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STOCHASTIC RAY PROPAGATION IN STRATIFIED RANDOM LATTICES — COMPARATIVE ASSESSMENT OF TWO MATHEMATICAL APPROACHES

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Abstract—In this paper, ray propagation in stratified semi-infinite percolation lattices consisting of a succession of uniform density layers is considered. Two different mathematical approaches for analytically evaluating the penetration depth are presented. In order to compare performances and to assess the range of validity of the two approaches, an exhaustive set of numerical Monte-Carlo-like experiments is presented.

1. INTRODUCTION

Wave propagation and scattering in random media is a challenging topic because of the large number of applications and involved research areas [1–9]. By considering the percolative model proposed in [10], where authors described the urban environment in terms of a uniform random lattice [11], this paper is focused on the analysis of propagation in stratified random media consisting of a discontinuous succession of layers with uniform density. In particular, such a work is aimed at providing an exhaustive numerical validation of two mathematical approaches, the former based on the result in [10], the latter applying the theory of the Markov chains [12].

This paper is organized as follows. In Section 2, the problem is briefly described and two mathematical approaches aimed at evaluating the propagation depth are introduced. Section 3 provides the results of a representative set of numerical experiments, while final comments and conclusions are drawn in Section 4.

2. PROBLEM STATEMENT AND MATHEMATICAL FORMULATION

Let us consider a stratified random lattice (Fig. 1) described by the following obstacles density distribution

$$q(j) = \begin{cases} q_1, & r_0 = 0 < j \leq r_1 \Rightarrow j \in L_1 \\ q_2, & r_1 < j \leq r_2 \Rightarrow j \in L_2 \\ \vdots & \\ q_n, & r_{n-1} < j \leq r_n \Rightarrow j \in L_n \\ \vdots & \end{cases} \tag{1}$$

where $q(j)$ is the probability that a site belonging to the j-th row is occupied and L_n denotes the n-th layer characterized by an occupancy probability q_n and constituted by the rows between $(r_{n-1}+1)$ and r_n.

The electromagnetic source is assumed to be located in the above empty half-plane and to radiate a monochromatic plane wave impinging on the lattice with a known incidence angle θ. Each site is large compared to the wavelength, therefore the incident wave is modeled in terms of a collection of parallel rays. Such rays undergo specular reflection on obstacles, while other electromagnetic interactions are neglected. The propagation is then described by determining the probability that a single ray reaches a prescribed level k inside the lattice before being reflected back in the above empty half-plane, $\Pr\{0 \longmapsto k \prec 0\}$.

2.1. Martingale Approach (MTGA)

The first mathematical approach is based on the theory presented in [10], where $\Pr\{0 \longmapsto k \prec 0\}$ in a random uniform grid is evaluated by applying the theory of the Martingale random processes [13]. Let us notice that a description of the ray propagation in terms of a

Martingale random process requires that ray jumps following the first one are independent, identically distributed and zero mean. Such an assumption is generally verified provided that the incident angle is not too far from 45° and the lattice is dense [14]. An extension of such an approach to the inhomogeneous case has been proposed in [15]. However, mathematical considerations as well as numerical experiments have shown that the obtained solution is reliable in correspondence with obstacles' density profiles with small variations. Therefore, an ad-hoc formulation when dealing with stratified random lattice is mandatory.

A stratified environment can be modeled as a succession of uniform layers $\{L_n;\ n = 1, 2, 3, ...\}$ and at each layer the propagation is mathematically described through the solution proposed in [10]. In particular, the probability that a ray freely crosses layer L_n [i.e., the probability that a ray traveling with positive direction in the level $(r_{n-1} + 1)$ reaches level r_n before being reflected back in level $(r_{n-1} + 1)$] is equal to

$$P_n \hat{=} \Pr\{(r_{n-1}+1) \longmapsto r_n \prec (r_{n-1}+1)\} = \begin{cases} 1, & r_n = r_{n-1}+1, \\ \dfrac{p_n}{q_{e_n} N_n}\left[1 - p_{e_n}^{N_n}\right], & r_n > r_{n-1}+1, \end{cases} \quad (2)$$

where $p_{e_n} = 1 - q_{e_n} = p_n^{\tan\theta+1}$ is the effective probability that a ray crosses a level with occupancy probability q_n without any reflections and $N_n = (r_n - r_{n-1} - 1)$.

