + i\omega)\left(\int_{-\infty}^{+\infty} e^{i\omega t}\,dt\right)d\omega. \tag{13}$$

Using the substitutions $w = \alpha t$ and $u = 1/\alpha$ into (12), one obtains the Sumudu transform of the function $f(t)$,

$$\begin{aligned} \alpha q_1(\alpha) &= \alpha \int_0^{+\infty} f(t) e^{-\alpha t}\, dt \\ &= \int_0^{+\infty} f(w/\alpha) e^{-w}\, dw = G(1/\alpha) = G(u), \end{aligned} \tag{14}$$

where $G(u)$ is the Sumudu transform.

With the help of the complex-conjugate property of the Fourier transform, $\mathcal{F}[f^*(t)] = F^*(-\omega)$, and $\mathcal{F}[1] = 2\pi\delta(\omega)$, where $(*)$ denotes complex conjugate and $\delta(t)$ is the Dirac-delta function, one obtains

$$\begin{aligned} q_2(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\alpha + i\omega) \left( \int_{-\infty}^{+\infty} e^{-i\omega t}\, dt \right)^* d\omega \\ &= \int_{-\infty}^{+\infty} F(\alpha + i\omega) \delta(\omega)\, d\omega = F(\alpha) = F(s)\Big|_{s=\alpha+i0}. \end{aligned} \tag{15}$$

According to (14) and (15), the relationship between the Sumudu transform and the Laplace transform is expressed as follows,

$$G(u) = G\left(\frac{1}{\alpha}\right) = \alpha F(\alpha), \qquad Re[s] = \alpha > \alpha_a. \tag{16}$$

Two-dimensional Sumudu transform is described in [20], and can be useful for the applications of two-dimensional signal and image processing.

Let us list the Sumudu transform of certain functions that are of interest in practice:

*Unit-step function* $s(t)$,

$$\begin{aligned} \mathbb{S}[s(t)] = S(u) &= 1, && \text{for } u \geq 0 \\ &= 0, && \text{for } u < 0 \end{aligned} \tag{17}$$

*Ramp function* $r(t)$,

$$\begin{aligned} r(t) &= t, && \text{for } t \geq 0 \\ &= 0, && \text{for } t < 0 \end{aligned} \tag{18}$$

$$\begin{aligned} \mathbb{S}[r(t)] = R(u) &= u, && \text{for } u \geq 0 \\ &= 0, && \text{for } u < 0 \end{aligned} \tag{19}$$

*Exponential ramp function* $e(t)$,

$$e(t) = 1 - e^{-t/\tau}, \qquad \text{for } t \geq 0$$
$$= 0, \qquad \text{for } t < 0 \qquad (20)$$

$$\mathbb{S}[e(t)] = \mathcal{E}(u) = \frac{u/\tau}{1 + (u/\tau)}, \qquad \text{for } u \geq 0$$
$$= 0, \qquad \text{for } u < 0 \qquad (21)$$

*Gaussian Pulse* $g(t)$,

$$g(t) = \exp\{-a^2(t - t_0)^2\}, \qquad \text{for } t \geq 0$$
$$= 0, \qquad \text{for } t < 0 \qquad (22)$$

$$\mathbb{S}[g(t)] = G(u) = \frac{\sqrt{\pi}}{2au} \operatorname{erfc}\left[\frac{1}{2au} - at_0\right] \times \exp\left\{\left(\frac{1}{2au}\right)^2 - \frac{t_0}{u}\right\}, \qquad \text{for } u \geq 0$$
$$= 0, \qquad \text{for } u < 0 \qquad (23)$$

In (23), erfc[.] is the complementary error function, and $g(t)$ is a Gaussian pulse with the spread parameter $a$ and unity peak amplitude positioned at the time instant $t = t_0$. The exponential ramp function $e(t)$ is shown in Fig. 1a as a function of normalized time $t/\tau$, and its Sumudu transform $\mathcal{E}(u)$ is shown in Fig. 1b as a function of $u/\tau$. The two plots in Fig. 1 are of a similar time variation. The Gaussian pulse $g(t)$ is plotted in Fig. 2a as a function of normalized time $t/\Delta T$, and its Sumudu transform $G(u)$ is plotted in Fig. 2b as a function of $u/\Delta T$. The spread parameter for the Gaussian pulse is $a = 2\sqrt{\pi}/\Delta T$, where $\Delta T$ is the nominal duration, and $t_0/\Delta T = 1$.

