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INVERSE FOR THE SKEWON MEDIUM DYADIC

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Abstract—In four-dimensional differential-form representation linear medium relations can be expressed in terms of a medium dyadic mapping the electromagnetic two-form involving the $\boldsymbol{B}$ and $\boldsymbol{E}$ fields to the two-form involving the $\boldsymbol{D}$ and $\boldsymbol{H}$ fields. There does not seem to exist a method to invert the medium dyadic in a coordinate-free manner for the general bi-anisotropic medium. Such an inversion is introduced here for the special class of skewon media which is a 15 parameter subclass of previously studied IB media. The resulting compact analytic expression is verified through two simple tests and an expansion in eigenvectors.

1. INTRODUCTION

Differential-form formalism offers a coordinate-free alternative for the analysis of electromagnetic problems[1–7]. When compared to the classical three-dimensional Gibbsian vector and dyadic formalism, four-dimensional differential forms allow a more compact representation of expressions. In particular, when working with general linear (bi-anisotropic) media, the medium parameters can be represented in terms of bivector dyadics in simple form, see, e.g., [8, 9]. The simplification is based on an extension of the three-dimensional dyadic algebra introduced by Gibbs [10, 11] to four dimensions [6]. A short introduction to the notation applied here is also given in [7, 12]

In differential-form representation the electromagnetic fields are represented as two-forms $\boldsymbol{\Phi}, \boldsymbol{\Psi}$, elements of the six-dimensional space $\mathbb{F}_2$,

$$\boldsymbol{\Phi} = \boldsymbol{B} + \boldsymbol{E} \wedge \boldsymbol{d}\tau, \qquad \boldsymbol{\Psi} = \boldsymbol{D} - \boldsymbol{H} \wedge \boldsymbol{d}\tau, \tag{1}$$

where $\tau = ct$ is the normalized time variable. In a linear (bi-anisotropic) medium the electromagnetic two-forms are related through the medium dyadic $\overline{\overline{\mathsf{M}}}$ mapping two-forms to two-forms:

$$\mathbf{\Psi} = \overline{\overline{\mathsf{M}}}|\mathbf{\Phi}. \tag{2}$$

The dyadic $\overline{\overline{\mathsf{M}}}$ corresponds to a collection of four Gibbsian medium dyadics $\overline{\overline{\epsilon}}, \overline{\overline{\xi}}, \overline{\overline{\zeta}}, \overline{\overline{\mu}}$. The simplest possible medium is represented as $\overline{\overline{\mathsf{M}}} = M\overline{\overline{\mathsf{I}}}^{(2)T}$ where $\overline{\overline{\mathsf{I}}}^{(2)T}$ is the unit dyadic mapping two-forms to themselves and M is a scalar admittance factor. Properties of such a medium were recently studied and the medium was labeled as PEMC, perfect electromagnetic conductor because of it is a generalization of PEC and PMC [13, 14].

A useful classification of bi-anisotropic media can be made when instead of the medium dyadic $\overline{\overline{\mathsf{M}}}$ we consider the modified medium dyadic $\overline{\overline{\mathsf{M}}}_g$ defined by the transformation through a quadrivector $\boldsymbol{e}_N = \boldsymbol{e}_{1234}$ as

$$\overline{\overline{\mathsf{M}}}_g = \boldsymbol{e}_N\lfloor\overline{\overline{\mathsf{M}}}. \tag{3}$$

$\overline{\overline{\mathsf{M}}}_g$ is an element of the dyadic space $\mathbb{E}_2\mathbb{E}_2$ and it maps two-forms to bivectors [6]. In fact, a decomposition of $\overline{\overline{\mathsf{M}}}_g$ in three parts was defined by Hehl and Obukhov in [5] as

$$\overline{\overline{\mathsf{M}}}_g = \overline{\overline{\mathsf{M}}}_{g0} + \overline{\overline{\mathsf{M}}}_{g1} + \overline{\overline{\mathsf{M}}}_{g2}. \tag{4}$$

