Progress In Electromagnetics Research, PIER 71, 85–94, 2007

FRACTIONAL DUALITY AND PERFECT ELECTROMAGNETIC CONDUCTOR (PEMC)

A. Hussain, Q. A. Naqvi, and M. Abbas

Electronics Department
Quaid-i-Azam University
Islamabad, Pakistan

Abstract—Using fractional curl operator, impedance of the surface which may be regarded as intermediate step between the perfect electromagnetic conductor (PEMC) and dual to the perfect electromagnetic conductor (DPEMC) has been determined. The results are compared with the situation which is intermediate step of perfect electric conductor (PEC) and perfect magnetic conductor (PMC).

1. INTRODUCTION

Perfect electric conductor (PEC) and perfect magnetic conductor (PMC) are basic concepts in electromagnetics. Lindell has recently introduced perfect electromagnetic conductor (PEMC) as generalization of the perfect electric conductor (PEC) and perfect magnetic conductor (PMC) [1–4]. It is well known that PEC boundary may be defined by the conditions

$$\mathbf{n} \times \mathbf{E} = 0, \qquad \mathbf{n}.\mathbf{B} = 0$$

While PMC boundary may be defined by the boundary conditions

$$\mathbf{n} \times \mathbf{H} = 0, \qquad \mathbf{n}.\mathbf{D} = 0$$

The PEMC boundary conditions are of the more general form

$$\mathbf{n} \times (\mathbf{H} + M\mathbf{E}) = 0, \qquad \mathbf{n}.(\mathbf{D} - M\mathbf{B}) = 0$$

where M denotes the admittance of the PEMC boundary. It is obvious that PMC corresponds to $M = 0$, while PEC corresponds to $M = \pm\infty$. It may be noted that PEMC boundary is non-reciprocal. Non-reciprocity of the PEMC boundary can be demonstrated by showing that the polarization of the plane wave reflected from PEMC surface is rotated. Problems involving PEMC boundaries with the admittance parameter M and air or other isotropic medium can be transformed to problems involving PEC or PMC boundaries using duality transformation [3].

Another generalization of PEC and PMC reveals from the concept of fractional curl operator, i.e., $(\nabla\times)^\alpha$ [6]. The boundary is known as fractional dual interface with PEC and PMC as the two special situations of the fractional dual interface [6–11]. The surface impedance of the interface which may be regarded as intermediate step between the PEC and PMC may be written as [6]

$$Z_{fd} = j\eta_0 \tan\left(k_0 z + \frac{\alpha\pi}{2}\right)$$

where fd stands for fractional dual. It may be noted that for normal incidence, impedance of the interface corresponding to fractional situations is isotropic. It is also known that for oblique incidence, impedance of the interface, which describes intermediate situations, becomes anisotropic [7, 10, 11]. PEC corresponds to value of fractional parameter $\alpha = 0$ while PMC corresponds to $\alpha = 1$. These results may be obtained using the following relations [6]

$$\mathbf{E}_{fd} = \frac{1}{(jk_0)^\alpha}(\nabla\times)^\alpha \mathbf{E}$$

$$\eta_0\mathbf{H}_{fd} = \frac{1}{(jk_0)^\alpha}(\nabla\times)^\alpha \eta_0\mathbf{H}$$

It may be noted that above two equations are Faraday-Ampere's Maxwell equations with fractionalized curl operator. Above relations yield solutions which may be regarded as intermediate step between the solution $(\mathbf{E}, \eta\mathbf{H})$ and $(\eta\mathbf{H}, -\mathbf{E})$, when the value of fractional parameter changes between zero and one.

In present discussion, our interest is to explore intermediate situations between the PEMC boundary and dual to PEMC boundary (DPEMC) using the idea of fractional curl operator. Behavior of the impedance dealing with intermediate situations is of interest. Time dependency is time harmonic, e^{jwt}, and has been suppressed throughout the dissscussion.

