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TOWARDS THE CHARACTERISTIC DISPERSION RELATION FOR STEP-INDEX HYPERBOLIC WAVEGUIDE WITH CONDUCTING HELICAL WINDING

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Abstract—A study is presented of the light wave propagation in a new type of dielectric optical waveguide with hyperbolic kind of cross-section. Further, the waveguide is assumed to have a conducting helical winding. The analysis essentially requires the use of elliptical coordinate system, which finally results into Mathieu and modified Mathieu functions as the representatives of the electromagnetic fields within the lightguide. Field components in the different sections of the guide are deduced, and the characteristic dispersion equation for the system is derived. The preliminary investigation on such type of waveguide throws the idea that the presence of helix pitch angle (which serves the purpose of additional controlling parameter for the guide) in the dispersion relation would greatly affect the propagation characteristics of the guide, and this can be of great practical importance.

1. INTRODUCTION

Multifarious prominent applications of optical waveguides in communications and integrated optical (IO) devices have been investigated over the past three decades. The transmission properties of optical waveguides essentially depend on the cross-sectional geometry of the guide. Waveguides with non-circular cross-sections sections, like planar, rectangular etc., form the basic building blocks of IO systems as these can be used as mode detectors etc. Therefore, the electromagnetic (EM) wave propagation through guides having various forms of symmetrical and non-symmetrical cross-sectional geometries has been an interesting area of research among the R&D community for a long time. In this context, guides with varieties of geometrical cross-sections have been reported, yielding important and interesting results. Most of these guides are now widely recognized for their use in the fabrication of IO circuits and laser beam technology. A few examples of such guides are tapered and elliptical fibers [1–5], rectangular and planar [6–10], triangular [11–14], parabolic cylindrical [15–18], polygonal [19] and Piet Hein [20, 21] type guides, the study of which have been presented using numerical and partially analytical techniques.

In the context of noncircular guides, a hyperbolic cross-sectional guide would be rather interesting. Obviously, such a waveguide structure is rather unconventional. However, Choudhury and Singh presented the detailed study of many types of unconventional structures in ref. [22]. The purpose of the present communication is to report the preliminary investigation of such a hyperbolic waveguide with conducting helical winding. The implementation of helical winding essentially makes the analysis much rigorous. However, such a winding is purposely introduced as it essentially controls the dispersion characteristics of the guide [23, 24]. In this context, it is important to mention that nearly half a century has elapsed since Watkins [25] wrote his classic work on topics in EM theory dealing with periodic transmission system including wire helix. Watkins himself cited the works of Brillouin [26], Kornhauser [27], Mathers and Kino [28].

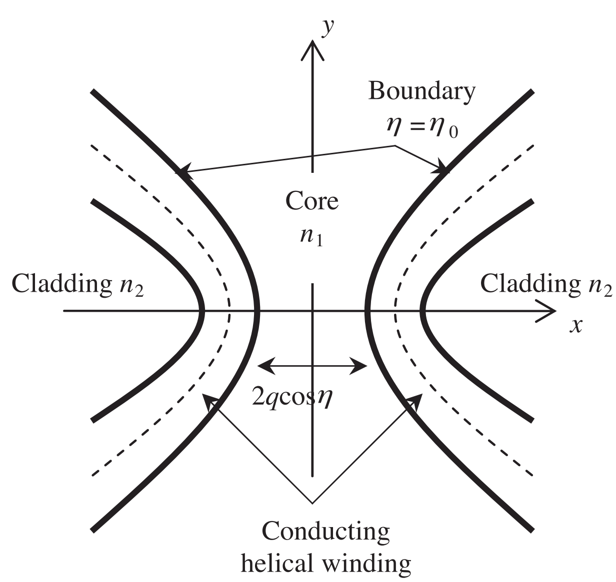

Figure 1. Illustration of the guide and the helical winding.

Kumar and Singh [29–31] have also presented earlier the study of elliptical fibers with conducting helical windings, and it was reported that the effect of the helical pitch angle was almost negligible for the guides having small eccentricity values. Further, in that work the parameter related to the helical pitch angle was not present in all terms of the dispersion relation. In the present paper, the study incorporates the development of the characteristic dispersion equation of the hyperbolic structure with helical winding, which essentially requires the use of the elliptical coordinate system for the analysis. We find that the pitch angle of the helix plays an important role as it is present in all terms of the dispersion relation.