Here $\overline{\overline{\mathsf{M}}}_{g0} + \overline{\overline{\mathsf{M}}}_{g1}$ equals the symmetric part and $\overline{\overline{\mathsf{M}}}_{g2}$ the antisymmetric part of $\overline{\overline{\mathsf{M}}}_g$. The transformed multiple of the unit dyadic

$$\overline{\overline{\mathsf{M}}}_{g0} = \frac{1}{6}\mathrm{tr}\overline{\overline{\mathsf{M}}}\ \boldsymbol{e}_N\lfloor\overline{\overline{\mathsf{I}}}^{(2)T} \tag{5}$$

was called the axion part while

$$\overline{\overline{\mathsf{M}}}_{g1} = \frac{1}{2}(\overline{\overline{\mathsf{M}}}_g + \overline{\overline{\mathsf{M}}}_g^T) - \overline{\overline{\mathsf{M}}}_{g0} \tag{6}$$

was called the principal part. The corresponding $\overline{\overline{\mathsf{M}}}_1$ dyadic is trace free. Finally, the antisymmetric part

$$\overline{\overline{\mathsf{M}}}_{g2} = \frac{1}{2}(\overline{\overline{\mathsf{M}}}_g - \overline{\overline{\mathsf{M}}}_g^T) \tag{7}$$

was called the skewon part. Physically, the skewon part is responsible to the Faraday-rotation and chiral properties of the medium. For

$\overline{\overline{\mathsf{M}}}_{g1} = 0$ and $\overline{\overline{\mathsf{M}}}_{g2} = 0$ the medium consists of its bare axion part and coincides with the PEMC medium.

To be able to perform all possible operations with the medium dyadic in terms of the four-dimensional dyadic algebra, an analytic expression for the inverse of the medium dyadic is required. The inverse allows one to write the medium equation (2) as

$$\mathbf{\Phi} = \overline{\overline{\mathsf{M}}}^{-1}|\mathbf{\Psi}. \tag{8}$$

It is known that the inverse of a dyadic $\overline{\overline{\mathsf{D}}} \in \mathbb{E}_1\mathbb{F}_1$ mapping one-forms to one-forms has the analytic form [6]

$$\overline{\overline{\mathsf{D}}}^{-1} = \frac{\bar{\bar{\mathsf{I}}}^{(4)T}\lfloor\lfloor\overline{\overline{\mathsf{D}}}^{(3)T}}{\mathrm{tr}\overline{\overline{\mathsf{D}}}^{(4)}}. \tag{9}$$

Unfortunately, it appears that there does not exist an analytic coordinate-free expression for the inverse of the general medium dyadic mapping two-forms to two-forms. In the previously obtained results the temporal coordinate has been separated and the inverse is expressed in terms of the four three-dimensional medium dyadics [6]. In the present paper a first step towards the general result is taken, and the inverse is derived for the skewon medium characterized by 15 scalar parameters.

2. IB-MEDIA

A medium with vanishing principal part, i.e., consisting of the axion and skewon parts only, was labeled as an IB-medium in another approach [15]. It was shown that the corresponding medium dyadic involving 16 parameters can can be represented as

$$\overline{\overline{\mathsf{M}}} = (\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})^T \tag{10}$$

in terms of a dyadic $\overline{\overline{\mathsf{B}}} \in \mathbb{E}_1\mathbb{F}_1$. Because of the form (10), the class was labeled as that of IB media. Instead of having $6 \times 6 = 36$ free parameters corresponding to the general dyadic $\overline{\overline{\mathsf{M}}}$, IB media are defined by a dyadic $\overline{\overline{\mathsf{B}}}$ possessing only $4 \times 4 = 16$ free parameters. Expressing

$$\overline{\overline{\mathsf{B}}} = B\bar{\bar{\mathsf{I}}} + \overline{\overline{\mathsf{B}}}_o, \tag{11}$$

where $\overline{\overline{\mathsf{B}}}_o$ is the trace-free part of $\overline{\overline{\mathsf{B}}}$:

$$\mathrm{tr}\overline{\overline{\mathsf{B}}}_o = \overline{\overline{\mathsf{B}}}_o||\bar{\bar{\mathsf{I}}}^T = 0, \qquad \mathrm{tr}\overline{\overline{\mathsf{B}}} = 4B, \tag{12}$$

the IB-medium dyadics can be expanded as

$$\overline{\overline{\mathsf{M}}} = M\bar{\bar{\mathsf{I}}}^{(2)T} + \overline{\overline{\mathsf{M}}}_o, \qquad M = 2B, \tag{13}$$

where $\overline{\overline{\mathsf{M}}}_o$ is a trace-free dyadic satisfying

$$\overline{\overline{\mathsf{M}}}_o = (\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^T, \qquad \mathrm{tr}\overline{\overline{\mathsf{M}}}_o = 0. \tag{14}$$

The class of IB media is split in (13) to two parts corresponding to those called axion (multiple of the unit dyadic) and skewon (trace-free part) media in [5].