By assuming that the level k belongs to the layer L_K and $N_K = (k - r_{K-1} - 1)$, the ray propagation inside the whole lattice is modeled through the Markov chain [12] shown in Figure 2, where states j^+ and j^- denote a ray traveling inside the level j with positive and negative direction, respectively, and $Q_n \hat{=} 1 - P_n$. Accordingly, the following solution is obtained

$$\Pr\{0 \longmapsto k \in L_K \prec 0\} = \frac{p_1}{\dfrac{1}{P_1} + p_1 \displaystyle\sum_{n=2}^{K} \left[\dfrac{1-P_n}{p_n P_n} + \dfrac{q_n}{p_n p_{n-1}}\right]}. \quad (3)$$

2.2. Markov Approach (MKVA)

In such an approach the original 2D ray-propagation problem is recast as a simple 1D random walk problem where the dependence on the incidence angle θ is avoided [16]. As a matter of fact, whenever a ray hits a vertical face it does not change its vertical direction of

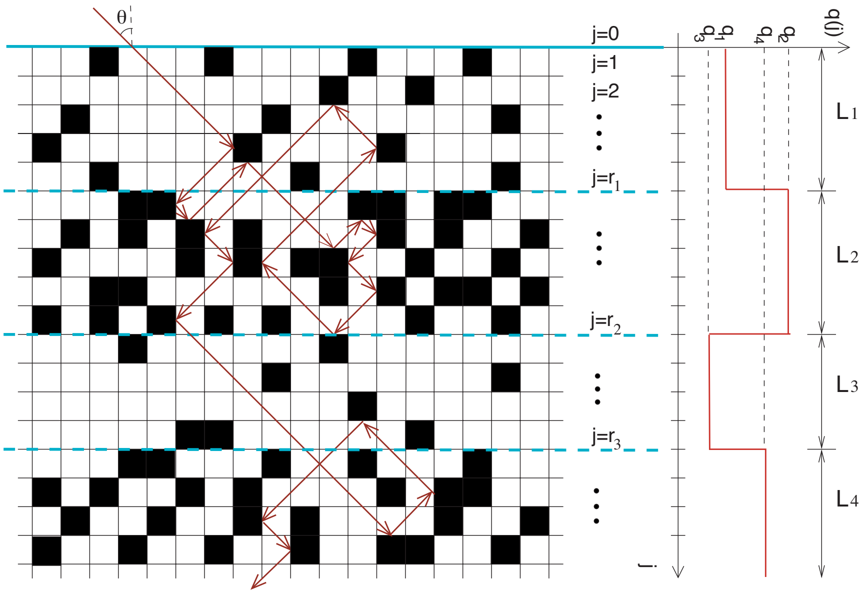


Figure 1. Sketch of ray propagation in a four-layers random lattice realization (left-hand side) and the obstacles' density distribution relative to the random lattice realization (right-hand side).

propagation. Thus, just reflections on horizontal faces, whose number is independent from θ, are taken into account. Accordingly, under the assumption that the propagating ray never crosses cells that it has already encountered along its path (verified when θ is not too far from 45° and in the case of sparse random lattices), ray propagation in a generic nonuniform random lattice is modeled through the Markov chain shown in Fig. 3 whose solution is

$$\Pr\{0 \longmapsto k \prec 0\} = \frac{p_1 p_2}{1 + p_1 p_2 \sum_{i=0}^{k-3} \left[\frac{q_{k-i}}{p_{k-i} p_{k-i-1}} \right]}, \; k \geq 1, \qquad (4)$$

where $q_j = 1 - p_j$ denotes the occupancy probability of the j-th level.