## 3. SOLUTION OF MAXWELL'S EQUATIONS

Laplace transform method has been used for solving Maxwell's equations based on transient excitation functions [12]. Here, we shall apply the Sumudu transform for solving the problem of transient propagation in an unbounded lossy medium with conductivity $\sigma > 0$.

Let a planar, transverse electromagnetic (TEM) wave propagate in the direction $z$ in a lossy medium with constant permittivity $\epsilon$, permeability $\mu$, and conductivity $\sigma$. The electric field vector $\mathbf{E}$ and

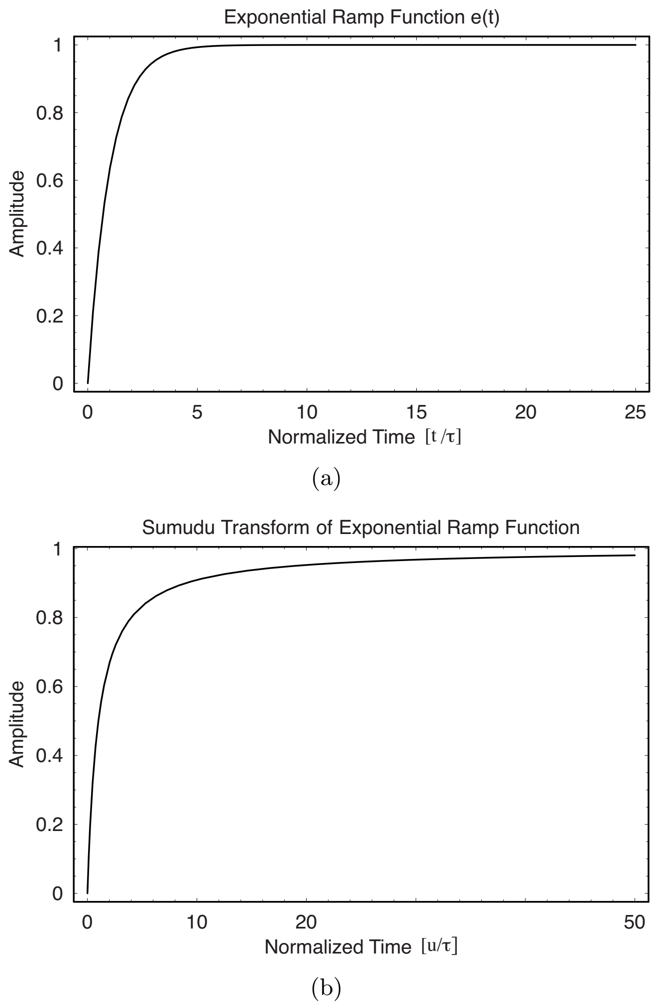



**Figure 1.** Exponential-ramp function $e(t)$ (a), and its Sumudu transform $\mathcal{E}$ (b).

the magnetic field vector $\mathbf{H}$ are related to one another by Maxwell's equations,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{24}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E}. \tag{25}$$