2. FRACTIONAL DUALITY FOR PEMC AND DPEMC BOUNDARIES

Consider reflection of a normal incidence TEM plane wave from a planer PEMC interface which is located at $z = 0$. For $z < 0$ as a region of interest, the expression for the normal incident plane wave is given by

$$\mathbf{E}^i = \hat{x} E_0 \exp(-jk_0 z) \tag{1a}$$

$$\eta_0 \mathbf{H}^i = \hat{y} E_0 \exp(-jk_0 z) \tag{1b}$$

Above expressions may be written as

$$\mathbf{E}^i = [\mathbf{A}_1 + \mathbf{A}_2] E_0 \exp(-jk_0 z)$$

$$\eta_0 \mathbf{H}^i = -j[\mathbf{A}_1 - \mathbf{A}_2] E_0 \exp(-jk_0 z)$$

where

$$\mathbf{A}_1 = \left(\frac{\hat{x} + j\hat{y}}{2}\right)$$

$$\mathbf{A}_2 = \left(\frac{\hat{x} - j\hat{y}}{2}\right)$$

Incident fields $(\mathbf{E}^i, \eta_0 \mathbf{H}^i)$ must satisfy the Maxwell's equations, therefore we can write

$$\hat{z} \times \mathbf{E}^i = [-j\mathbf{A}_1 + j\mathbf{A}_2] E_0 \exp(-jk_0 z)$$

$$\hat{z} \times \eta_0 \mathbf{H}^i = -j[-j\mathbf{A}_1 - j\mathbf{A}_2] E_0 \exp(-jk_0 z)$$

It may be noted that for time harmonic fields given in (1), quantity $\left(-\frac{1}{jk_0}\nabla\times\right)$ in Maxwell equations is equivalent to cross product operator $(\hat{z}\times)$. Two vectors $\mathbf{A}_1$ and $\mathbf{A_2}$ are the eigen vectors of the cross product operator $(\hat{z}\times)$ while $(-j)$ and $(+j)$ are the respective eigen values. According to the recipe for the fractionalization of a linear operator [6, 12], fractionalization of the curl operator means fractionalization of the cross product operator. Fractionalization of cross product operator means fractionalization of corresponding eigen values. In order words, fractional dual solutions corresponding to original solutions given in (1) may be written as

$$(\hat{z}\times)^\alpha \mathbf{E}^i = [(-j)^\alpha \mathbf{A}_1 + (j)^\alpha \mathbf{A}_2] E_0 \exp(-jk_0 z)$$

$$(\hat{z}\times)^\alpha \eta_0 \mathbf{H}^i = -j[(-j)^\alpha \mathbf{A}_1 - (j)^\alpha \mathbf{A}_2] E_0 \exp(-jk_0 z)$$

Above may be simplified as

$$(\hat{z}\times)^\alpha \mathbf{E}^i = \left[\hat{x}\cos\left(\frac{\alpha\pi}{2}\right) + \hat{y}\sin\left(\frac{\alpha\pi}{2}\right)\right] E_0 \exp(-jk_0 z)$$

$$(\hat{z}\times)^\alpha \eta_0 \mathbf{H}^i = \left[-\hat{x}\sin\left(\frac{\alpha\pi}{2}\right) + \hat{y}\cos\left(\frac{\alpha\pi}{2}\right)\right] E_0 \exp(-jk_0 z) \tag{2}$$

Field reflected from the planer PEMC surface located at $z = 0$ may be obtained using the following relations [3]

$$
\begin{aligned}
\mathbf{E}^r &= \bar{\bar{R}}(\theta).\mathbf{E}^i = [\cos 2\theta \bar{\bar{I}} - \sin 2\theta \hat{z} \times \bar{\bar{I}}].\mathbf{E}^i \\
\eta_0 \mathbf{H}^r &= -\bar{\bar{R}}(\theta).\eta_0 \mathbf{H}^i = [-\cos 2\theta \bar{\bar{I}} + \sin 2\theta \hat{z} \times \bar{\bar{I}}].\eta_0 \mathbf{H}^i
\end{aligned} \tag{3}
$$