Unlike that in [15], such a formulation satisfactorily works in dealing also with high discontinuities [14] and therefore, it holds true for stratified profiles, as well. Thus, it is enough to customize (4) to

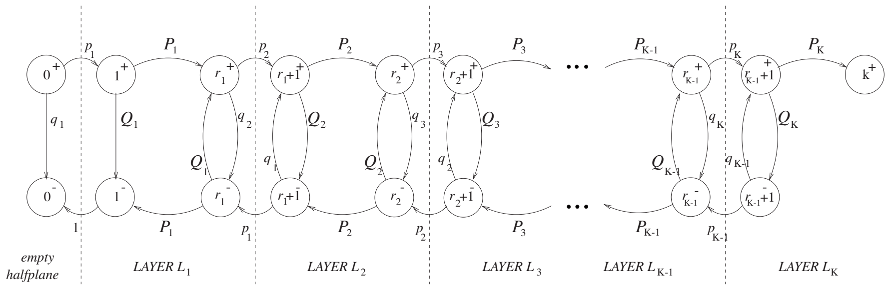


Figure 2. Martingale approach — Markov chain mathematically modeling the ray propagation towards level k in a stratified random lattice.

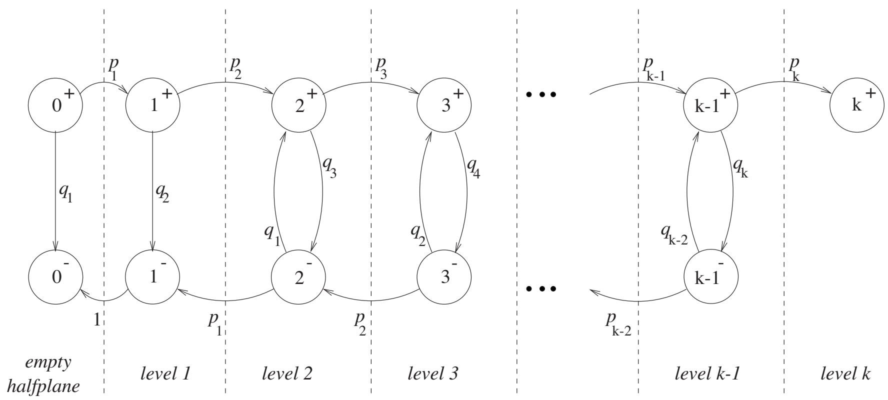


Figure 3. Markov approach — Markov chain mathematically modeling the ray propagation in a generic non-uniform half-plane random lattice with obstacles' density distribution $q(j) = q_j$, j being the level index.

stratified profile detailed in (1). After some algebra we get

$$\Pr\{0 \longmapsto k \in L_K \prec 0\} = \frac{p_1^2}{1 + p_1^2 \left[\frac{q_1}{p_1^2} (N_1 - 1) + \sum_{n=2}^{K} \left(\frac{q_n}{p_n p_{n-1}} + \frac{q_n N_n}{p_n^2} \right) \right]}. \tag{5}$$

3. NUMERICAL VALIDATION

In order to validate and compare the proposed solutions, an exhaustive set of numerical experiments has been carried out. In particular, both three- and four-layers configurations have been taken into account by varying the occupation probability of each layer $\{q_n;\ n = 1, ..., K\}$ between 0.05 and 0.35 with step 0.1. No higher occupancy probability values have been considered since, in order to ensure propagation, the occupancy probability must be lower than the so-called percolation threshold q_c [11] ($q_c \approx 0.40725$ for the two-dimensional case). Moreover, for completeness, different values of the incidence angle have been evaluated, $\theta = \{15°, 30°, 45°, 60°, 75°\}$.

As a reference, the propagation depth has been estimated by Monte-Carlo-like ray-tracing experiments. In particular, for each density profile and incidence angle, 100 random grids have been generated and for each of them 500 rays have been launched from different entry positions. Then, $\Pr\{0 \longmapsto k \in L_K \prec 0\}$ has been estimated from the collection of paths in the first $k_{max} = 32$ levels.