If the electric field vector is polarized along the $x$ direction such

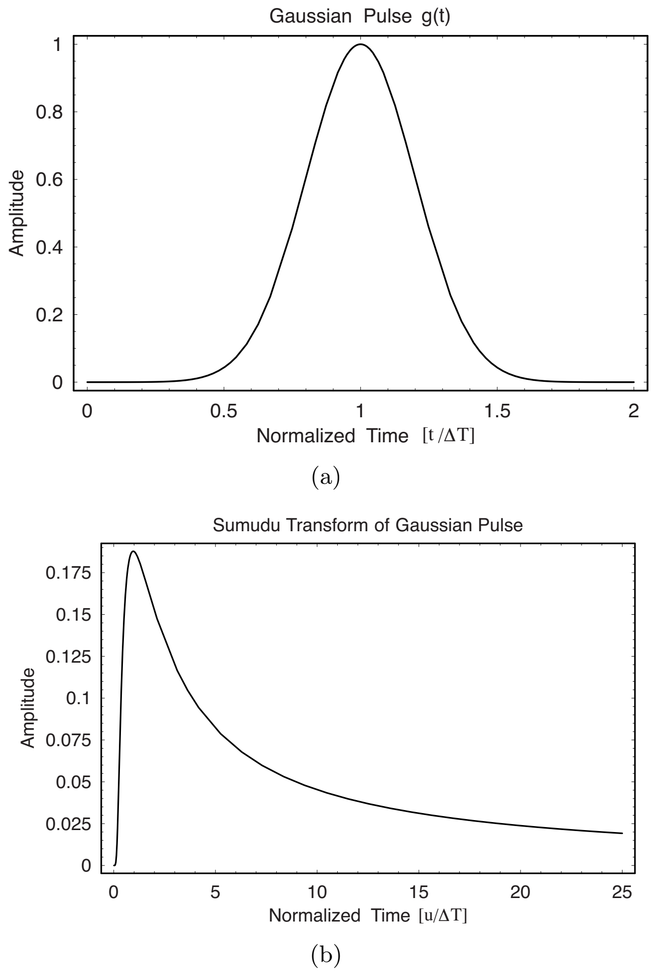



**Figure 2.** Gaussian pulse $g(t)$ with nominal duration $\Delta T$ (a), and its Sumudu transform $G(u)$ (b).

that $E = E_x(z, t)$ and the magnetic field $H = H_y(z, t)$, the Maxwell's equations given in (24) and (25) yield the following pair of differential equations,

$$\frac{\partial E_x}{\partial z} + \mu \frac{\partial H_y}{\partial t} = 0, \tag{26}$$

$$\frac{\partial H_y}{\partial z} + \epsilon \frac{\partial E_x}{\partial t} + \sigma E_x = 0. \tag{27}$$

With the help of the Sumudu transform property

$$\mathbb{S}\left[\frac{d^n f(t)}{dt^n}\right] = \frac{F(u)}{u^n} - \frac{f(0)}{u^n} - \cdots - \frac{d^{(n-1)} f(0)}{u}, \tag{28}$$

the Sumudu transform of (26) and (27) with respect to $t$ results in the differential equations

$$\frac{\partial F(z,u)}{\partial z} + \mu \frac{G(z,u)}{u} - \mu \frac{H(z,0)}{u} = 0, \tag{29}$$

$$\frac{\partial G(z,u)}{\partial z} + \epsilon \frac{F(z,u)}{u} + \sigma F(z,u) - \epsilon \frac{E(z,0)}{u} = 0, \tag{30}$$

where $F(z,u) = \mathbb{S}[E_x(z,t)]$ and $G(z,u) = \mathbb{S}[H_y(z,t)]$. In order to eliminate $G(z,u)$ and obtain a differential equation for $F(z,u)$ only, take the partial derivative of (29) with respect to $z$,

$$\frac{\partial^2 F(z,u)}{\partial z^2} + \frac{\mu}{u} \frac{\partial G(z,u)}{\partial z} - \frac{\mu}{u} \frac{\partial H(z,0)}{\partial z} = 0 \tag{31}$$

Upon elimination of $G(z,u)$ an ordinary inhomogeneous differential equation for $F(z,u)$ is obtained in which $u$ enters only as a parameter,

$$\frac{\partial^2 F(z,u)}{\partial z^2} - \left(\frac{\mu\epsilon}{u^2} + \frac{\mu\sigma}{u}\right) F(z,u) = \frac{\mu}{u}\left[\frac{\partial H(z,t)}{\partial z}\right]_{t=0} - \frac{\mu\epsilon}{u^2} E(z,0) \tag{32}$$