In above equations admittance M of the PEMC interface has been expressed in terms of another parameter θ by the relation $M\eta_0 = \tan\theta$. It is obvious that PEC means $\theta = \pi/2$ and PMC means $\theta = 0$. Above may be written as

$$
\begin{aligned}
\mathbf{E}^r &= (\cos 2\theta \hat{x} - \sin 2\theta \hat{y}) E_0 \exp(jk_0 z) \\
\eta_0 \mathbf{H}^r &= (-\sin 2\theta \hat{x} - \cos 2\theta \hat{y}) E_0 \exp(jk_0 z)
\end{aligned}
$$

Above expressions may be expanded in terms of eigen functions of the cross product operator $(\hat{z}\times)$ as

$$
\begin{aligned}
\mathbf{E}^r &= \left[\frac{\exp(j2\theta)}{2} \mathbf{A}_1 + \frac{\exp(-j2\theta)}{2} \mathbf{A}_2 \right] E_0 \exp(jk_0 z) \\
\eta_0 \mathbf{H}^r &= j \left[\frac{\exp(j2\theta)}{2} \mathbf{A}_1 - \frac{\exp(-j2\theta)}{2} \mathbf{A}_2 \right] E_0 \exp(jk_0 z)
\end{aligned}
$$

Above fields must satisfy the Maxwell equations. Substitution of above equations in Maxwell equations yields the following

$$
(-\hat{z}\times)\mathbf{E}^r = \left[\frac{\exp(j2\theta)}{2} (j) \mathbf{A}_1 + \frac{\exp(-j2\theta)}{2} (-j) \mathbf{A}_2 \right] E_0 \exp(jk_0 z)
$$

$$
(-\hat{z}\times)\eta_0 \mathbf{H}^r = (-1) j \left[\frac{\exp(j2\theta)}{2} (-j) \mathbf{A}_1 - \frac{\exp(-j2\theta)}{2} (j) \mathbf{A}_2 \right] E_0 \exp(jk_0 z)
$$

Fractionalization of the cross product gives

$$
\begin{aligned}
(-\hat{z}\times)^\alpha \mathbf{E}^r &= (-1)^\alpha \left[\frac{\exp(j2\theta)}{2} (-j)^\alpha \mathbf{A}_1 + \frac{\exp(-j2\theta)}{2} (j)^\alpha \mathbf{A}_2 \right] \\
&\quad \times E_0 \exp(jk_0 z) \\
(-\hat{z}\times)^\alpha \eta_0 \mathbf{H}^r &= (-1)^\alpha j \left[\frac{\exp(j2\theta)}{2} (-j)^\alpha \mathbf{A}_1 - \frac{\exp(-j2\theta)}{2} (j)^\alpha \mathbf{A}_2 \right] \\
&\quad \times E_0 \exp(jk_0 z)
\end{aligned}
$$

Simplification yields fractional dual fields for reflected fields

$$\begin{aligned}(-\hat{z}\times)^{\alpha}\mathbf{E}^{r} &= \exp(j\alpha\pi)\left[\hat{x}\cos\left(2\theta-\frac{\alpha\pi}{2}\right)-\hat{y}\sin\left(2\theta-\frac{\alpha\pi}{2}\right)\right]\\ &\quad\times E_0\exp(jk_0z)\\ (-\hat{z}\times)^{\alpha}\eta_0\mathbf{H}^{r} &= -\exp(j\alpha\pi)\left[\hat{x}\sin\left(2\theta-\frac{\alpha\pi}{2}\right)+\hat{y}\cos\left(2\theta-\frac{\alpha\pi}{2}\right)\right]\\ &\quad\times E_0\exp(jk_0z)\end{aligned} \tag{4}$$

Fractional dual fields corresponding to the total fields, means sum of incident and reflected, is

