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A RIGOROUS AND COMPLETED STATEMENT ON HELMHOLTZ THEOREM

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Abstract—There are some limitations on the statement of classic Helmholtz theorem although it has broad applications. Actually, it only applies to simply connected domain with single boundary surface and does not provide any conclusion about the domain where discontinuities of field function exist. However, discontinuity is often encountered in practice, for example, the location of surface sources or interface of two kinds of medium. Meanwhile, most existing versions of Helmholtz theorem are imprecise and imperfect to some extent. This paper not only tries to present a precise statement and rigorous proof on classic Helmholtz theorem with the accuracy of mathematical language and logical strictness, but also generalizes it to the case of multiply connected domain and obtains a generalized Helmholtz theorem in the sense of Lebesgue measure and Lebesgue integral defined on three-dimensional Euclidean space. Meanwhile, our proof and reasoning are more sufficient and perfect.

As an important application of the generalized Helmholtz theorem, the concepts of irrotational and solenoidal vector function are emphasized. The generalized Helmholtz theorem and the present conclusion should have important reference value in disciplines including vector analysis such as electromagnetics.

1. INTRODUCTION

It is well-known that Helmholtz theorem decomposes an arbitrary vector function into two parts: one is an irrotational component which can be expressed by the gradient of a scalar function and the other is a rotational part which can be expressed by the curl of vector function. As a particular decomposition form of a vector function, the theorem has very important applications in electromagnetics

because its decomposition terms have been closely related to the scalar potential and vector potential of a vector field.

Many authors have presented their own versions of statement and proof on Helmholtz theorem in books and literatures. Recently, it caused much attentions and some statements on it is also published [17–19]. In this paper, we try to give a rigorous statement on it, which is to some extent different from that in Ref. [19]. Some of the most influential versions are cited as follows with the form of proposition.

Proposition 1. If $\vec{F}$ is an arbitrary continuous vector function with all the second order partial derivatives in free space, and its surface integration or its any partial derivative is zero at infinitive, then this vector function must be able to be uniquely separated into the sum of the gradient of a scalar function and the curl of a vector function, that is

$$\vec{F} = -\nabla\varphi + \nabla \times \vec{A} \tag{1}$$

in which

$$\varphi = \int_V \frac{\nabla' \cdot \vec{F}(\vec{r}')}{4\pi R} dV' \quad \vec{A} = \int_V \frac{\nabla' \times \vec{F}(\vec{r}')}{4\pi R} dV'$$

Proposition 2. If $\vec{F}$ is an arbitrary first order continuously differentiable vector function with $R^2 \left|\vec{F}\right|$ bounded at infinitive, then this vector function must be able to be completely and uniquely decomposed into the sum of the gradient of a scalar function and the curl of a vector function as shown in formula (1).

Proposition 3. If an arbitrary field function in bounded domain satisfies the condition that field function is bounded on the boundary, then this vector function must be able to be completely decomposed into the sum of the gradient of a scalar function and the curl of a vector function.

Proposition 4. An arbitrary vector function in bounded domain satisfied vector homogeneous boundary condition must be able to be completely and uniquely decomposed into the sum of an irrotational field and a solenoidal field as shown in formula (1). Meanwhile, irrotational field and solenoidal field are mutually orthogonal and satisfy the same kind boundary condition.

Proposition 5. For an arbitrary vector field in bounded domain V, if the divergence, the curl of the field in domain V and the value of the field on boundary S are given, then the vector field can be able to be uniquely ascertained and expressed by the vector sum of an irrotational field and a solenoidal field.

All of above statements have been published in the form of theorem or corollary in books and literatures. However, through deeply consideration and inference, we can find that all above versions of Helmholtz theorem are imprecise and imperfect to some extent.

Firstly, they lack precise and proper depiction about the condition of proposition. It is well-known that the statement of theorem must be rigorous and objective. The condition of theorem should be terse, unrepeatable and as weak as possible. But the condition is a little strong in some of above propositions, for example, Proposition 2 and Proposition 4. In some propositions, there is no obvious distinction between the existence of decomposition and the uniqueness of decomposition, which easily make reader misconceive.

Secondly, the applicable range of above propositions is restricted to simply connected domain with single boundary surface, and propositions is failed when the domain is a simply connected domain with multiply boundary surface or multiply connected domain. Nevertheless, discontinuity is often encountered in practice, for example, the location of surface sources or interface of two kinds of medium.

Finally, it is must be emphasized that Helmholtz theorem is not equivalent to the uniqueness theorem of a vector function. Helmholtz theorem is an operator-based decomposition theorem of a vector function and does not indicate directly any uniqueness theorem for boundary value problem. So Proposition 5 is not a kind form of statement on Helmholtz theorem.

Actually, there exist other imprecise even wrong statements and proofs in other books and literatures.

It is easily to find that our comments on above-mentioned Helmholtz theorem are objective and rational after reading this paper.