2. ANALYTICAL TREATMENT

Figures 1 and 2 illustrate the structure of the waveguide to be considered in this paper. As stated above, in order to analyze such a structure, it becomes essential to use the elliptical coordinate system (ξ, η, z) with z as the direction of propagation of the EM wave; the coordinate system [32] is shown in Fig. 3. It becomes evident

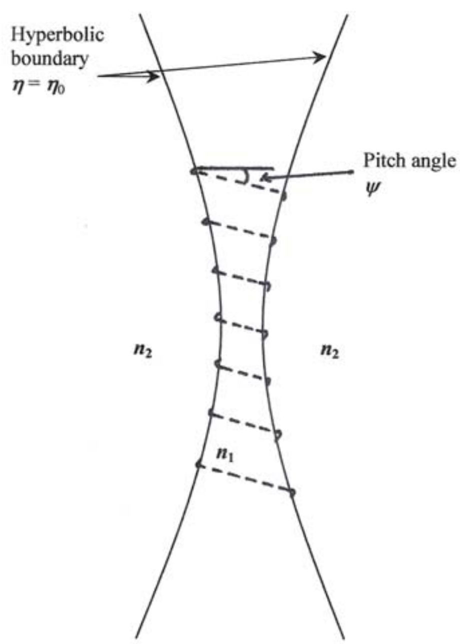

Figure 2. Hyperbolic guide and the helical winding.

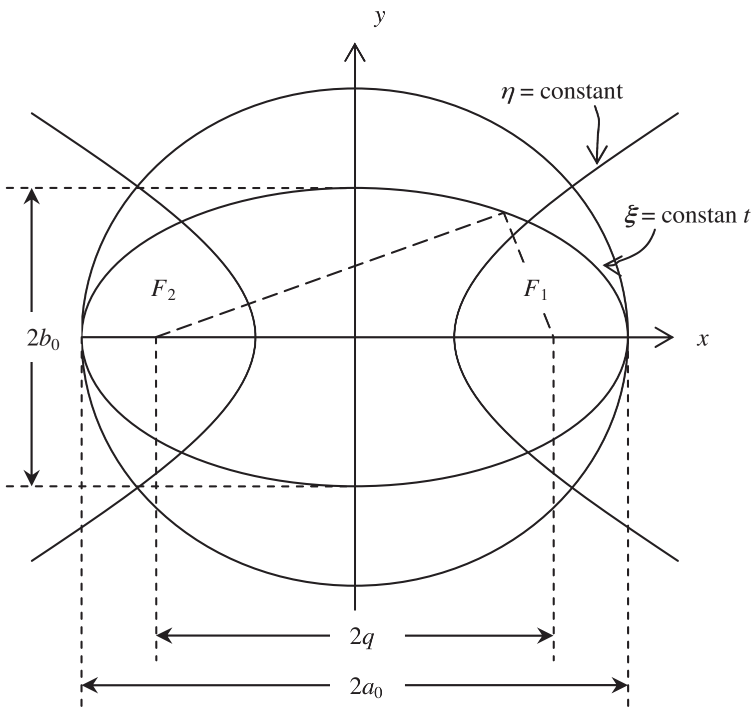

Figure 3. The elliptic coordinate system.

from Fig. 1 that the core-cladding cross-sectional boundary will be represented by a hyperbola $\eta = \eta_0$, where η_0 is the parametric value representing the boundary of the hyperbolic guide. The equation of hyperbola can be represented in Cartesian coordinates as

$$\left(\frac{x}{q\cos\eta}\right)^2 - \left(\frac{y}{q\sin\eta}\right)^2 = 1 \tag{1}$$