In order to quantitatively evaluate the accuracy of the proposed methods, let us define the following error indexes, namely the prediction error δ_k

$$\delta_k \triangleq \frac{|\Pr_R\{0 \longmapsto k\} - \Pr_P\{0 \longmapsto k\}|}{\max_k [\Pr_R\{0 \longmapsto k\}]} \times 100, \tag{6}$$

and the mean error $\langle\delta\rangle$

$$\langle\delta\rangle \triangleq \frac{1}{k_{max}} \sum_{k=1}^{k_{max}} \delta_k, \tag{7}$$

where the sub-scripts R indicates the value estimated with the reference approach and P stands for the same value evaluated through either (3) or (5). Moreover, let us define the global mean error Δ

$$\Delta \triangleq \frac{1}{S} \sum_{s=1}^{S} \langle\delta\rangle_s, \tag{8}$$

S being the total number of considered obstacles' density profiles and $\langle\delta\rangle_s$ the mean error relative to the s-th distribution.

Furthermore, in order to easily identify a profile, let us use the indexes 1, 2, 3, 4 for indicating an occupation probability equal to 0.05, 0.15, 0.25, and 0.35, respectively. Accordingly, a sequence of N indexes denote a N-layers stratified profile, each element of the sequence indicating the occupation probability of the corresponding $n-$th layers. As an example, profile 3423 identifies a four-layers profile where $q_1 = 0.25$, $q_2 = 0.35$, $q_3 = 0.15$ and $q_4 = 0.25$.

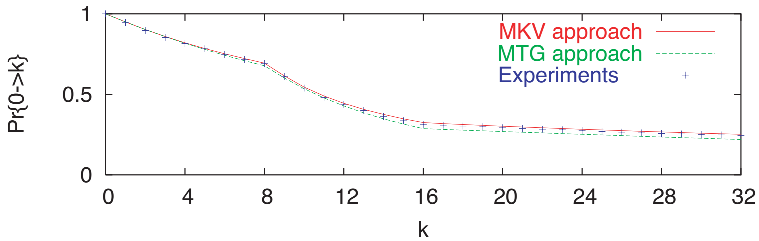
3.1. On the Role of the Obstacles Density

Let us refer to an incidence angle θ equal to 45° and consider *low density profiles* (i.e., $q_n = \{0.05, 0.15\}$, $n = 1, ..., K$) and *high density profiles* (i.e., $q_n = \{0.25, 0.35\}$, $n = 1, ..., K$). With reference to Figures 4 and 5 and as expected, it can be noticed that the MKVA outperforms the MTGA when dealing with *low density profiles,* while the MTGA is better in correspondence with *high density profiles*. This is further confirmed by the mean error values. As far as the three-layers profiles are concerned, $\left[\frac{\langle\delta\rangle_{MTGA}}{\langle\delta\rangle_{MKVA}}\right]_{121} = 2.22$, $\left[\frac{\langle\delta\rangle_{MTGA}}{\langle\delta\rangle_{MKVA}}\right]_{212} = 17.44$, $\left[\frac{\langle\delta\rangle_{MKVA}}{\langle\delta\rangle_{MTGA}}\right]_{343} = 1.66$, and $\left[\frac{\langle\delta\rangle_{MKVA}}{\langle\delta\rangle_{MTGA}}\right]_{343} = 2.47$. Concerning the four-layers *low density profiles*, $\langle\delta\rangle_{MKVA} = 0.71\%$ vs. $\langle\delta\rangle_{MTGA} = 1.55\%$ (profile 1212) and $\langle\delta\rangle_{MKVA} = 0.17\%$ vs. $\langle\delta\rangle_{MTGA} = 2.98\%$ (profile 2121). On the other hand, $\langle\delta\rangle_{MKVA} = 1.42\%$ vs. $\langle\delta\rangle_{MTGA} = 0.92\%$ (profile 3434) and $\langle\delta\rangle_{MKVA} = 1.7\%$ vs. $\langle\delta\rangle_{MTGA} = 0.69\%$ (profile 4343).