Strictly speaking, Helmholtz theorem has various formulae and statements according to the difference in topological property of the discussed region, decomposition method, the dimension of vector function and the function space with which vector function is affiliated. It is much emphasized that this paper bases its discussion on Lebesgue measure and Lebesgue integral defined on Euclidean space, which make our adoptive mathematical statements and foundation of theory more general than traditional method.

Generally, our discussed domain is a connected domain. A disconnected domain can be decompounded into the union of some connected domains and boundaries. Connected domain can be divided into two categories: simply connected domain and multiply connected domain. The boundary face of simply connected domain can be a simple surface or a simple closed surface. According to the difference

of boundary face, a simply connected domain can be classified into the case with single boundary surface and the case with multiply boundary surface. Correspondingly, the boundary face of multiply connected domain is named multiply connected boundary surface.

This paper firstly presents precise statements and rigorous proofs on bounded and unbounded domain cases of Helmholtz theorem in simply connected domain, then we generalize it to multiply connected domain and obtain a generalized Helmholtz theorem. Meanwhile, the physical meaning and application of the theorem are indicated in electromagnetics.

2. HELMHOLTZ THEOREM IN SIMPLY CONNECTED DOMAIN

A simply connected domain can be classified into the case of single boundary surface and multiply boundary surface. We only need to discuss the case of single boundary surface because the case of multiply boundary surface can be divided into several simply connected domains with single boundary surface by means of appropriate auxiliary curves or surfaces.

Theorem 1. Helmholtz theorem in bounded domain

Let $G \subset R^3$ and G is a open domain, S is an arbitrary closed curved surface that is smooth or piecewise smooth, V is the volume surrounded by the surface S, $\overline{V}(V \cup \partial V = V \cup S)$ represents a closure of domain V.

For a single-valued vector function of several variables $\vec{F}(\vec{r})$ which is defined in domain $\overline{V}(\vec{r} \in \overline{V})$, if it satisfies the following conditions:

(1) $\forall \vec{r} \in V$, $\vec{F}(\vec{r}) \in C^1$,
(2) $\vec{F}(\vec{r})$ is almost everywhere continual and bounded on boundary S,

then $\vec{F}(\vec{r})$ must be able to be completely decomposed into the sum of an irrotational field and a solenoidal field. Decomposition form is written as follow:

$$\begin{aligned}\vec{F}(\vec{r}) \;=\; & -\nabla\left[\int\limits_V \frac{\nabla' \cdot \vec{F}(\vec{r}\,')}{4\pi R} dV' - \oint\limits_S \frac{\vec{F}(\vec{r}\,') \cdot \hat{n}}{4\pi R} dS'\right] \\ & +\nabla \times \left[\int\limits_V \frac{\nabla' \times \vec{F}(\vec{r}\,')}{4\pi R} dV' + \oint\limits_S \frac{\vec{F}(\vec{r}\,') \times \hat{n}}{4\pi R} dS'\right] \qquad (2)\end{aligned}$$

Let

$$\varphi = \int_V \frac{\nabla' \cdot \vec{F}(\vec{r}\,')}{4\pi R} dV' - \oint_S \frac{\vec{F}(\vec{r}\,') \cdot \hat{n}}{4\pi R} dS' \tag{3}$$

$$\vec{A} = \int_V \frac{\nabla' \times \vec{F}(\vec{r}\,')}{4\pi R} dV' + \oint_S \frac{\vec{F}(\vec{r}\,') \times \hat{n}}{4\pi R} dS' \tag{4}$$

then (2) is abbreviated to

$$\vec{F} = -\nabla\varphi + \nabla \times A \tag{5}$$

in which $R = |\vec{r} - \vec{r}\,'|$, $\hat{n}$ is a exterior unit normal vector of closed surface S.

Proof: For an arbitrary given vector function $\vec{F}(\vec{r})$, a transform can be introduced as below:

$$\vec{F}(\vec{r}) = -\nabla^2 \vec{W}(\vec{r}) \tag{6}$$

where vector function $\vec{W}(\vec{r})$ is first order continuously differentiable.

In Euclidean space, vector operator ∇^2 can be projected into three scalar operators corresponding three directions $\hat{x}$, $\hat{y}$ and $\hat{z}$, So (6) can be decomposed into three scalar equations as follows:

$$F_x = -\nabla^2 W_x \tag{7.1}$$

$$F_y = -\nabla^2 W_y \tag{7.2}$$

$$F_z = -\nabla^2 W_z \tag{7.3}$$

Each equation in (7) is a scalar Poisson equation and has a unique solution when $\vec{W}_i(\vec{r})$ $(i = x, y, z)$ is a first order continuously differentiable function and has proper boundary condition. Meanwhile, (6) is virtually a vector Poisson equation with respect to $\vec{W}(\vec{r})$ and the introduction of the above transform aims to establish a constructed problem of $\vec{W}(\vec{r})$ based on given $\vec{F}(\vec{r})$. Therefore different kinds of boundary conditions of $\vec{W}(\vec{r})$ can be applied and $\vec{W}(\vec{r})$ satisfied the first kind homogeneous boundary condition is chosen as the base of following discussion. So adopting the Green's function satisfied Dirichlet boundary condition, we can get the expression of $\vec{W}_i(\vec{r})$ as