where $2q\cos\eta$ is the length of the interception made by the hyperbola on the x-axis. The core-cladding refractive indices are assumed to be n_1 and n_2, respectively, so that $(n_1 - n_2)/n_1$ is positive, and much smaller than 1. This indicates that we are considering the case of weakly guiding step-index waveguide. The helical windings on the waveguide are shown in the Fig. 2, where ψ represents the helical pitch angle. We now write the appropriate expressions for the axial components of the electric (E-) and the magnetic (H-) fields E_{z1}, H_{z1}, E_{z2} and H_{z2} for the different sections of the guide. The tangential components of the E- and H-fields $E_{\xi1}$, $H_{\xi1}$, $E_{\xi2}$ and $H_{\xi2}$ are also stated. Here the subscripts 1 and 2 refer to the core and the cladding regions of the guide, respectively.

Since the hyperbola is symmetrical with respect to both the x- and y-axes, we are showing the axial field components for the first quadrant only. The same procedure will be followed for the other quadrants too. Thus, the region $(\pi/2) > \eta > \eta_0$ represents the core section, and $0 < \eta < \eta_0$ the cladding region. The field components for the even and odd modes [33] are as follows:

2.1. Even Modes

In the core section (i.e., $(\pi/2) > \eta > \eta_0$),

$$E_{z1} = KSe_\nu\left(\xi, \gamma_1^2\right) se_\nu\left(\eta_0, \gamma_1^2\right) \tag{2}$$

$$H_{z1} = LCe_\nu\left(\xi, \gamma_1^2\right) ce_\nu\left(\eta_0, \gamma_1^2\right) \tag{3}$$

In the cladding section (i.e., $0 < \eta < \eta_0$),

$$E_{z2} = MGek_\nu\left(\xi, -\gamma_2^2\right) se_\nu(\eta_0, -\gamma_2^2) \tag{4}$$

$$H_{z2} = NFek_\nu\left(\xi, -\gamma_2^2\right) ce_\nu(\eta_0, -\gamma_2^2) \tag{5}$$

2.2. Odd Modes

In the core section (i.e., $(\pi/2) > \eta > \eta_0$),

$$E_{z1} = KCe_\nu(\xi, \gamma_1^2) ce_\nu(\eta_0, \gamma_1^2) \tag{6}$$

$$H_{z1} = LSe_\nu(\xi, \gamma_1^2) se_\nu(\eta_0, \gamma_1^2) \tag{7}$$

In the cladding section (i.e., $0 < \eta < \eta_0$),

$$E_{z2} = MFek_\nu\left(\xi, -\gamma_2^2\right) ce_\nu(\eta_0, -\gamma_2^2) \tag{8}$$

$$H_{z2} = NGek_\nu\left(\xi, -\gamma_2^2\right) se_\nu(\eta_0, -\gamma_2^2) \tag{9}$$

In Eqs. (2)–(9), K, L, M and N are arbitrary constants to be determined from the boundary conditions.

Now, we are dealing with the even modes. However, the analysis for the odd modes can also be carried out in the same way with the interchanging of the relevant field functions. For the analytical treatment, we first convert Maxwell's equations into the elliptical coordinate system [34]. After a few mathematical steps, the transverse components of E- and H- fields for the even modes can be derived follows:

$$E_{\eta 1} = \frac{i}{(k^2 n_1^2 - \beta^2).q.(\cos h^2\xi - \cos^2\eta)^{1/2}}[\beta K.Se_\nu(\xi,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2) -\omega\mu_0.L.Ce_\upsilon(\xi,\gamma_1^2).ce_\upsilon(\eta_0,\gamma_1^2)] \tag{10}$$

$$H_{\eta 1} = \frac{i}{(k^2 n_1^2 - \beta^2).q.(\cosh^2\xi - \cos^2\eta)^{1/2}}[\beta.L.Ce_\nu(\xi,\gamma_1^2).ce'_\nu(\eta_0,\gamma_1^2) +\omega n_1^2\varepsilon_0.K.Se'_\nu(\xi,\gamma_1^2).se_\nu(\eta_0,\gamma_1^2)] \tag{11}$$