$$\begin{aligned}\mathbf{E}_{fd} &= E_0\exp\left(\frac{j\alpha\pi}{2}\right)\\ &\quad\times\left[\hat{x}\left\{\left(\cos\left(\frac{\alpha\pi}{2}\right)+\cos\left(2\theta-\frac{\alpha\pi}{2}\right)\right)\cos\left(k_0z+\frac{\alpha\pi}{2}\right)\right.\right.\\ &\quad\left.+j\left(-\cos\left(\frac{\alpha\pi}{2}\right)+\cos\left(2\theta-\frac{\alpha\pi}{2}\right)\right)\sin\left(k_0z+\frac{\alpha\pi}{2}\right)\right\}\\ &\quad+\hat{y}\left\{\left(\sin\left(\frac{\alpha\pi}{2}\right)-\sin\left(2\theta-\frac{\alpha\pi}{2}\right)\right)\cos\left(k_0z+\frac{\alpha\pi}{2}\right)\right.\\ &\quad\left.\left.-j\left(\sin\left(\frac{\alpha\pi}{2}\right)+\sin\left(2\theta-\frac{\alpha\pi}{2}\right)\right)\sin\left(k_0z+\frac{\alpha\pi}{2}\right)\right\}\right]\end{aligned} \tag{5a}$$

$$\begin{aligned}\eta_0\mathbf{H}_{fd} &= E_0\exp\left(\frac{j\alpha\pi}{2}\right)\\ &\quad\times\left[\hat{x}\left\{-\left(\sin\left(\frac{\alpha\pi}{2}\right)+\sin\left(2\theta-\frac{\alpha\pi}{2}\right)\right)\cos\left(k_0z+\frac{\alpha\pi}{2}\right)\right.\right.\\ &\quad\left.+j\left(\sin\left(\frac{\alpha\pi}{2}\right)-\sin\left(2\theta-\frac{\alpha\pi}{2}\right)\right)\sin\left(k_0z+\frac{\alpha\pi}{2}\right)\right\}\\ &\quad+\hat{y}\left\{\left(\cos\left(\frac{\alpha\pi}{2}\right)-\cos\left(2\theta-\frac{\alpha\pi}{2}\right)\right)\cos\left(k_0z+\frac{\alpha\pi}{2}\right)\right.\\ &\quad\left.\left.-j\left(\cos\left(\frac{\alpha\pi}{2}\right)+\cos\left(2\theta-\frac{\alpha\pi}{2}\right)\right)\sin\left(k_0z+\frac{\alpha\pi}{2}\right)\right\}\right]\end{aligned} \tag{5b}$$

Impedance of the corresponding surface may be obtained as

$$Z_{fd}^{xy} = \eta_0\left[\frac{(C_\alpha+C_\theta)\cos\left(k_0z+\frac{\alpha\pi}{2}\right)+j(-C_\alpha+C_\theta)\sin\left(k_0z+\frac{\alpha\pi}{2}\right)}{(C_\alpha-C_\theta)\cos\left(k_0z+\frac{\alpha\pi}{2}\right)-j(C_\alpha-C_\theta)\sin\left(k_0z+\frac{\alpha\pi}{2}\right)}\right] \tag{6a}$$