below:

$$\begin{aligned}\vec{W}_i(\vec{r}) &= \int_V \vec{F}_i\left(\vec{r}'\right) G\left(\vec{r}', \vec{r}\right) dV' - \oint_S \vec{W}_i\left(\vec{r}'\right) \frac{\partial G\left(\vec{r}', \vec{r}\right)}{\partial n'} dS' \\ &= \int_V \vec{F}_i\left(\vec{r}'\right) G\left(\vec{r}', \vec{r}\right) dV' \end{aligned} \tag{8.1}$$

where

$$\nabla^2 G\left(\vec{r}', \vec{r}\right) = -\delta\left(\vec{r}' - \vec{r}\right) \tag{8.2}$$

$$G\left(\vec{r}', \vec{r}\right)\big|_S = 0 \tag{8.3}$$

Prolong the function $\vec{F}(\vec{r})$ defined in domain V to whole space by assuming $\vec{F}(\vec{r}) \equiv 0$ outside the volume V. Noting that (6) is just a constructed problem, we can discuss the boundary value problem in especial case, for example, $\vec{W}_i(\vec{r})$ in free space. Correspondingly we obtain Green's function of a free space $G_0\left(\vec{r}', \vec{r}\right)$.

$$G_0\left(\vec{r}', \vec{r}\right) = \frac{1}{4\pi R} \tag{8.4}$$

Firstly, substituted (8.4) into (8.1), (8.1) becomes:

$$\vec{W}_i = \int \frac{\vec{F}_i\left(\vec{r}'\right)}{4\pi R} dV', \quad i = x, y, z \tag{9}$$

After compounding the above solution of three directions, we gain

$$\vec{W} = \int \frac{\vec{F}\left(\vec{r}'\right)}{4\pi R} dV' \tag{10}$$

where $R = |\vec{r} - \vec{r}'|$, $\vec{r}$ represents the coordinate of field point in $\vec{W}$, $\vec{r}'$ represents the coordinate of the source point in $\vec{F}$.

We quote a vector identity

$$\nabla^2 = -\nabla \times \nabla \times + \nabla\nabla\cdot$$

The above identity holds for any first order continuously differentiable vector function. Therefore (6) becomes:

$$\vec{F} = -\nabla^2 \vec{W} = -\nabla\nabla \cdot \vec{W} + \nabla \times \nabla \times \vec{W} \tag{11}$$

Secondly, substituted (10) into (11), (11) becomes:

$$\vec{F}(\vec{r}) = -\nabla\left(\nabla\cdot\int\limits_V \frac{\vec{F}(\vec{r}\,')}{4\pi R}dV'\right) + \nabla\times\left(\nabla\times\int\limits_V \frac{\vec{F}(\vec{r}\,')}{4\pi R}dV'\right) \quad (12)$$

Since ∇ operator has no effect on variables with prime sign and the following property formula holds.

$$\nabla\frac{1}{R} = -\nabla'\frac{1}{R}$$

Simplifying (12), we obtain

$$\begin{aligned}\vec{F}(\vec{r}) \;=\; & -\nabla\int\limits_V\left[\frac{\nabla'\cdot\vec{F}(\vec{r}\,')}{4\pi R} - \nabla'\cdot\frac{\vec{F}(\vec{r}\,')}{4\pi R}\right]dV' \\ & +\nabla\times\left[\int\limits_V \frac{\nabla'\times\vec{F}(\vec{r}\,')}{4\pi R} - \nabla'\times\left(\frac{\vec{F}(\vec{r}\,')}{4\pi R}\right)dV'\right] \quad (13)\end{aligned}$$

Finally, based on classic Gauss theorem and Stokes theorem in single connected domain, (13) becomes (2).

The decomposition form of (2) is really existent because scalar Poisson Equations (7) have unique solution when the domain V and proper boundary condition are given. However, it is much emphasized that as a decomposition form the uniqueness of (2) can not be assured under present conditions. The uniqueness of decomposition is determined by independence, completeness and orthogonality in decomposition terms. Under the present conditions, orthogonality of decomposition terms can not be ensured, therefore we can not confirm the uniqueness of decomposition.

Smoothness of boundary surface S, that is, the equation of surface is continuous differentiable, confirms the existence of surface integral of the first kind, the second kind and corresponding volume integral, thus the derivation of (9), (10), (12), (13) and (2) is guarantied.

Closure of surface S and continuity of one order partial difference of $\vec{F}(\vec{r})$ in domain V are the precondition of the utilization of Gauss theorem and Stokes formula. Strictly speaking, classic Gauss theorem and Stokes formula only indicate the property of single connected domain. It can be applied to multiply connected domain after the theorem is modified.

For the condition (1), it ensures not only the feasibility of divergence and curl operation of vector function, but also the

integrability of the result of divergence and curl operation of vector function. Accordingly it is a most proper restriction condition up to now.

The condition (2) implies that all discontinuity points on boundary S are discontinuity points of the first kind and its area measure is zero. It ensures not only the utilization of Gauss theorem and Stokes formula, but also surface integral in (2) exists and the value of surface integral is invariant. This completes the proof.