From (20), one obtains the relationship,

$$\left[\frac{\partial H(z,t)}{\partial z}\right]_{t=0} = -\epsilon \left[\frac{\partial E(z,t)}{\partial t}\right]_{t=0} - \sigma [E(z,t)]_{t=0}. \tag{33}$$

Insertion of (33) into (32) yields

$$\begin{aligned} \frac{\partial^2 F(z,u)}{\partial z^2} &- \left(\frac{\mu\epsilon}{u^2} + \frac{\mu\sigma}{u}\right) F(z,u) \\ &= -\left(\frac{\mu\epsilon}{u^2} + \frac{\mu\sigma}{u}\right) E(z,0) - \frac{\mu\epsilon}{u}\left[\frac{\partial E(z,t)}{\partial z}\right]_{t=0} \end{aligned} \tag{34}$$

Let us assume that the following initial conditions are known,

$$\lim_{t\to 0} E(z,t) = f_0(z), \qquad \lim_{t\to 0} \frac{\partial E(z,t)}{\partial t} = f_0'(z), \tag{35}$$

where the initial condition $E(z,0) = f_0(z)$ means strictly the limit of $E(z,t)$ as $t \to 0$. With the help of the substitution

$$\gamma^2 = \frac{\mu\epsilon}{u^2} + \frac{\mu\sigma}{u} \tag{36}$$

we obtain finally the differential equation

$$\frac{d^2F(z,u)}{dz^2} - \gamma^2 F(z,u) = -\gamma^2 f_0(z) - \frac{\mu\epsilon}{u} f_0'(z). \tag{37}$$

The right side of (37) can be defined by the function $W(z,u)$,

$$W(z,u) = -\gamma^2 f_0(z) - \frac{\mu\epsilon}{u} f_0'(z), \tag{38}$$

so that (37) can be expressed as follows,

$$\frac{d^2F(z,u)}{dz^2} - \gamma^2 F(z,u) = W(z,u). \tag{39}$$

The general solution of (39) is composed of two parts: *complementary function* $F_c(z,u)$ that is a solution of the differential equation

$$\frac{d^2F(z,u)}{dz^2} - \gamma^2 F(z,u) = 0, \tag{40}$$

and *particular solution* $F_p(z,u)$ of (39). The two solutions are of the forms,

$$F_c(z,u) = A(u)e^{\gamma z} + B(u)e^{-\gamma z} \tag{41}$$

$$F_p(z,u) = \frac{e^{\gamma z}}{2\gamma} \int e^{-\gamma z} W(z,u)\, dz + \frac{e^{-\gamma z}}{2\gamma} \int e^{\gamma z} W(z,u)\, dz \tag{42}$$

Consider the boundary condition

$$\lim_{z \to 0} E(z,t) = E(0,t) = f(t), \qquad t \geq 0, \tag{43}$$

and the assumption that for $z > 0$, the wave $f(t)$ is traveling in a lossy medium with conductivity $\sigma > 0$. In this case, one obtains from (41) $A(u) = 0$ and

$$F(0,u) = f(u) = \mathbb{S}[f(t)] = B(u). \tag{44}$$

With the substitutions $a = 1/\sqrt{\mu\epsilon}$ and $b = \sigma/2\epsilon$, the exponential function $e^{-\gamma z}/\gamma$ can be expressed as follows [12]:

$$\frac{e^{-\gamma z}}{\gamma} = a \int_{z/a}^{\infty} e^{-bt} J_0 \left( \frac{b}{a} \sqrt{z^2 - a^2 t^2} \right) e^{-t/u} \, dt \tag{45}$$

where $J_0(.)$ is the Bessel function of order zero. Differentiation of (45) with respect to $z$ yields:

$$e^{-\gamma z} = e^{-\frac{b}{a} z} e^{-\frac{1}{au} z} - a \int_{z/a}^{\infty} e^{-bt} \frac{\partial}{\partial z} J_0 \left( \frac{b}{a} \sqrt{z^2 - a^2 t^2} \right) e^{-t/u} \, dt. \tag{46}$$