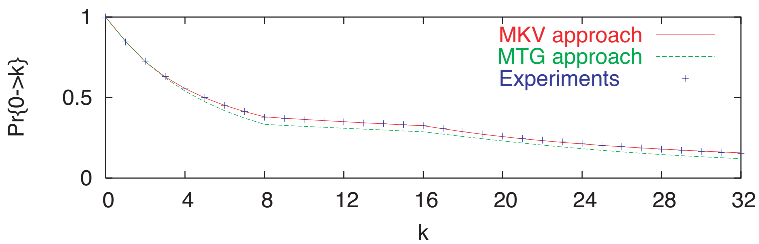
For completeness, also *mixed profiles* [i.e., stratified random grids made up in part of dense layers ($q_n \geq 0.25$) and in part of sparse layers ($q_n \leq 0.15$)] have been considered. However, since similar conclusions can be drawn both for three- and four-layers configurations, only the mean error values of the three-layers profiles are reported (Fig. 6).

Thanks to such results, some rules-of-use of the two approaches can be drawn. When the first layer is neither too sparse nor too dense (i.e, $0.15 \leq q_1 \leq 0.25$), the MKVA outperforms the MTGA unless the second layer has high occupancy probability (i.e., $q_2 = 0.35$) or both $q_2 \geq 0.25$ and $q_3 \geq 0.25$, whatever the occupancy probability value of the remaining layers. When $q_1 = 0.05$, the MTGA gives better results when either q_2 or q_3 are equal to 0.35. On the other hand, when $q_1 = 0.35$, the MTGA outperforms the MKVA. Summarizing, the MKVA outperforms the MTGA in 15 cases over 36 when we deal with a three-layers profile and in 45 cases over 108 when four-layers profiles are considered.

Another interesting observation is concerned with the role of the discontinuities. By analyzing the mean error values, it is evident that the range of validity of both approaches does not depend on the difference between the values of the occupation probabilities in adjacent layers, but it is affected only by the density of the obstacles at each layer.



(a)



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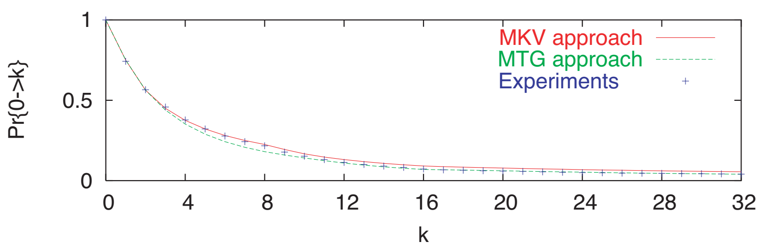

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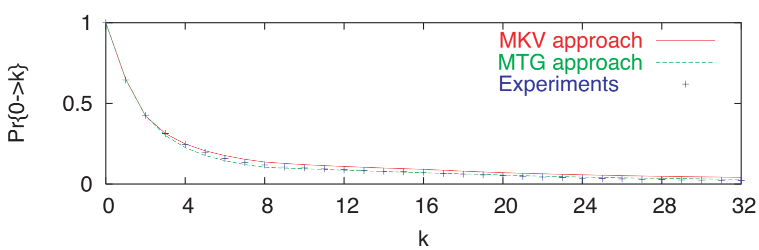
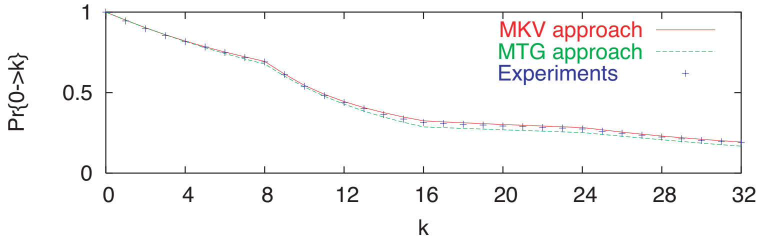
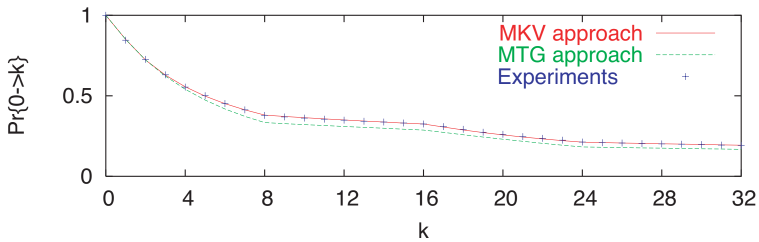


(d)

Figure 4. Three-layers profile, $\theta = 45°$, $r_1 = 8$ and $r_2 = 16$ — Estimated values of $\Pr\{0 \longmapsto k\}$ versus k when $\theta = 45°$ for the profiles (a) 121, (b) 212, (c) 343, and (d) 434.