$$E_{\eta 2} = \frac{i}{(k^2 n_2^2 - \beta^2).q.(\cosh^2\xi - \cos^2\eta)^{1/2}} [\beta.M.Gek_\nu(\xi,-\gamma_2^2).se'(\eta_0,-\gamma_2^2) +\omega\mu_0.N.Fek'_\nu(\xi,-\gamma_2^2).ce_\nu(\eta_0,-\gamma_2^2)] \tag{12}$$

$$H_{\eta_2} = \frac{i}{(k^2 n_2^2 - \beta^2).q.(\cosh^2\xi - \cos^2\eta)^{1/2}} [\beta.N.Fek_\nu(\xi,-\gamma_2^2).ce'_\nu(\eta_0,-\gamma_2^2) +\omega n_2^2\varepsilon_0.M.Gek'_\nu(\xi,-\gamma_2^2).se_\nu(\eta_0,-\gamma_2^2)] \tag{13}$$

$$E_{\xi 1} = \frac{i}{\omega n_1^2\varepsilon_0.q.(\cosh^2\xi - \cos^2\eta)^{1/2}}.L.Ce_\nu(\xi,\gamma_1^2).ce'_\nu(\eta,\gamma_1^2) +\frac{i\beta}{\omega n_1^2\varepsilon_0(k^2 n_1^2-\beta^2)ql}\Big\{\beta.L.Ce_\nu(\xi,\gamma_1^2).ce'_\nu(\eta_0,\gamma_1^2) +\omega n_1^2\varepsilon_0.K.Se'_\nu(\xi,\gamma_1^2).se_\nu(\eta_0,\gamma_1^2)\Big\} \tag{14}$$

$$H_{\xi 1} = \frac{-i}{\omega\mu_0.ql}.K.Se_\nu(\xi,\gamma_1^2).se'_\nu(\eta_0,\gamma_1^2) -\frac{i\beta}{\omega\mu_0(k^2 n_1^2-\beta^2)ql}\{\beta.K.Se_\nu(\xi,\gamma_1^2).se'_\nu(\eta_0,\gamma_1^2) -\omega\mu_0.L.Ce'_\nu(\xi,\gamma_1^2).ce_\nu(\eta_0,\gamma_1^2)\} \tag{15}$$

$$E_{\xi 2} = \frac{i}{\omega n_2^2\varepsilon_0.q.(\cosh^2\xi - \cos^2\eta)^{1/2}}.N.Fek_\nu(\xi,-\gamma_2^2).ce'_\nu(\eta_0,-\gamma_2^2) +\frac{i\beta}{\omega n_2^2\varepsilon_0(k^2 n_2^2-\beta^2)ql}\{\beta.N.Fek_\nu(\xi,-\gamma_2^2).ce'_\nu(\eta_0,-\gamma_2^2) +\omega n_2^2\varepsilon_0.M.Gek'_\nu(\xi,-\gamma_2^2).se_\nu(\eta_0,-\gamma_2^2)\} \tag{16}$$

$$\begin{aligned} H_{\xi 2} \;=\; & \frac{-i}{\omega\mu_0.ql}.M.Gek_\nu(\xi,-\gamma_2^2).se'_\nu(\eta_0,-\gamma_2^2) \\ & -\frac{i\beta}{\omega\mu_0(k^2n_2^2-\beta^2)ql}\{\beta.M.Gek_\nu(\xi,-\gamma_2^2).se'_\nu(\eta_0,-\gamma_2^2) \\ & -\omega\mu_0.N.Fek'_\nu(\xi,-\gamma_2^2).ce_\nu(\eta_0,-\gamma_2^2)\} \end{aligned} \tag{17}$$

In Eqs. (10)–(17), the symbols γ_1 and γ_2 are as follows:

$$\gamma_1 = uq/2$$

$$\gamma_2 = wq/2$$

with u and w having their meanings as

$$k^2n_1^2 - \beta^2 = u^2,$$

$$k^2n_2^2 - \beta^2 = -w^2.$$

Also, $l = \left(\cosh^2\xi - \cos^2\eta\right)^{1/2}$

As stated before, in the above equations, the suffix 1 refers to the situations in the core (i.e., $(\pi/2) > \eta > \eta_0$) section, whereas the suffix 2 stands for the situations in the cladding (i.e., $0 < \eta < \eta_0$).