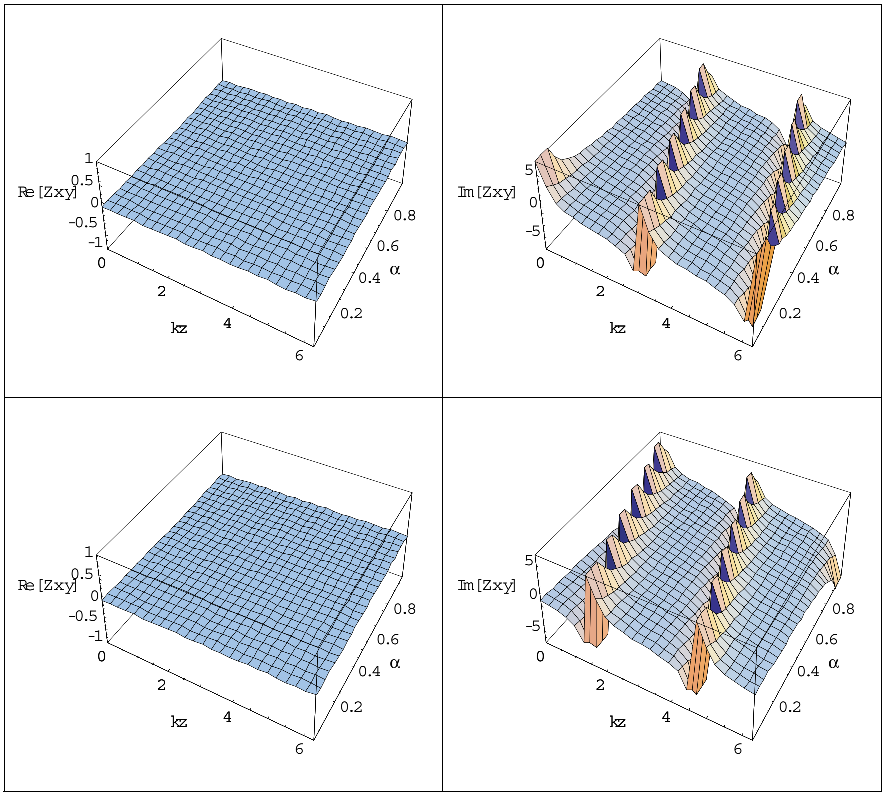

Figure 1. Variation of Zxy w.r.t. kz and α (for $\theta = 0,\ \pi/4,\ \pi/2$).

$$Z_{fd}^{yx} = -\eta_0 \left[\frac{(S_\alpha - S_\theta)\cos\left(k_0 z + \dfrac{\alpha\pi}{2}\right) - j(S_\alpha + S_\theta)\sin\left(k_0 z + \dfrac{\alpha\pi}{2}\right)}{-(S_\alpha + S_\theta)\cos\left(k_0 z + \dfrac{\alpha\pi}{2}\right) + j(S_\alpha - S_\theta)\sin\left(k_0 z + \dfrac{\alpha\pi}{2}\right)} \right] \tag{6b}$$

$$Z_{fd}^{xx} = \eta_0 \left[\frac{(C_\alpha + C_\theta)\cos\left(k_0 z + \dfrac{\alpha\pi}{2}\right) + j(-C_\alpha + C_\theta)\sin\left(k_0 z + \dfrac{\alpha\pi}{2}\right)}{-(S_\alpha + S_\theta)\cos\left(k_0 z + \dfrac{\alpha\pi}{2}\right) + j(S_\alpha - S_\theta)\sin\left(k_0 z + \dfrac{\alpha\pi}{2}\right)} \right] \tag{6c}$$

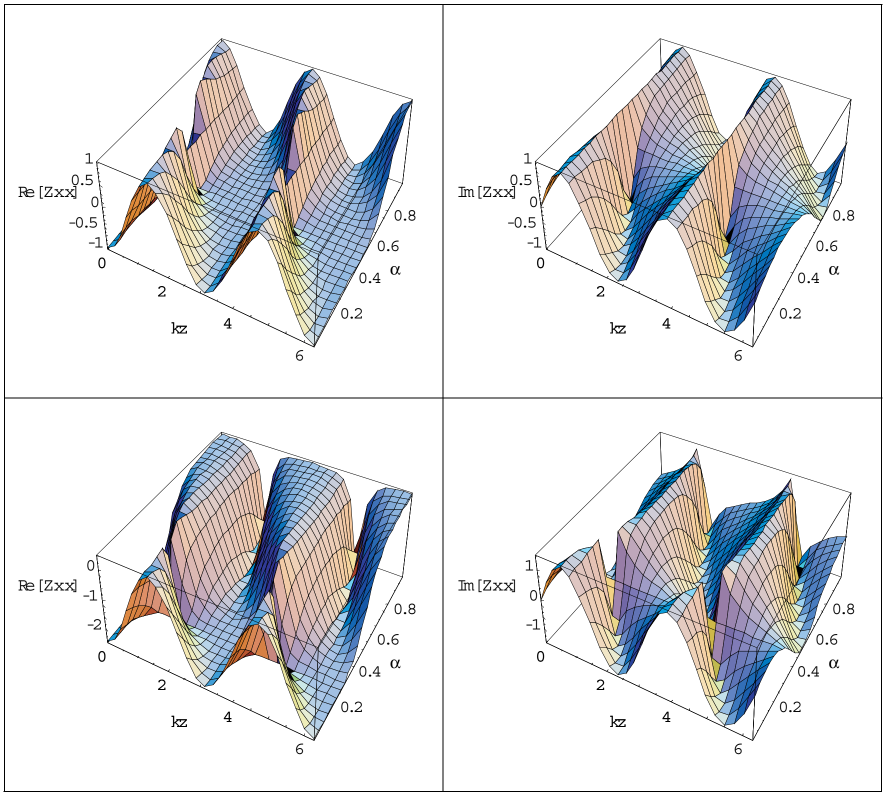

Figure 2. Variation of impedance Zxy w.r.t. α and θ ($kz = 0$).