Remarks on Theorem 1.

(1) Piecewise smooth surface in the theorem, which is also called regular surface, requires that not only every piece is smooth, but also its positive normal vector is uniquely determinate and continuous on every point of surface.

(2) For the condition (1), we have the following propositions.

Proposition A. $\forall \vec{r} \in V$, $\vec{F}(\vec{r}) \in C^1 \Leftrightarrow \vec{F}(\vec{r})$ has a continuous partial derivative of first order in domain V.

Proposition B. $\vec{F}(\vec{r})$ has a continuous partial derivative of first order in domain $V \Leftrightarrow$ its component function $\vec{F}_i(\vec{r})(i = 1, 2, 3)$ has a continuous partial derivative of first order in domain V, in which $\vec{F}(\vec{r}) = (\vec{F}_1(\vec{r}), \vec{F}_2(\vec{r}), \vec{F}_3(\vec{r}))$.

Proposition C. $\vec{F}(\vec{r})$ has a continuous partial derivative of first order in domain $V \Rightarrow \vec{F}(\vec{r})$ is a first order differentiable function in domain V.

Proposition D. $\vec{F}(\vec{r})$ has a continuous partial derivative of first order in domain $V \Rightarrow \vec{F}(\vec{r})$ have all the second order partial derivatives in domain V.

Proposition E. $\vec{F}(\vec{r})$ has a continuous partial derivative of first order in domain $V \Leftarrow \vec{F}(\vec{r})$ is a first order continuous differentiable function in domain V.

Proposition F. $\vec{F}(\vec{r})$ has a continuous partial derivative of first order in domain $V \Rightarrow \vec{F}(\vec{r})$ is bounded and integrable in domain V.

If $\vec{F}(\vec{r})$ is only a first order differentiable function in Theorem 1, it is not enough to ensure the integrability of the result of divergence and curl operation of $\vec{F}(\vec{r})$. So the condition is too weak to adopt.

If $\vec{F}(\vec{r})$ is a vector function with all the second order partial derivatives on some points in domain V, it is not able to insure the order interchange of the operation about mixed partial derivative of $\vec{F}(\vec{r})$. So the condition is also unfitting.

If the condition that $\vec{F}(\vec{r})$ is a first order continuous differentiable vector function in domain V is assumed, then the conclusion of Theorem 1 still holds, but it is a little strong compared with the present

condition (1).

From Proposition F and the condition (2), we can infer that $\vec{F}(\vec{r})$ is bounded in domain $\overline{V}$. Therefore it is redundant to restrict the boundedness and integrability of $\vec{F}(\vec{r})$ in domain V or $\overline{V}$ under the condition (1).

From the above discussion, the condition (1) is proper and precise for the theorem.

(3) The condition (2) implies that all discontinuity points on boundary S are discontinuity points of the first kind and its area measure is zero. If the measure of all discontinuity points is not equal to zero, then we must define some new concepts such as surface gradient, surface divergence and surface curl to ensure Gauss theorem and Stokes formula validated. If there exist some singular points in discontinuity points, we can discuss it with the theory of singular integral. These cases are not explicitly stated here.

(4) There are many proof methods on Helmholtz theorem, for example, with impulse function and vector identity or with Green vector identity of the second kind. Compared with the proof based on impulse function our proof is more precise and more natural. And our condition for the theorem is weaker compared to the proof based on Green vector identity of the second kind.

(5) The conclusion of Theorem 1 implies a fact that the uniqueness of decomposition can not be confirmed because the two parts of decomposition form are not mutually orthogonal and complete. If we want to obtain the uniqueness, the proper boundary condition of vector function must be appended.

In Theorem 1, $\vec{r}$ represents the coordinate of field point, $\vec{r}'$ represents the coordinate of the source point, $\nabla' \cdot \vec{F}(\vec{r}')$ and $\nabla' \times \vec{F}(\vec{r}')$ are the density of flux and the density of curl source individually. The part of surface integral indicates the contribution of surface sources in boundary S which can be sources or induced sources. So the Theorem 1 presents a kind of quantitative relation between field and source. The vector function satisfied the condition of the theorem can be completely decomposed into the sum of the gradient of a scalar function and the curl of a vector function.

Theorem 2. Helmholtz theorem in unbounded domain

Let $V = R^3$. For a single-valued vector function of several variables $\vec{F}(\vec{r})$, which is defined in domain $V(\vec{r} \in V)$, if it satisfies the following conditions:

(1) $\forall \vec{r} \in V$, $\vec{F}(\vec{r}) \in C^1$,
(2) $\vec{F}(\vec{r})$ is vanishing at infinitive at enough quick rate and make its surface integration approach zero,

then $\vec{F}(\vec{r})$ must be able to be completely and uniquely decomposed into the sum of an irrotational field and a solenoidal field. Decomposition form is written as follow:

$$\vec{F}(\vec{r}) = -\nabla \int\limits_V \frac{\nabla' \cdot \vec{F}(\vec{r}')}{4\pi R} dV' + \nabla \times \int\limits_V \frac{\nabla' \times \vec{F}(\vec{r}')}{4\pi R} dV' \tag{14}$$

Let

$$\varphi = \int\limits_V \frac{\nabla' \cdot \vec{F}(\vec{r}')}{4\pi R} dV' \tag{15}$$

$$\vec{A} = \int\limits_V \frac{\nabla' \times \vec{F}(\vec{r}')}{4\pi R} dV' \tag{16}$$

then (14) is abbreviated to

$$\vec{F} = -\nabla\varphi + \nabla \times \vec{A} \tag{17}$$

in which $R = |\vec{r} - \vec{r}'|$.