Dyadics $\overline{\overline{\mathsf{M}}}$ of the form (10) satisfy the identity (A4) in the Appendix, which for a trace-free dyadic $\overline{\overline{\mathsf{A}}} = \overline{\overline{\mathsf{B}}}_o$ reduces to the particularly simple form

$$\bar{\bar{\mathsf{I}}}^{(4)}\lfloor\lfloor(\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^T = -\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o \quad \Rightarrow \quad \bar{\bar{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{M}}}_o = -\overline{\overline{\mathsf{M}}}_o^T. \tag{15}$$

Considering the modified dyadic of the IB medium,

$$\overline{\overline{\mathsf{M}}}_g = \boldsymbol{e}_N\lfloor\overline{\overline{\mathsf{M}}} = \boldsymbol{e}_N\lfloor(\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})^T \in \mathbb{E}_2\mathbb{E}_2, \tag{16}$$

one can easily show that for $\overline{\overline{\mathsf{B}}} = \overline{\overline{\mathsf{B}}}_o$ it is antisymmetric. In fact, (15) can be rewritten as

$$\boldsymbol{e}_N\lfloor\overline{\overline{\mathsf{M}}}_o = -\overline{\overline{\mathsf{M}}}_o^T\rfloor\boldsymbol{e}_N = -(\boldsymbol{e}_N\lfloor\overline{\overline{\mathsf{M}}}_o)^T \quad \Rightarrow \quad \overline{\overline{\mathsf{M}}}_{go} = -\overline{\overline{\mathsf{M}}}_{go}^T. \tag{17}$$

Conversely, any antisymmetric dyadic $\overline{\overline{\mathsf{D}}} \in \mathbb{E}_2\mathbb{E}_2$ can be expressed in the form $\overline{\overline{\mathsf{D}}} = \boldsymbol{e}_N\lfloor(\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^T$ for some trace-free dyadic $\overline{\overline{\mathsf{B}}}_o \in \mathbb{E}_1\mathbb{F}_1$.

It is the purpose of this paper to find the inverse of the dyadic $\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o$ satisfying

$$(\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)|(\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^{-1} = (\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^{-1}|(\bar{\bar{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o) = \bar{\bar{\mathsf{I}}}^{(2)}. \tag{18}$$

The derivation is based on a set of dyadic identities, a list of which is given in the Appendix.

3. CONSTRUCTING THE INVERSE

Let us start from the identity (A6) which for the trace-free dyadic $\overline{\overline{\mathsf{A}}} = \overline{\overline{\mathsf{B}}}_o$ reduces to

$$\overline{\overline{\mathsf{B}}}_o^{(3)}\lfloor\lfloor\bar{\bar{\mathsf{I}}}^T = -\overline{\overline{\mathsf{B}}}_o{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o^2. \tag{19}$$

Combination of (A13) and (A14) for $\overline{\overline{\mathsf{X}}} = \overline{\overline{\mathsf{B}}}_o^{(3)}$ can be expressed as

$$\overline{\overline{\mathsf{B}}}_o^{(3)} = \overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor(\overline{\overline{\mathsf{I}}}^{(4)T}\lfloor\lfloor\overline{\overline{\mathsf{B}}}_o^{(3)}) = \overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor(\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)}\,\overline{\overline{\mathsf{I}}} - \overline{\overline{\mathsf{B}}}_o^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)T})^T, \tag{20}$$

which inserted in (19) yields

$$\begin{aligned}\overline{\overline{\mathsf{B}}}_o{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o^2 &= -(\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor(\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)}\,\overline{\overline{\mathsf{I}}}^T - \overline{\overline{\mathsf{B}}}_o^{(3)T}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)}))\lfloor\lfloor\overline{\overline{\mathsf{I}}} \\ &= -2\overline{\overline{\mathsf{I}}}^{(2)}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)} + \overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor((\overline{\overline{\mathsf{B}}}_o^{(3)T}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)}){}^\wedge_\wedge\overline{\overline{\mathsf{I}}}^T).\end{aligned} \tag{21}$$