By using the substitutions $\nu = t/u$, and $dt = u\, d\nu$, the integral in (46) reduces to the Sumudu transform of the function $\Phi(z, \nu)$,

$$\begin{aligned} e^{-\gamma z} &= e^{-\frac{b}{a} z} e^{-\frac{1}{au} z} - au \int_{z/au}^{\infty} \left[ e^{-b(u\nu)} \frac{\partial}{\partial z} J_0 \left( \frac{b}{a} \sqrt{z^2 - a^2 (u\nu)^2} \right) \right] e^{-\nu} \, d\nu \\ &= e^{-\frac{b}{a} z} e^{-\frac{1}{au} z} - au \, \mathbb{S}[\Phi(z, \nu)] \end{aligned} \tag{47}$$

where the function $\Phi(z, \nu)$ is defined as

$$\Phi(z, \nu) = \begin{cases} 0, & \text{for } 0 < \nu < z/a \\ e^{-b\nu} \frac{\partial}{\partial z} J_0 \left( \frac{b}{a} \sqrt{z^2 - (a\nu)^2} \right), & \text{for } \nu \geq z/a \end{cases} \tag{48}$$

Based on (41), (44), and (47), the Sumudu transform of the desired solution for the electric field strength $E(z, t)$ can be expressed as follows:

$$\begin{aligned} F(z, u) &= F(u) e^{-\gamma z} \\ &= F(u) e^{-\frac{b}{a} z} e^{-\frac{1}{au} z} - au F(u) \mathbb{S}[\Phi(z, \nu)] \end{aligned} \tag{49}$$

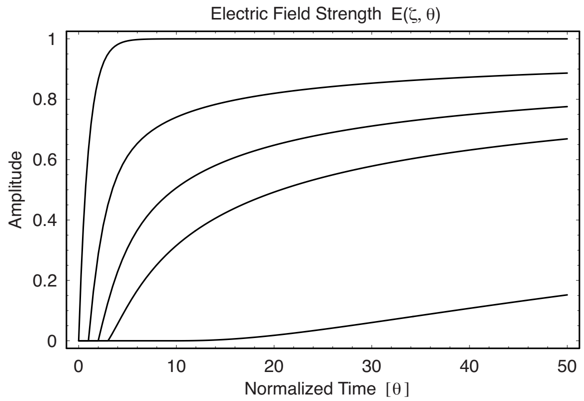
Consider the following properties of the Sumudu transform [6]:

$$\mathbb{S}[f(t - t_0)] = F(u) e^{-t_0/u} \tag{50}$$

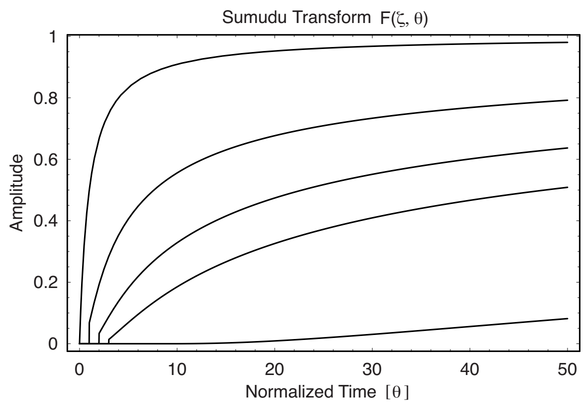
$$\mathbb{S}[f_1(t) \star f_2(t)] = u F_1(u) F_2(u) \tag{51}$$

where the star ($\star$) denotes convolution. Now, the inverse Sumudu transformation of (49) yields the solution

$$\begin{aligned} E(z, t) &= e^{-\frac{b}{a} z} f(t - z/a) \\ &\quad - a \int_{z/a}^{\infty} f(t - \lambda) e^{-b\lambda} \frac{\partial}{\partial z} J_0 \left( \frac{b}{a} \sqrt{z^2 - (a\lambda)^2} \right) d\lambda \end{aligned} \tag{52}$$




(a)