Proof: Infinite domain can be looked upon as the limit case of finite domain, which is equivalent to the case of $R \to \infty$. Along the proving thread of Theorem 1, we let the closed surface S be infinite surface. If the convergence of integral in (2) is ensured, we can obtain the feasibility and the utilization of limit operation which is applied to the expression (2) of the case of finite domain.

The condition, which $\exists\delta > 0$, such that $R^{1+\delta}\left|\vec{F} \cdot \hat{n}\right|$ and $R^{1+\delta}\left|\vec{F} \times \hat{n}\right|$ bounded, is the sufficient condition of identity $\vec{F}(\vec{r}) = -\nabla^2 \int\limits_V \frac{\vec{F}(\vec{r}')}{4\pi R} dV'$, and ensures that Green theorem and Stokes formula can be applied to the case of infinite domain without any amendment. Of course the condition (2) includes the above condition. So

$$\begin{aligned}
\vec{F}(\vec{r}) &= -\nabla\left(\nabla \cdot \int\limits_V \frac{\vec{F}(\vec{r}')}{4\pi R} dV'\right) + \nabla \times \left(\nabla \times \int\limits_V \frac{\vec{F}(\vec{r}')}{4\pi R} dV'\right) \\
&= -\nabla\left[\int\limits_V \frac{\nabla' \cdot \vec{F}(\vec{r}')}{4\pi R} dV' - \oint\limits_{S\to\infty} \frac{\vec{F}(\vec{r}') \cdot \hat{n}}{4\pi R} dS'\right] \\
&\quad + \nabla \times \left[\int\limits_V \frac{\nabla' \times \vec{F}(\vec{r}')}{4\pi R} dV' + \oint\limits_{S\to\infty} \frac{\vec{F}(\vec{r}') \times \hat{n}}{4\pi R} dS'\right]
\end{aligned}$$

$$= -\nabla \int_V \frac{\nabla' \cdot \vec{F}(\vec{r}')}{4\pi R} dV' + \nabla \times \int_V \frac{\nabla' \times \vec{F}(\vec{r}')}{4\pi R} dV' \tag{14}$$

At fact the condition (2) implies the homogeneous boundary condition of $\vec{F}(\vec{r})$, which makes the decomposition components mutual orthogonal. Therefore the decomposition form (14) of $\vec{F}(\vec{r})$ must be unique. This completes the proof.

Remarks on Theorem 2

(1) For the condition (2), the qualitative description has to adopted because it is very difficult to strictly determine the quantitative condition that surface integration approach zero at infinitive. Many authors have tried to find the quantitative condition of condition (2). We cite them as below:

Condition A. $R^2 \left|\vec{F}\right|$ is bounded when $R \to \infty$.

Condition B. $R^2 \left|\vec{F} \cdot \hat{n}\right|$ and $R^2 \left|\vec{F} \times \hat{n}\right|$ is bounded when $R \to \infty$.

Condition C. $\forall \varepsilon > 0$, $R^{1+\varepsilon} \left|\vec{F}\right|$ is bounded when $R \to \infty$.

Condition D. $R^1 \left|\vec{F} \cdot \hat{n}\right| \to 0$ and $R^1 \left|\vec{F} \times \hat{n}\right| \to 0$ when $R \to \infty$.

Condition E. $\vec{F}(\vec{r}) \to 0$ when $R \to \infty$.

Actually, Condition A and B are theoretically feasible, but it is too strong to adopt, Condition C is practically feasible but there is no theoretical proof. Condition D and E are too weak to be applied in the theorem. Therefore it is not proper to adopt any kind of above conditions as a strict and universal condition.

Condition (2) has abundant connotation and can derive radiation boundary condition of far field. However, it is cursory to consider them equivalent.

3. HELMHOLTZ THEOREM IN MULTIPLY CONNECTED DOMAIN

Boundary surface of a domain has two categories: boundary surface of simply connected domain and boundary surface of multiply connected domain. The latter can be classified into many cases.

In simply connected domain, boundary surface can be single boundary surface or multiply boundary surface. However for multiply connected domain, boundary surfaces are consisted of boundary surfaces of several simply connected domains, boundary surface of multiply connected domain or composite surfaces of them. For simplification, this paper only discusses the domain with composite

surface which is consisted of several simply connected sub-domains because multiply boundary surface can be always divided into several single boundary surfaces of simply connected domain by means of appropriate auxiliary surfaces. The argument in the general case is similar.