Applying now the expansion (A4), (21) can be expressed as

$$\overline{\overline{\mathsf{B}}}_o{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o^2 = -2\overline{\overline{\mathsf{I}}}^{(2)}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)} + \mathrm{tr}(\overline{\overline{\mathsf{B}}}_o^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)T})\overline{\overline{\mathsf{I}}}^{(2)} - (\overline{\overline{\mathsf{B}}}_o^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)T}){}^\wedge_\wedge\overline{\overline{\mathsf{I}}}. \tag{22}$$

Here we must insert from (A5), (A6) and (A2)

$$\begin{aligned}\overline{\overline{\mathsf{B}}}_o^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)T} &= \frac{1}{2}(\overline{\overline{\mathsf{B}}}_o^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T)\lfloor\lfloor\overline{\overline{\mathsf{I}}} = -\frac{1}{2}(\overline{\overline{\mathsf{B}}}_o{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o^2)\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T \\ &= -\frac{1}{2}(\overline{\overline{\mathsf{B}}}_o\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^2 - 2\overline{\overline{\mathsf{B}}}_o^3) = \overline{\overline{\mathsf{B}}}_o\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)} + \overline{\overline{\mathsf{B}}}_o^3\end{aligned} \tag{23}$$

with

$$\mathrm{tr}(\overline{\overline{\mathsf{B}}}_o^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)T}) = \mathrm{tr}\overline{\overline{\mathsf{B}}}_o^3. \tag{24}$$

Combining these we have

$$\overline{\overline{\mathsf{B}}}_o{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o^2 = -2\overline{\overline{\mathsf{I}}}^{(2)}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)} + \mathrm{tr}\overline{\overline{\mathsf{B}}}_o^3\overline{\overline{\mathsf{I}}}^{(2)} - (\overline{\overline{\mathsf{B}}}_o\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)} + \overline{\overline{\mathsf{B}}}_o^3){}^\wedge_\wedge\overline{\overline{\mathsf{I}}}. \tag{25}$$

Finally we make use of the expansion

$$(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)|(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}^2) = \overline{\overline{\mathsf{B}}}_o{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o^2 + \overline{\overline{\mathsf{B}}}_o^3{}^\wedge_\wedge\overline{\overline{\mathsf{I}}}, \tag{26}$$

which is a special case of the more general identity (A1). With this (25) is reduced to

$$\begin{aligned}(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)|(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o^2) &= \overline{\overline{\mathsf{B}}}_o^3{}^\wedge_\wedge\overline{\overline{\mathsf{I}}} - 2\overline{\overline{\mathsf{I}}}^{(2)}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)} + \mathrm{tr}\overline{\overline{\mathsf{B}}}_o^3\overline{\overline{\mathsf{I}}}^{(2)} - (\overline{\overline{\mathsf{B}}}_o\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)} + \overline{\overline{\mathsf{B}}}_o^3){}^\wedge_\wedge\overline{\overline{\mathsf{I}}} \\ &= -2\overline{\overline{\mathsf{I}}}^{(2)}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)} + \mathrm{tr}\overline{\overline{\mathsf{B}}}_o^3\overline{\overline{\mathsf{I}}}^{(2)} - \mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)}\,\overline{\overline{\mathsf{B}}}_o{}^\wedge_\wedge\overline{\overline{\mathsf{I}}} \\ &= \overline{\overline{\mathsf{I}}}^{(2)}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)} - \mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)}\,\overline{\overline{\mathsf{B}}}_o{}^\wedge_\wedge\overline{\overline{\mathsf{I}}}.\end{aligned} \tag{27}$$

Reformulating this as

$$(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)|(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o^2 + \mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)}\,\overline{\overline{\mathsf{I}}}^{(2)}) = \overline{\overline{\mathsf{I}}}^{(2)}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)}, \tag{28}$$

and assuming $\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)} \neq 0$ we can identify the expression for the inverse as

$$(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^{-1} = \frac{\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o^2 + \mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)}\,\overline{\overline{\mathsf{I}}}^{(2)}}{\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)}}. \tag{29}$$