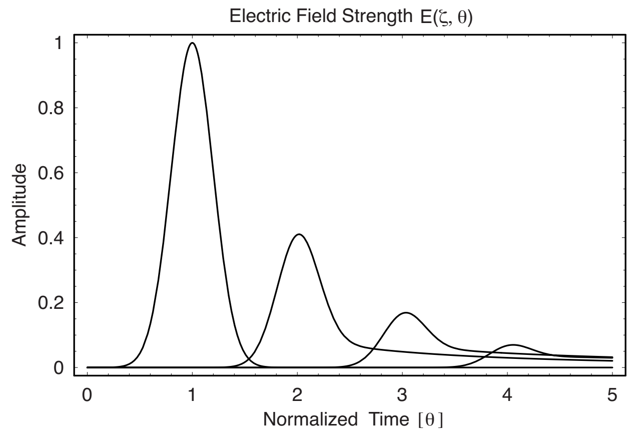

(b)

**Figure 3.** Electric field strength $E(\zeta, \theta)$ due to an exponential-ramp function (a), and its Sumudu transform $F(\zeta, \theta)(b)$. The excitation function is applied at the boundary of a lossy medium, i.e., sea surface, with the ratio $2\epsilon/\sigma = \tau$. The plots hold for the normalized distance $\zeta = 0, 1, 2, \text{and } 3$.

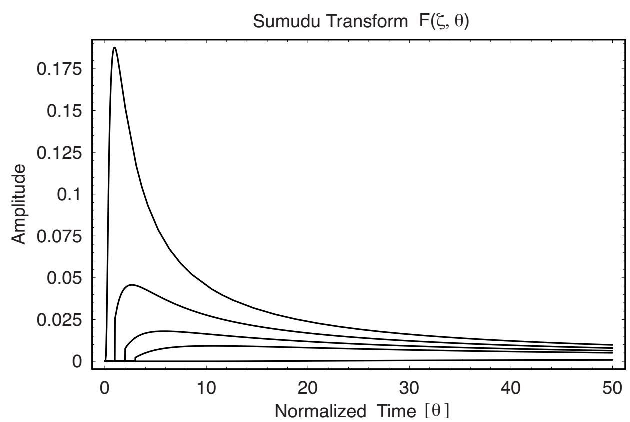
With the change of variable $\beta = t - \lambda$, (52) becomes

$$E(z,t) = e^{-\frac{b}{a}z} f(t - z/a) - ae^{-bt} \int_0^{t-z/a} f(\beta) e^{b\beta} \frac{\partial}{\partial z} J_0 \left( \frac{b}{a} \sqrt{z^2 - a^2(t-\beta)^2} \right) d\beta \quad (53)$$

The solution $E(z,t)$ given in (53) can be expressed in terms of a normalized time variable $\theta$ and a normalized space variable $\zeta$ for the




(a)




(b)

**Figure 4.** Electric field strength $E(\zeta, \theta)$ due to a Gaussian pulse of peak amplitude unity centered at $t_0/\Delta T = 1$, and nominal duration $\Delta T = 0.354\,\text{ns}$ (a), and its Sumudu transform $F(\zeta, \theta)$ (b). The Gaussian excitation function is applied at the boundary of a lossy medium, i.e., sea surface, with the ratio $2\epsilon/\sigma = \Delta T$. The plots hold for the normalized distance $\zeta = 0, 1, 2,$ and 3.

convenience of numerical computation. Consider the substitutions

$$\theta = bt = \frac{\sigma}{2\epsilon}t; \qquad \zeta = \frac{b}{a}z = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}z; \qquad \eta = b\beta = \frac{\sigma\beta}{2\epsilon} \tag{54}$$

and the following properties of the Bessel functions:

$$I_p(x) = (i)^{-p}J_p(ix), \qquad p = 0, 1, 2, \ldots, \tag{55}$$

$$\frac{d}{dx}I_0[f(x)] = I_1[f(x)]\frac{df(x)}{dx}, \tag{56}$$

where $I_p(.)$ is the modified Bessel function of order $p$. Based on (54)–(56), one obtains the normalized form for the electric field strength propagating in the lossy medium with conductivity $\sigma > 0$:

$$E(\zeta,\theta) = e^{-\zeta}f\left(\frac{2\epsilon}{\sigma}(\theta-\zeta)\right) + \zeta e^{-\theta}\int_0^{\theta-\zeta} e^{\eta}f\left(\frac{2\epsilon}{\sigma}\eta\right)\frac{I_1\left(\sqrt{(\theta-\eta)^2-\zeta^2}\right)}{\sqrt{(\theta-\eta)^2-\zeta^2}}\,d\eta \tag{57}$$