$$Z_{fd}^{yy} = \eta_0 \left[\frac{(S_\alpha - S_\theta)\cos\left(k_0 z + \frac{\alpha\pi}{2}\right) - j(S_\alpha + S_\theta)\sin\left(k_0 z + \frac{\alpha\pi}{2}\right)}{(C_\alpha - C_\theta)\cos\left(k_0 z + \frac{\alpha\pi}{2}\right) - j(C_\alpha + C_\theta)\sin\left(k_0 z + \frac{\alpha\pi}{2}\right)} \right] \tag{6d}$$

where

$$C_\alpha = \cos\left(\frac{\alpha\pi}{2}\right)$$
$$C_\theta = \cos\left(2\theta - \frac{\alpha\pi}{2}\right)$$
$$S_\alpha = \sin\left(\frac{\alpha\pi}{2}\right)$$

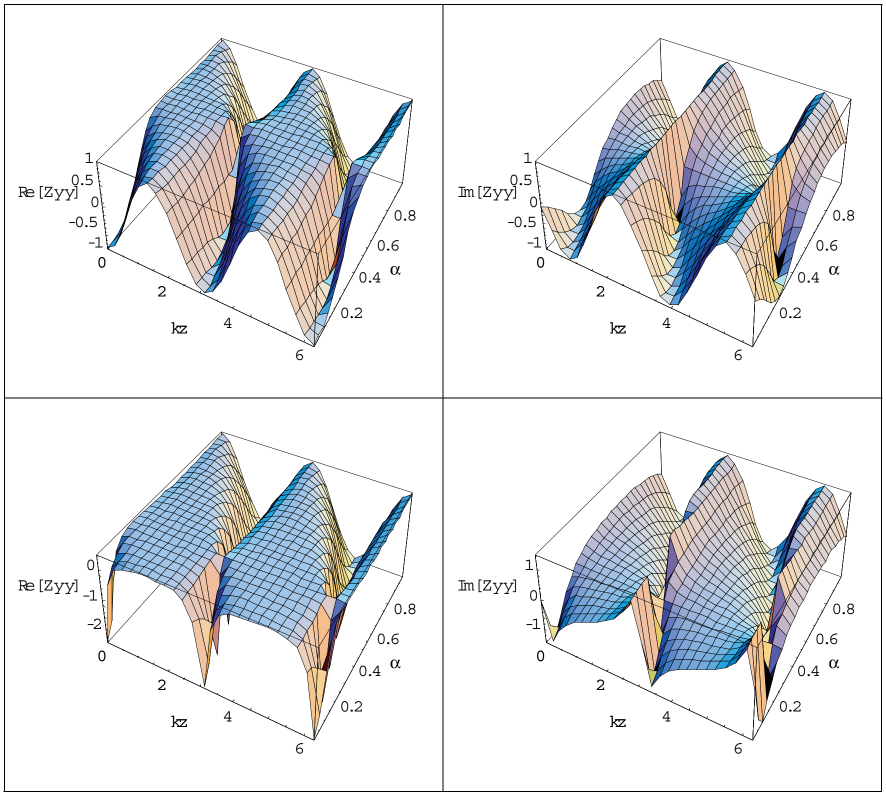

Figure 3. Zoom view of the variation of impedance Zxy w.r.t. α and θ ($kz = 0$).

$$S_\theta = \sin\left(2\theta - \frac{\alpha\pi}{2}\right)$$

As PEC means parameter $\theta = \pi/2$ and PMC means $\theta = 0$. Therefore for $\theta = \pi/2$, following is obtained

$$Z_{fd}^{xy}(\theta = \pi/2) = Z_{fd}^{yx}(\theta = \pi/2) = -j\eta_0 \tan\left(k_0 z + \frac{\alpha\pi}{2}\right)$$