In open domain, at most countable discontinuity points or point sets of the first kind can constitute isolated set, curve line or surface according to geometry. Based on these discontinuity points or point set, multiply connected domain can be partitioned into at most countable simply connected domain. The continuous function defined in the above domain is called piecewise continuous function.

Lemma 1. Assume $f(x)$ is a bounded function defined on a measurable set E, and the measure of E is bounded, then $f(x)$ is integrable in the sense of Lebesgue if and only if $f(x)$ is a measurable function.

Theorem 3. Generalized Helmholtz theorem in bounded domain

Let $G \subset R^3$ and V is a multiply connected domain in domain G, boundary S of domain V is piecewise smooth. A single-valued vector function of several variables $\vec{F}(\vec{r})$ which is defined in domain $V(\vec{r} \in \overline{V})$ has at most countable discontinuity points or point sets of the first kind. Based on the piecewise continuous vector function $\vec{F}(\vec{r})$, V can be partitioned into the form $V = \bigcup\limits_{i \in I} V_i$, where I represents at most counted index set, $\vec{F}(\vec{r})$ is a continuous function for simply connected domain V_i $(\forall i \in I)$, meanwhile, $\forall i, j \in I$, $V_i \cap V_j = \emptyset$ and as closed surface of domain V_i, S_i $(\forall i \in I)$ is at least piecewise smooth. If $\vec{F}(\vec{r})$ satisfies the following conditions:

(1) $\forall i \in I$, $\forall \vec{r} \in V_i$, $\vec{F}(\vec{r}) \in C^1$,
(2) $\vec{F}(\vec{r})$is bounded in every surface S_i $(\forall i \in I)$, meanwhile, both $\frac{\vec{F}(\vec{r}\,')\cdot\hat{n}_i}{4\pi R}$ and $\frac{\vec{F}(\vec{r}\,')\times\hat{n}_i}{4\pi R}$ are measurable function,

then $\vec{F}(\vec{r})$ must be able to be completely decomposed into the sum of an irrotational field and a solenoidal field. Decomposition form is written as follow:

$$\begin{aligned}\vec{F}(\vec{r}) \;=\; & -\nabla\left[\int\limits_V \frac{\nabla'\cdot\vec{F}(\vec{r}\,')}{4\pi R}dV' - \sum_{i\in I}\oint\limits_{S_i}\frac{\vec{F}(\vec{r}\,')\cdot\hat{n}_i}{4\pi R}dS_i'\right] \\ & +\nabla\times\left[\int\limits_V \frac{\nabla'\times\vec{F}(\vec{r}\,')}{4\pi R}dV' + \sum_{i\in I}\oint\limits_{S_i}\frac{\vec{F}(\vec{r}\,')\times\hat{n}_i}{4\pi R}dS_i'\right] \quad (18)\end{aligned}$$

$$
\begin{aligned}
= & -\nabla \left[\int_V \frac{\nabla' \cdot \vec{F}(\vec{r}')}{4\pi R} dV' - \oint_S \frac{\vec{F}(\vec{r}') \cdot \hat{n}}{4\pi R} dS' \right. \\
& \left. - \sum_{i,j \in I} \int_{S_{ij}} \frac{(\vec{F}_i(\vec{r}') - \vec{F}_j(\vec{r}')) \cdot \hat{n}_{ij}}{4\pi R} dS'_{ij} \right] \\
& + \nabla \times \left[\int_V \frac{\nabla' \times \vec{F}(\vec{r}')}{4\pi R} dV' + \oint_S \frac{\vec{F}(\vec{r}') \times \hat{n}}{4\pi R} dS' \right. \\
& \left. + \sum_{i,j \in I} \int_{S_{ij}} \frac{(\vec{F}_i(\vec{r}') - \vec{F}_j(\vec{r}')) \times \hat{n}_{ij}}{4\pi R} dS'_{ij} \right]
\end{aligned} \tag{19}
$$

where both $\vec{F}_i$ and $\vec{F}_j$ represent vector functions defined on surface $S_{ij}(S_{ij} = S_i \cap S_j)$, meanwhile, they are located in two sides of S_{ij}, $\widehat{n_{ij}}$ is the unit normal vector of closed surface S_{ij} and directed to V_j from V_i. Summation appeared in (19) is a kind of unilateral sum, that is, unrepeated sum in interfaces.

Let

$$
\varphi = \int_V \frac{\nabla' \cdot \vec{F}(\vec{r}')}{4\pi R} dV' - \sum_{i \in I} \oint_{S_i} \frac{\vec{F}(\vec{r}') \cdot \hat{n}_i}{4\pi R} dS'_i \tag{20}
$$

$$
\vec{A} = \int_V \frac{\nabla' \times \vec{F}(\vec{r}')}{4\pi R} dV' + \sum_{i \in I} \oint_{S_i} \frac{\vec{F}(\vec{r}') \times \hat{n}_i}{4\pi R} dS'_i \tag{21}
$$

then (18) is abbreviated to

$$
\vec{F} = -\nabla\varphi + \nabla \times \vec{A} \tag{22}
$$

in which $R = |\vec{r} - \vec{r}'|$, $\hat{n}_i$ is an exterior unit normal vector of closed surface S_i.