We can now state the equations representing the boundary conditions for the waveguide structure under consideration. Remembering that the conductivity of the helix is zero in a direction normal to the conduction [35], the tangential components of the electric and the magnetic fields must satisfy the following equations at the boundary $\eta = \eta_0$:

$$(E_{Z_1} - E_{Z_2})\cos\psi - (E_{\xi_1} - E_{\xi_2})\sin\psi \;=\; 0 \tag{18}$$

$$(H_{Z_1} - H_{Z_2})\sin\psi + (H_{\xi_1} - H_{\xi_2})\cos\psi \;=\; 0 \tag{19}$$

Further, considering the tangential field components in the direction of conductivity, we must have these components as zero. Thus, at $\eta = \eta_0$, we can have

$$E_{z1}\sin\psi + E_{\xi 1}\cos\psi \;=\; 0 \tag{20}$$

$$E_{z2}\sin\psi + E_{\xi 2}\cos\psi \;=\; 0 \tag{21}$$

Equations (18)–(21) essentially represent the set of boundary conditions.

Substituting the values of E_{z1}, H_{z1}, E_{z2}, H_{z2}, from Eqs. (2)–(5), and $E_{\xi 1}$, $H_{\xi 1}$, $E_{\xi 2}$, $H_{\xi 2}$ from Eqs. (14)–(17) in the boundary condition

Eqs. (18)–(21), we finally get

$$
\begin{aligned}
&K\left[Se_\nu(\xi,\gamma_1^2).se_\nu(\eta_0,\gamma_1^2)\cos\psi-\frac{i\beta}{u^2ql}.Se'_\nu(\xi,\gamma_1^2).se_\nu(\eta_0,\gamma_1^2)\sin\psi\right]\\
&+L\left[-\frac{i\beta^2}{\omega n_1^2\varepsilon_0u^2ql}Ce_\nu(\xi,\gamma_1^2).ce'_\nu(\eta_0,\gamma_1^2)\right.\\
&\left.+\frac{i\beta^2}{\omega n_1^2\varepsilon_0u^2ql}.Ce_\nu(\xi,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\right]\sin\psi\\
&+M\left[-Gek_\nu(\xi,-\gamma_2^2).se_\nu(\eta_0,-\gamma_2^2)\cos\psi\right.\\
&\left.+\frac{i\beta}{w^2ql}.Gek'_\nu(\xi,-\gamma_2^2).se_\nu(\eta_0,-\gamma_2^2)\sin\psi\right]\\
&+N\left[-\frac{i}{ql\omega n_2^2\varepsilon_0}Fek_\nu(\xi,-\gamma_2^2)ce'_\nu(\eta_0,-\gamma_2^2)\right.\\
&\left.-\frac{i\beta^2}{\omega n_2^2\varepsilon_0w^2ql}Fek_\nu(\xi,-\gamma_2^2)ce'_\nu(\eta_0,-\gamma_2^2)\right]\sin\psi=0
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&K\left[-\frac{i}{\omega\mu_0ql}.Se_\nu(\xi,\gamma_1^2).se'_\nu(\eta_0,\gamma_1^2)-\frac{i\beta^2}{\omega\mu_0u^2ql}.Se_\nu(\xi,\gamma_1^2).se'_\nu(\eta_0,\gamma_1^2)\right]\cos\psi\\
&+L\left[Ce_\nu(\xi,\gamma_1^2).ce_\nu(\eta_0,\gamma_1^2)\sin\psi+\frac{i\beta}{u^2ql}Ce'_\nu(\xi,\gamma_1^2).ce_\nu(\eta_0,\gamma_1^2)\cos\psi\right]\\
&+M\left[\frac{i}{\omega\mu_0ql}Gek_\nu(\xi,-\gamma_2^2)se'_\nu(\eta_0,-\gamma_2^2)\right.\\
&\left.-\frac{i\beta^2}{\omega\mu_0w^2ql}Gek_\nu(\xi,-\gamma_2^2)se'_\nu(\eta_0,-\gamma_2^2)\right]\cos\psi\\
&+N\left[-Fek_\nu(\xi,-\gamma_2^2)ce_\nu(\eta_0,-\gamma_2^2)\sin\psi\right.\\
&\left.+\frac{i\beta}{w^2ql}Fek'_\nu(\xi,-\gamma_2^2)ce_\nu(\eta_0,-\gamma_2^2)\cos\psi\right]=0
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&K\left[Se_\nu(\xi,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)\sin\psi+\frac{i\beta}{u^2ql}Se'_\nu(\xi,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)\cos\psi\right]\\
&+L\left[\frac{i}{ql\omega n_1^2\varepsilon_0}Ce_\nu(\xi,\gamma_1^2)ce'_\nu(\eta_0,\gamma_1^2)\right.\\
&\left.+\frac{i\beta^2}{\omega n_1^2\varepsilon_0u^2ql}Ce_\nu(\xi,\gamma_1^2)ce'_\nu(\eta_0,\gamma_1^2)\right]\cos\psi=0
\end{aligned}
\tag{24}
$$