(a)



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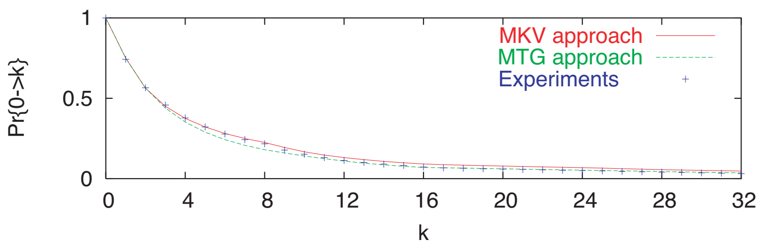

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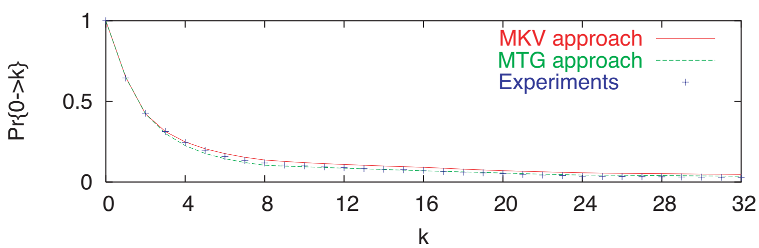

(d)

Figure 5. Four-layers profile, $\theta = 45°$, $r_1 = 8$ and $r_2 = 16$ and $r_3 = 24$ — Estimated values of $\Pr\{0 \longmapsto k\}$ versus k when $\theta = 45°$ for the profiles (a) 1212, (b) 2121, (c) 3434, and (d) 4343.

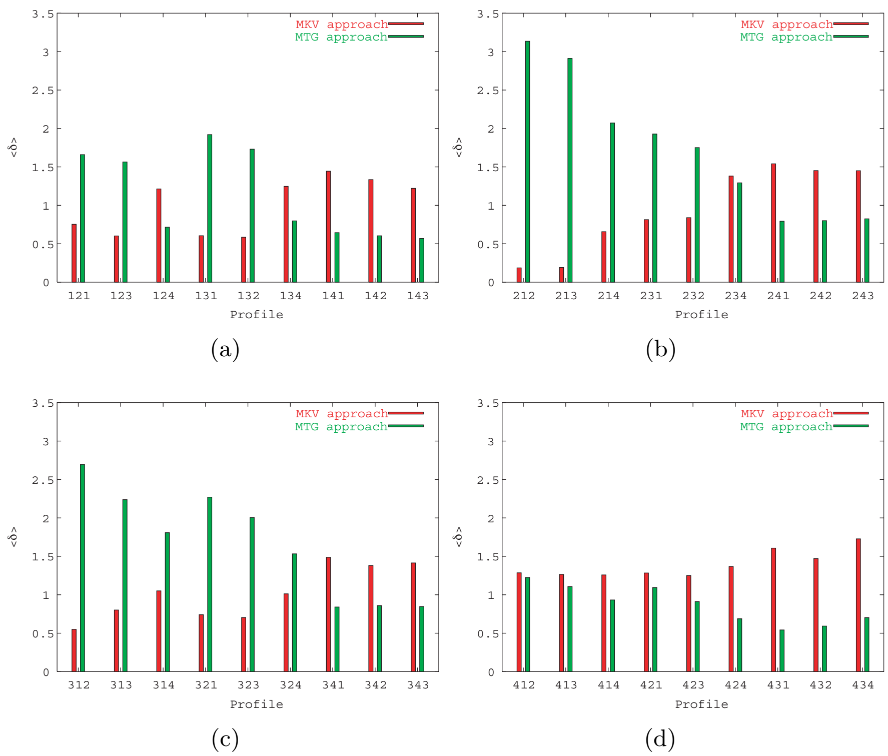

Figure 6. Three-layers profile, $\theta = 45°$, $r_1 = 8$ and $r_2 = 16$ — Mean error $\langle\delta\rangle$ for different values of q_n, $n = 1, 2, 3$, being (a) $q_1 = 0.05$, (b) $q_1 = 0.15$, (c) $q_1 = 0.25$, and (d) $q_1 = 0.35$.