The solution for the associated magnetic field strength $H(\zeta,\theta)$ can be derived from either (26) or (27). Detailed derivation of $H(\zeta,\theta)$, based on the Laplace transformation method, is given in [18]. General solutions for $E(\zeta,\theta)$ and $H(\zeta,\theta)$ for electric as well as magnetic transient excitation functions are derived in [14] based on the modification of Maxwell's equations.

Analogous to (57), the Sumudu transform given in (49) can also be expressed in terms of normalized time variable $\theta = bu$ and space variable $\zeta = bz/a$ for the convenience of numerical computation and plotting,

$$F(\zeta,\theta) = \mathbb{S}[E(\zeta,\theta)] = e^{-\zeta}F\left(\frac{2\epsilon}{\sigma}\theta\right)e^{-\zeta/\theta} + \zeta e^{-\theta}\int_0^{\theta-\zeta} e^{\eta}F\left(\frac{2\epsilon}{\sigma}\theta\right)e^{-(\theta-\eta)/\theta}\frac{I_1\left(\sqrt{(\theta-\eta)^2-\zeta^2}\right)}{\sqrt{(\theta-\eta)^2-\zeta^2}}\,d\eta \tag{58}$$

Plots of the electric field strength $E(\zeta,\theta)$ due to the exponential-ramp function $f(t) = e(t)$ given in (20), applied at the boundary of a lossy medium with the ratio $2\epsilon/\sigma = \tau$, are shown in Fig. 3a for the normalized distances $\zeta = 0, 1, 2, 3,$ and 10. At the boundary where $\zeta = 0$, the electric field strength $E(0,\theta) = e(\theta)$. The corresponding

Sumudu transform $F(\zeta, \theta)$ of the electric field strength $E(\zeta, \theta)$ is shown in Fig. 3b. According to the plots in Fig. 3, $E(\zeta, \theta) = F(\zeta, \theta) = 0$ for $\theta < \zeta$, and for $\theta \geq \zeta$, the electric field $E(\zeta, \theta)$ and its Sumudu transform $F(\zeta, \theta)$ have similar characteristics for the different values of normalized distance $\zeta$.

Plots of the electric field strength $E(\zeta, \theta)$ due to a Gaussian pulse excitation function $g(t)$ given in (22), with $a = 2\sqrt{\pi}/\Delta T$, $\Delta T = 0.354\,\text{ns}$, are shown in Fig. 4a, for the normalized distances $\zeta = 0, 1, 2, 3,$ and 10. The lossy conducting medium in which the Gaussian pulse is propagating is *seawater* with relative permittivity $\epsilon_r = 80$, permittivity $\epsilon = 705 \times 10^{-12}\,\text{F/m}$, and conductivity $\sigma = 4$ S/m. In this case, the ratio $2\epsilon/\sigma = \Delta T$ in (57). The corresponding Sumudu transform $F(\zeta, \theta)$ of the Gaussian electric field strength $E(\zeta, \theta)$ is shown in Fig. 4b.

According to Fig. 3 and Fig. 4, the set of plots for the solution $E(\zeta, \theta)$ given in (57) and the plots for the Sumudu transform $F(\zeta, \theta)$ given in (58) reveal equal information regarding the characteristics of transient-wave propagation in a lossy conducting medium. In this case, performing an inverse Sumudu transform to obtain the time-domain solution $E(z, t)$ from its Sumudu transform $F(z, u)$ is a redundant step for the purpose of investigating the problem of transient-wave propagation in lossy media.