$$\begin{aligned}&M\left[Gek_\nu(\xi,-\gamma_2^2)se_\nu(\eta_0,-\gamma_2^2)\sin\psi\right.\\&\left.-\frac{i\beta}{w^2ql}Gek'_\nu(\xi,-\gamma_2^2)se_\nu(\eta_0,-\gamma_2^2)\cos\psi\right]\\&+N\left[\frac{i}{ql\omega n_2^2\varepsilon_0}Fek_\nu(\xi,-\gamma_2^2)ce'(\eta_0,-\gamma_2^2)\right.\\&\left.-\frac{i\beta^2}{\omega n_2^2\varepsilon_0 w^2ql}Fek_\nu(\xi,-\gamma_2^2)ce'_\nu(\eta_0,-\gamma_2^2)\right]\cos\psi=0\qquad(25)\end{aligned}$$

In Eqs. (21)–(25) the prime represents differentiation with respect to η or ξ, depending on whichever variable is present in the argument. If the argument contains η_0, we have

$$\left.\frac{d}{d\eta}se_\nu(\eta,\gamma_1^2)\right|_{\eta=\eta 0}=se'_\nu(\eta_0,\gamma_1^2)$$

The same notation is used in the case of other functions.

In order that Eqs. (22), (23), (24) and (25) to be consistent, and yield a non-trivial solution, the determinant formed by the coefficients of the constants must vanish. Essentially that determinant will be a 4×4 one (not incorporated into the text), and the vanishing condition will yield the characteristic eigenvalue equation for real and imaginary parts, as follows:

2.3. The Equation for Real Part

$$\begin{aligned}&-\frac{1}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)\\&se_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi\\&+\frac{2\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)\\&se_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi\\&-\frac{\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)\\&se_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi\end{aligned}$$

$$+\frac{2\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)$$
$$se_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^4q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)Ce_\nu(\xi,\gamma_1^2)$$
$$se_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{\beta^6}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^4q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)$$
$$se_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{1}{n_1^2w^2}Se_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)Ce_\nu(\xi,\gamma_1^2)Gek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{\beta^2}{n_1^2u^2w^2}Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)Gek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{1}{n_1^2w^2}Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)Gek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{\beta^2}{n_1^2u^2w^2}Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)Gek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{1}{n_1^2u^2}Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)Gek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$
$$+\frac{\beta^2}{n_1^2u^4}Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)Gek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$
$$-\frac{\beta^2}{n_1^2u^2w^4ql}Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)Gek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{\beta^4}{n_1^2u^4w^4q^2l^2}Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)Gek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$

$$+\frac{1}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2q^2l^2}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^2q^2l^2}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^4q^2l^2}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{2\beta^6}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^4q^2l^2}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4q^2l^2}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^2q^2l^2}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^2q^2l^2}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Gek_\nu(\xi,-\gamma_2^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{1}{n_1^2w^2}Ce_\nu(\xi,\gamma_1^2)Se_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)Gek'_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$
$$-\frac{1}{n_1^2u^2}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)Gek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$
$$-\frac{1}{n_1^2w^2}Ce_\nu(\xi,\gamma_1^2)Se_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)Gek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$