For $\theta = 0$, we have

$$Z_{fd}^{xy}(\theta = 0) = j\eta_0 \cot\left(k_0 z + \frac{\alpha\pi}{2}\right)$$

$$Z_{fd}^{yx}(\theta = 0) = j\eta_0 \cot\left(k_0 z + \frac{\alpha\pi}{2}\right)$$

Above results are same as had been obtained by Engheta [6]. It means that for $\theta = 0$ and $\theta = \pi/2$ fractional impedance with Z_{fd}^{xy}, Z_{fd}^{yx} (Z_{fd}^{xy}=Z_{fd}^{yx}) will be intermediate between PEC and PMC. This is because when $\theta = 0, \pi/2$, then there is no polarization rotation.

When $\theta \neq 0, \pi/2$, then fractional impedance with Z^{xx} and Z^{yy} will be intermediate between PEMC and DPEMC.

3. DISCUSSION AND CONCLUSIONS

It is obvious from the results given in the last section that even for normal incidence, impedance of the fractional dual surface between PEMC and DPEMC is anisotropic. On the other side, when dealing with fractional dual situation between the PEC and PMC, we obtain isotropic impedance for normal incidence. Behavior of the fractional dual impedance with respect to fractional parameter α and kz is studied. The plots are presented in Figure 1 to Figure 3. It is known that for PMC ($\theta = 0$) and PEC ($\theta = \pi/2$), change in values of fractional parameter α is equivalent to change in values of kz. It is also noted from the plots that for non-limiting values of θ, variation in values of fractional parameter α is not always equivalent to variation in values of kz.

REFERENCES

1. Lindell, I. V. and A. H. Sihvola, "Realization of the PEMC boundary," *IEEE Trans. Antennas Propag.*, Vol. 53, 3012–3018, Sep. 2005.
2. Lindell, I. V. and A. H. Sihvola, "Perfect electromagnetic conductor," *J. Electromagn. Waves Appl.*, Vol. 19, No. 7, 861–869, 2005.
3. Lindell, I. V. and A. H. Sihvola, "Transformation method for problems involving perfect electromagnetic conductor (PEMC) structures," *IEEE Trans. on Antennas and Propagation*, Vol. 53, No. 9, 3005–3011, Sep. 2005.
4. Lindell, I. V. and A. H. Sihvola, "Losses in PEMC boundary," *IEEE Trans. on Antennas and Propagation*, Vol. 54, No. 9, 2553–2558, Sept. 2006.
5. Lindell, I. V. "Electromagnetic fields in self dual media in differential-form representation," *Progress In Electromagnetics Research*, PIER, 58, 319–333, 2006.
6. Engheta, N., "Fractional curl operator in electromagnetics," *Microwave Opt. Tech. Lett.*, Vol. 17, 86–91, 1998.

7. Veliev, E. I. and N. Engheta, "Fractional curl operator in reflection problems," *10th Int. Conf. on Mathematical Methods in Electromagnetic Theory*, Ukraine, Sept. 14–17, 2004.
8. Hussain, A. and Q. A. Naqvi, "Fractional curl operator in chiral medium and fractional nonsymmetric transmission line," *Progress In Electromagnetics Research*, PIER 59, 199–213, 2006.
9. Hussain, A., S. Ishfaq, and Q. A. Naqvi, "Fractional curl operator and Fractional waveguides," *Progress In Electromagnetics Research*, PIER 63, 319–335, 2006.
10. Ivakhnychenko, M. V., E. I. Veliev, and T. M. Ahmedov, "New generalized electromagnetic boundaries fractional operators approach," *Proceedings of MMET*06, Kharkiv*, 434–437, 2006.
11. Hussain, A., M. Faryad, and Q. A. Naqvi, "Fractional curl operator and Fractional chiro-waveguide," *Journal of Electromagnetic Waves and Applications*, Vol. 21, No. 8, 1119–1129, 2007.
12. Ozaktas, H. M., Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, Wiley, New York, 2001.