Finally, let us point out that the MKVA returns mean error values lower than 2%. As a matter of fact, by considering both four- and three-layers profiles, $\langle\delta\rangle$ ranges from 0.17% to 1.73%, while it grows up to 3.13% when we apply the MTGA. As far as the global mean error is concerned, we obtain $\Delta_{MKVA} = 1.09\%$ vs. $\Delta_{MTGA} = 1.35\%$ and $\Delta_{MKVA} = 1.06\%$ vs. $\Delta_{MTGA} = 1.28\%$ for the three- and four-layers configurations, respectively.

3.2. On the Role of the Incidence Angle

As far as the role of the dependence on the incidence angle θ is concerned, Figure 7 plots the behavior of the global mean error Δ versus θ for a three-layers scenario. Similar results arise when dealing with four-layers profiles.

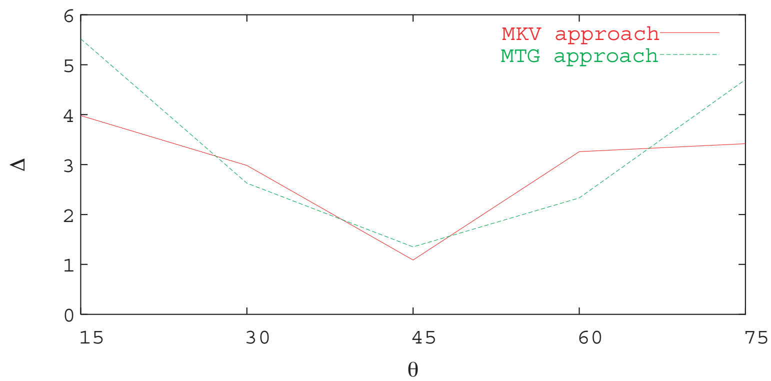

Figure 7. Three-layers profile, $r_1 = 8$ and $r_2 = 16$ — Global mean error Δ versus the incidence angle θ.

As expected, both approaches give better performances when θ is close to 45°. The plots are almost symmetric with respect to the optimal value $\theta_{opt} = 45°$. Such a behavior points out that it does not matter the value of the incidence angle, but only the distance $|\theta - \theta_{opt}|$.

Moreover, it is interesting to observe that, although the MKVA does not take into account the incidence angle, on average it outperforms the MTGA when $\theta = 15°$, $\theta = 45°$ and $\theta = 75°$, since the ratio $\frac{\Delta_{MTGA}}{\Delta_{MKVA}}$ ranges from 1.24 to 1.39. On the other hand, when $\theta = 30°$ and $\theta = 60°$, $\frac{\Delta_{MKVA}}{\Delta_{MTGA}} = 1.14$ and $\frac{\Delta_{MKVA}}{\Delta_{MTGA}} = 1.39$, respectively.

Finally, it should be noticed that, despite its independence on θ, the MKVA is less sensitive to the incidence angle value, as confirmed by the following indexes, $\left[\frac{\max\Delta}{\min\Delta}\right]_{MKVA} = 3.65$ vs. $\left[\frac{\max\Delta}{\min\Delta}\right]_{MTGA} = 4.09$.

4. CONCLUSIONS

Dealing with ray propagation in stratified random lattices, two different mathematical models, namely the Martingale approach (MTGA) and the Markov approach (MKVA), have been presented and compared through an exhaustive numerical analysis.

The obtained results are: (a) both approaches give more faithful estimate in correspondence with incidence angles close to 45° and the mean error increases with the distance $|\theta - 45°|$, (b) both approaches are not affected by the value of the discontinuities, (c) the MKVA satisfactorily performs when dealing with sparse media, while the MTGA works better in correspondence with dense lattices, (d) on average, the MKVA returns lower mean error than the MTGA.

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