Proof: After the decomposition of domain V, we can infer that surface integrals on every surface S_i and S are exist from Lemma 1 and condition (2). Then we obtain the decomposition form of every simply connected domain based on Theorem 1. And then the superposition of decomposition forms is done to obtain (18). Finally, we separate the integral domain of surface integral in (18) into two parts: one part is outside surface S of whole domain V and the other part is the interface

between sub-domains. So the surface integral is simplified to (19). This completes the proof.

Remarks on Theorem 3.

In the sense of the Lebesgue integral, countable additivity always holds. But in order to ensure the validity of inference, the integrability of every domain or surface must be satisfied because the measure of discontinuous sets in every domain may be not zero. So condition (2) in the case of multiply connected domain is stronger than the condition of Theorem 1.

Corollary 1.

Let $G \subset R^3$ and V is a bounded multiply connected domain in domain G, boundary S of domain V is at least piecewise smooth. A single-valued vector function of several variables $\vec{F}(\vec{r})$, which is defined in domain V ($\vec{r} \in \overline{V}$), has at most countable discontinuity points or point sets of the first kind. Based on the piecewise continuous vector function $\vec{F}(\vec{r})$, V can be partitioned into the form $V = \bigcup\limits_{i \in I} V_i$, where I represents at most counted index set , $\vec{F}(\vec{r})$ is a continuous function in simply connected domain V_i ($\forall i \in I$), meanwhile, $\forall i, j \in I$, $V_i \cap V_j = \emptyset$ and as closed surface of domain V_i, S_i ($\forall i \in I$) is at least piecewise smooth. If $\vec{F}(\vec{r})$ satisfies the following conditions:

(1) $\forall i \in I$, $\forall \vec{r} \in V_i$, $\vec{F}(\vec{r}) \in C^1$,
(2) $\vec{F}(\vec{r})$ is almost everywhere continual and bounded in every surface S_i ($\forall i \in I$),

then $\vec{F}(\vec{r})$ must be able to be completely decomposed into the sum of an irrotational field and a solenoidal field. Decomposition form is written as follow:

$$\begin{aligned}\vec{F}(\vec{r}) \;=\; & -\nabla\left[\int\limits_V \frac{\nabla' \cdot \vec{F}(\vec{r}\,')}{4\pi R} dV' - \oint\limits_S \frac{\vec{F}(\vec{r}\,') \cdot \hat{n}}{4\pi R} dS'\right] \\ & +\nabla \times \left[\int\limits_V \frac{\nabla' \times \vec{F}(\vec{r}\,')}{4\pi R} dV' + \oint\limits_S \frac{\vec{F}(\vec{r}\,') \times \hat{n}}{4\pi R} dS'\right] \qquad (23)\end{aligned}$$

Let

$$\varphi = \int\limits_V \frac{\nabla' \cdot \vec{F}(\vec{r}\,')}{4\pi R} dV' - \oint\limits_S \frac{\vec{F}(\vec{r}\,') \cdot \hat{n}}{4\pi R} dS' \qquad (24)$$

$$\vec{A} = \int\limits_V \frac{\nabla' \times \vec{F}(\vec{r}\,')}{4\pi R} dV' + \oint\limits_S \frac{\vec{F}(\vec{r}\,') \times \hat{n}}{4\pi R} dS' \qquad (25)$$

then (23) is abbreviated to

$$\vec{F} = -\nabla\varphi + \nabla \times \vec{A} \tag{26}$$

in which $R = |\vec{r} - \vec{r}\,'|$, $\hat{n}_i$ is an exterior unit normal vector of closed surface S_i.

Proof: Condition (2) implies that the geometry which is consisted of the discontinuous points in every interface is at most curve line. Consequently area measure of every surface S_{ij} in (19) is equal to zero and surface integral on every interface S_{ij} is also zero.

So,

$$\begin{aligned}
\vec{F}(\vec{r}) &= -\nabla\left[\int_V \frac{\nabla' \cdot \vec{F}(\vec{r}\,')}{4\pi R} dV' - \sum_{i\in I} \oint_{S_i} \frac{\vec{F}(\vec{r}\,') \cdot \hat{n}_i}{4\pi R} dS_i'\right] \\
&\quad + \nabla \times \left[\int_V \frac{\nabla' \times \vec{F}(\vec{r}\,')}{4\pi R} dV' + \sum_{i\in I} \oint_{S_i} \frac{\vec{F}(\vec{r}\,') \times \hat{n}_i}{4\pi R} dS_i'\right] \\
&= -\nabla\left[\int_V \frac{\nabla' \cdot \vec{F}(\vec{r}\,')}{4\pi R} dV' - \oint_S \frac{\vec{F}(\vec{r}\,') \cdot \hat{n}}{4\pi R} dS'\right] \\
&\quad + \nabla \times \left[\int_V \frac{\nabla' \times \vec{F}(\vec{r}\,')}{4\pi R} dV' + \oint_S \frac{\vec{F}(\vec{r}\,') \times \hat{n}}{4\pi R} dS'\right]
\end{aligned} \tag{27}$$

This completes the proof.