$$-\frac{\beta^2}{n_1^2u^2w^4q^2l^2}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)Gek'_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$

$$-\frac{\beta^2}{n_1^2u^2w^2}Ce_\nu(\xi,\gamma_1^2)Se_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)Gek'_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$

$$+\frac{\beta^2}{n_1^2u^4}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)Gek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$

$$-\frac{\beta^2}{n_1^2u^2w^2}Ce_\nu(\xi,\gamma_1^2)Se_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)Gek_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$

$$-\frac{\beta^4}{n_1^2u^4w^4q^2l^2}Ce_\nu(\xi,\gamma_1^2)Se'_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)Gek'_\nu(\xi,-\gamma_2^2)$$
$$ce'_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)se_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$

$$+\frac{1}{n_2^2u^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$

$$-\frac{\beta^2}{n_2^2u^2w^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$

$$+\frac{1}{n_2^2u^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$

$$-\frac{\beta^2}{n_2^2u^2w^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$

$$-\frac{1}{n_2^2w^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$

$$+\frac{\beta^2}{n_2^2w^4}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$

$$+\frac{\beta^2}{n_2^2u^4w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{\beta^4}{n_2^2u^4w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{1}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{2\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{2\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{\beta^6}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{1}{n_2^2w^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$
$$-\frac{1}{n_2^2u^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$

$$-\frac{1}{n_2^2u^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$
$$-\frac{\beta^2}{n_2^2u^4w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$+\frac{\beta^2}{n_2^2w^4}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$
$$-\frac{\beta^2}{n_2^2u^2w^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$
$$-\frac{\beta^2}{n_2^2u^2w^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi$$
$$-\frac{\beta^4}{n_2^2u^4w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{1}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{2\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$
$$-\frac{\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$

$$-\frac{2\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi$$

$$-\frac{\beta^6}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^2\psi=0 \qquad (26)$$

2.4. The Equation for Imaginary Part

$$\frac{1}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$

$$+\frac{2\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$

$$-\frac{\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$

$$+\frac{2\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$

$$-\frac{\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2w^4q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$

$$-\frac{\beta^6}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^4q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$

$$+\frac{1}{n_1^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^2}{n_1^2u^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^2}{n_1^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$
$$-\frac{\beta^4}{n_1^2u^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$
$$-\frac{\beta^2}{n_1^2u^2w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^4}{n_1^2u^4w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^2}{n_1^2u^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^4}{n_1^2u^4w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{1}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{2\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$

$$+\frac{2\beta^6}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^4q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se'_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{1}{n_1^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^4\psi$$
$$+\frac{\beta^2}{n_1^2u^2w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^2}{n_1^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^2}{n_1^2u^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^2}{n_1^2u^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^4\psi$$
$$-\frac{\beta^4}{n_1^2u^4w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^4}{n_1^2u^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^4}{n_1^2u^4w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek'_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$

$$-\frac{1}{n_2^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^2}{n_2^2w^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^2}{n_2^2u^4q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$
$$-\frac{\beta^4}{n_2^2u^4w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$
$$-\frac{\beta^2}{n_2^2u^2w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^4}{n_2^2u^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^2}{n_2^2u^2w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^4}{n_2^2u^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{1}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$
$$-\frac{\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$
$$+\frac{2\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$

$$-\frac{2\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$
$$+\frac{\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$
$$-\frac{\beta^6}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\cos^4\psi$$
$$+\frac{1}{n_2^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^4\psi$$
$$+\frac{\beta^2}{n_2^2u^2w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^2}{n_2^2u^2w^2q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^2}{n_2^2u^4q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^2}{n_2^2w^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^4\psi$$
$$+\frac{\beta^4}{n_2^2u^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$+\frac{\beta^4}{n_2^2u^2w^4q^2l^2}Gek'_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^4}{n_2^2u^4w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se'_\nu(\xi,\gamma_1^2)Ce'_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$

$$-\frac{1}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{2\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^2}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{2\beta^4}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^2w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi$$
$$-\frac{\beta^6}{\frac{4\pi^2}{\lambda^2}n_1^2n_2^2u^4w^2q^2l^2}Gek_\nu(\xi,-\gamma_2^2)Se_\nu(\xi,\gamma_1^2)Ce_\nu(\xi,\gamma_1^2)Fek_\nu(\xi,-\gamma_2^2)$$
$$se'_\nu(\eta_0,\gamma_1^2)ce'_\nu(\eta_0,-\gamma_2^2)ce'_\nu(\eta_0,\gamma_1^2)se_\nu(\eta_0,-\gamma_2^2)\sin^2\psi\cos^2\psi=0 \qquad (27)$$