## 4. CONCLUSIONS

The origin of the Sumudu transform is traced back to the classical Fourier integral. The Sumudu transform is a convenient tool for solving differential equations in the time domain without the need for performing an inverse Sumudu transform. For transient excitation functions, the time-domain solution of Maxwell's differential equations and its Sumudu transform yield equal information regarding the phenomenon of wave propagation. This property is referred as the Sumudu reciprocity which is useful in engineering applications that involve solving differential equations.

## REFERENCES

1. Watugala, G. K., "Sumudu transform—an integral transform to solve differential equations and control engineering problems," *Inter. J. Math. Ed. Sci. Tech.,* Vol. 24, 35–42, 1993.
2. Watugala, G. K., "Sumudu transform a new integral transform to solve differential equations and control engineering problems,"

*Mathematical Engineering in Industry,* Vol. 6, No. 4, 319–329, 1998.

3. Belgacem, F. B. M., "Applications of the Sumudu transform to Indefinite Periodic Parabolic Equations," *Proceedings of the International Conference on Nonlinear Problems and Aerospace Applications,* 1–10, Budapest, Hungary, 2007.
4. Belgacem, F. B. M., "Introducing and analysing deeper Sumudu properties," *Nonlinear Studies Journal,* Vol. 13, No. 1, 23–41, 2006.
5. Belgacem, F. B. M. and A. A. Karaballi, "Sumudu transform fundamental properties investigations and applications," *Journal of Applied Mathematics and Stochastic Analysis,* 1–23, 2006.
6. Belgacem, F. B. M., A. A Karaballi, and L. S. Kalla, "Analytical investigations of the Sumudu transform, and applications to integral production equations," *Mathematical Problems in Engineering,* No. 3, 103–118, 2003.
7. Asiru, M. A., "Sumudu transform and solution of integral equations of convolution type," *Internat. J. Educ. Sci. Technol,* Vol. 32, No. 6, 906–910, 2001.
8. Asiru, M. A., "Further properties of the Sumudu transform and its applications," *Inter. J. Math. Ed. Sci. Tech.,* Vol. 33, No. 2, 441–449, 2002.
9. Asiru, M. A., "Applications of the Sumudu transform to discrete dynamicsystem," *Inter. J. Math. Ed. Sci. Tech.,* Vol. 34, No. 6, 944–949, 2003.
10. Weerakoon, S., "Application of Sumudu transform to partial differential equations," *Inter. J. Math. Ed. Sci. Tech.,* Vol. 25, No. 2, 277–283, 1994.
11. Weerakoon, S., "Complex inversion formula for Sumudu transform," *Inter. J. Math. Ed. Sci. Tech.,* Vol. 29, No. 4, 618–621, 1998.
12. Stratton, J. A., *Electomagnetic Theory,* McGraw-Hill Book Company, New York, 1941.
13. Kong, J. A., *Maxwell Equations,* EMW Publishing, Cambridge, MA, 2002.
14. Harmuth, H. F. and M. G. M. Hussain, *Propagation of Electromagnetic Signals,* World Scientific, Singapore, 1994.
15. Zhou, X., "On independence, completeness of Maxwell's equations and uniqueness theorems in electromagnetics," *Progress In Electromagnetics Research,* PIER 64, 117–134, 2006.
16. Hussain, M. G. M., "Mathematical model for the electromagnetic

conductivity of lossy materials," *Journal of Electromagnetic Waves and Applications,* Vol. 19, No. 2, 271–279, 2005.

17. Shen, J., "Time harmonic electromagnetic fields in an biaxial anisotropic medium," *Journal of Electromagnetic Waves and Applications,* Vol. 9, No. 6, 753–767, 2005.
18. El-Shandwily, M. E., "Solutions of Maxwell's equations for general nonperiodic waves in lossy media," *IEEE Trans. Electromagn. Compat.*, Vol. 30, No. 4, 577–582, Nov. 1988.
19. Thomson, W. T., *Laplace Transformation Theory and Engineering Applications,* Prentice-Hall Engineering Design Series, Prentice-Hall, Inc., New York, 1950.
20. Watugala, G. K., "Sumudu transform for functions of two variables," *Mathematical Engineering in Industry,* Vol. 8, No. 4, 293–302, 2002.