Remark on Corollary 1

From condition (2) we can infer that $\vec{F}(\vec{r})$ is almost everywhere continuous and bounded in boundary S. Therefore Theorem 1 is a particular case of Corollary 1, where S is outside surface $S_i (i = 1)$, that is, the case of without interface.

Theorem 4. Generalized Helmholtz theorem in unbounded domain

Let $V = R^3$. A single-valued vector function of several variables $\vec{F}(\vec{r})$, which is defined in domain V ($\vec{r} \in \overline{V}$), has at most countable discontinuity points or point sets of the first kind. Based on the piecewise continuous vector function $\vec{F}(\vec{r})$, V can be partitioned into the form $V = (\bigcup_{i\in I} V_i) \cup (\bigcup_{j\in J} V_j)$, where V_i ($\forall i \in I$) is a bounded simply connect domain and V_j ($\forall j \in J$) is a unbounded simply connect domain, both I and J are at most counted index set, $\vec{F}(\vec{r})$ is a continuous function in every simply connected domain V_i and V_j,

meanwhile, $l \in I \cup J$, $m \in I \cup J$, $V_l \cap V_m = \emptyset$ and as closed surface of domain V_i, S_i ($\forall i \in I$) is at least piecewise smooth. S_j ($\forall j \in J$) is asymptotic closed surface of domain V_j, If $\vec{F}(\vec{r})$ satisfies the following conditions:

(1) $\forall l \in I \cup J$, $\vec{r} \in V_l$, $\vec{F}(\vec{r}) \in C^1$,
(2) $\vec{F}(\vec{r})$ is bounded in every surface S_i ($\forall i \in I$), meanwhile, both $\frac{\vec{F}(\vec{r}')\cdot\hat{n}_i}{4\pi R}$ and $\frac{\vec{F}(\vec{r}')\times\hat{n}_i}{4\pi R}$ in every surface S_i are measurable functions,
(3) $\vec{F}(\vec{r})$ is vanishing at infinitive at enough quick rate and make its surface integration approach zero,

then $\vec{F}(\vec{r})$ must be able to be completely and uniquely decomposed into the sum of an irrotational field and a solenoidal field. Decomposition form is written as follow:

$$\vec{F}(\vec{r}) = -\nabla\left[\int_V \frac{\nabla'\cdot\vec{F}(\vec{r}')}{4\pi R}dV' - \sum_{i,j\in I\cup J}\int_{S_{ij}} \frac{(\vec{F}_i(\vec{r}')-\vec{F}_j(\vec{r}'))\cdot\hat{n}_{ij}}{4\pi R}dS'_{ij}\right] + \nabla\times\left[\int_V \frac{\nabla'\times\vec{F}(\vec{r}')}{4\pi R}dV' + \sum_{i,j\in I\cup J}\int_{S_{ij}} \frac{(\vec{F}_i(\vec{r}')-\vec{F}_j(\vec{r}'))\times\hat{n}_{ij}}{4\pi R}dS'_{ij}\right] \quad (28)$$

in which $R = |\vec{r} - \vec{r}'|$, both $\vec{F}_i$ and $\vec{F}_j$ represent vector functions defined on surface S_{ij} ($S_{ij} = S_i \cap S_j$), meanwhile, they are located in two sides of S_{ij}, $\widehat{n_{ij}}$ is the unit normal vector of closed surface S_{ij} and directed to V_j from V_i. Summation in (28) is a kind of unilateral sum, that is, unrepeated sum in interfaces.

Proof: From Theorem 3, we know that the condition (2) can ensure that classic Green theorem and Stokes formula can be applied to the case of infinite domain without any amendment. More importantly, condition (3) makes the surface integral on S is vanished. So (28) is obtained from (19). This completes the proof.

4. CONCLUSION

Theorem 1 can be derived from Theorem 3 when the number of piecewise continuous function is only single, at the same time, Theorem 2 is also a particular case of Theorem 4 when the area measure of discontinuous points and point sets is equal to zero.

Theorem 1 and Theorem 3 is for the case of bounded domain, meanwhile, Theorem 2 and Theorem 4 is for the case of unbounded domain. From derivation we can obtain the uniqueness

of decomposition on the case of unbounded domain, however, for the case of bounded domain there is no similar conclusion because the decomposition terms are not mutual orthogonal. There are some important problems needed to be considered by us for the future such as how to realize the uniqueness and completeness of decomposition on the case of bounded domain, how to build the difference and connection between Helmholtz theorem and deterministic theorems of vector field, how to strictly define irrotational field and solenoidal field and how to ascertain the uniqueness of potential function.

In this paper we not only present a precise statement and rigorous proof on classic Helmholtz theorem, but also generalize it to the case of multiply connected domain and obtain a generalized Helmholtz theorem in sense of Lebesgue measure and Lebesgue integral defined on three-dimensional Euclidean space. Meanwhile, our proof and reasoning are more sufficient and perfect.

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