3. DISCUSSION

Equations (26) and (27) constitute the central new results of this analytical investigation. The authors want to point out that hyperbolic waveguide itself is a too much complex problem to tackle analytically, and the analysis has been brought to a point where the characteristic Eqs. (26) and (27) are obtained for the structure under consideration. The dispersion equations involve Mathieu and the modified Mathieu functions of the first and second kinds. This is a good starting point for numerical computation. We observe that, since the helical pitch angle ψ occurs in the characteristic equations, we have an additional parameter with which we can modify and control the dispersion characteristics, a result which may prove to be of great practical importance. Another aspect in this context is related to the use of the coordinate system. For the analysis, we are using the elliptical coordinate system, where there are two different types of curves – one is closed curve (i.e., ellipse) and other one is open curve (i.e., hyperbola). That is why we are using the field components in the directions of ξ and η. In the present analytical investigation, we considered the η-component too, and it is being introduced for the first time in our knowledge. Further, in our case, Mathieu functions cannot be converted into the sine and cosine functions owing to the large eccentricity of the guide. However, in the case of elliptical fibers, these functions can be converted into the circular harmonic functions for small eccentricity, making thereby the analysis a little simpler. In order to analyze the case of a hyperbolic waveguide with large eccentricity, there exist some series solutions. Further work in this direction is in progress, which the authors expect to take up in a future communication.

APPENDIX A. EXPLANATION OF BOUNDARY CONDITIONS

E_z and E_ξ are the components of the electric field in z- and ξ-directions, respectively. Therefore, the z-component of the electric field in the direction of helix can be written as

$$E_Z \cos(90^\circ - \psi) = E_Z \sin\psi$$

Also, the ξ-component of the electric field in the direction of helix will be $E_\xi \cos\psi$. Thus, the total electric field in the direction of helix will be given as

$$E_Z \sin\psi + E_\xi \cos\psi.$$

As the tangential component of the electric field is perpendicular to the direction of the helix, therefore,

$$E_Z \sin\psi + E_\xi \cos\psi = 0$$

Further, since the tangential component of the electric field is continuous across the sections of the hyperbolic waveguide, and is perpendicular to direction of helix, therefore, one can have

$$(E_{Z_1} - E_{Z_2})\cos\psi - (E_{\xi_1} - E_{\xi_2})\sin\psi = 0.$$

Following the same situation, as the tangential component of the magnetic field is parallel to the direction of helix, and is continuous across the hyperbolic waveguide, therefore, we can have

$$(H_{Z_1} - H_{Z_2})\sin\psi + (H_{\xi_1} - H_{\xi_2})\cos\psi = 0,$$

These equations follow from the condition that the components of the electric field in helical direction are zero.

APPENDIX B. SYMBOLS

$Se_\nu(\xi, \gamma_1^2)$ is the modified Mathieu function of the first kind.
$Ce_\nu(\xi, \gamma_1^2)$ is the modified Mathieu function of the first kind.
$Gek_\nu(\xi, -\gamma_2^2)$ is the modified Mathieu function of the second kind.
$Fek_\nu(\xi, -\gamma_2^2)$ is the modified Mathieu function of the second kind.
$se_\nu(\eta_0, \gamma_1^2)$ is Mathieu function of the first kind.
$ce_\nu(\eta_0, \gamma_1^2)$ is Mathieu function of the first kind.
$se_\nu(\eta_0, -\gamma_2^2)$ is Mathieu function of the second kind.
$ce_\nu(\eta_0, -\gamma_2^2)$ is Mathieu function of the second kind.

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