This is the main result of the present paper. The form of (29) suggests that the inverse can be written as

$$(\overline{\mathsf{I}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o)^{-1} = \overline{\mathsf{I}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}}_o \tag{30}$$

where $\overline{\overline{\mathsf{A}}}_o$ is a trace-free dyadic:

$$\overline{\overline{\mathsf{A}}}_o = \frac{\overline{\overline{\mathsf{B}}}_o^2 + \frac{1}{2}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)}\,\overline{\mathsf{I}}}{\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)}} = \frac{\overline{\overline{\mathsf{B}}}_o^2 - \frac{1}{4}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^2\,\overline{\mathsf{I}}}{\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)}}, \tag{31}$$

$$\mathrm{tr}\overline{\overline{\mathsf{A}}}_o = 0. \tag{32}$$

4. CHECKING THE RESULT

To obtain confidence in the inverse formula (29), let us make two verifying checks. Because from (30) we must have

$$(\overline{\mathsf{I}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}}_o)^{-1} = \overline{\mathsf{I}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o, \tag{33}$$

let us check this for inner consistency. (33) is equivalent to requiring that the expression (31) be valid with $\overline{\overline{\mathsf{B}}}_o$ and $\overline{\overline{\mathsf{A}}}_o$ interchanged:

$$\overline{\overline{\mathsf{B}}}_o = \frac{\overline{\overline{\mathsf{A}}}_o^2 - \frac{1}{4}\mathrm{tr}\overline{\overline{\mathsf{A}}}_o^2\,\overline{\mathsf{I}}}{\mathrm{tr}\overline{\overline{\mathsf{A}}}_o^{(3)}}. \tag{34}$$

Substituting $\overline{\overline{\mathsf{A}}}_o$ in terms of $\overline{\overline{\mathsf{B}}}_o$ in the right side of (34) yields

$$\frac{\overline{\overline{\mathsf{A}}}_o^2 - \frac{1}{4}\mathrm{tr}\overline{\overline{\mathsf{A}}}_o^2\,\overline{\mathsf{I}}}{\mathrm{tr}\overline{\overline{\mathsf{A}}}_o^{(3)}} = \mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)}\left(\frac{(\overline{\overline{\mathsf{B}}}_o^2 - \frac{1}{4}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^2\overline{\mathsf{I}})^2 - \frac{1}{4}\mathrm{tr}(\overline{\overline{\mathsf{B}}}_o^2 - \frac{1}{4}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^2\overline{\mathsf{I}})^2\overline{\mathsf{I}}}{\mathrm{tr}(\overline{\overline{\mathsf{B}}}_o^2 - \frac{1}{4}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^2\overline{\mathsf{I}})^{(3)}}\right). \tag{35}$$

In expanding the right-hand side we invoke the Cayley-Hamilton equation (A15) for $\overline{\overline{\mathsf{B}}}_o$ and (A8), (A7). After some algebraic steps we find

$$\mathrm{tr}(\overline{\overline{\mathsf{B}}}_o^2 - \frac{1}{4}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^2\overline{\mathsf{I}})^{(3)} = (\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)})^2, \tag{36}$$

and

$$(\overline{\overline{\mathsf{B}}}_o^2 - \frac{1}{4}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^2\overline{\mathsf{I}})^2 - \frac{1}{4}\mathrm{tr}(\overline{\overline{\mathsf{B}}}_o^2 - \frac{1}{4}\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^2\overline{\mathsf{I}})^2\overline{\mathsf{I}} = \overline{\overline{\mathsf{B}}}_o\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)}, \tag{37}$$

whence the right-hand side of (35) reduces to $\overline{\overline{\mathsf{B}}}_o$ as required.

As a second check let us substitute in (29) the simple trace-free dyadic

$$\overline{\overline{\mathsf{B}}}_o = \frac{B}{2}(\boldsymbol{e}_1\boldsymbol{\varepsilon}_1 + \boldsymbol{e}_2\boldsymbol{\varepsilon}_2 + \boldsymbol{e}_3\boldsymbol{\varepsilon}_3 - 3\boldsymbol{e}_4\boldsymbol{\varepsilon}_4) \tag{38}$$

satisfying

$$\overline{\overline{\mathsf{B}}}_o^2 = \frac{B^2}{4}(\boldsymbol{e}_1\boldsymbol{\varepsilon}_1 + \boldsymbol{e}_2\boldsymbol{\varepsilon}_2 + \boldsymbol{e}_3\boldsymbol{\varepsilon}_3 + 9\boldsymbol{e}_4\boldsymbol{\varepsilon}_4) = \frac{B\overline{\overline{\mathsf{B}}}_o}{2} + 3B^2\boldsymbol{e}_4\boldsymbol{\varepsilon}_4, \tag{39}$$

$$\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)} = -\frac{3B^2}{2}, \qquad \mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)} = -B^3. \tag{40}$$

After some steps we obtain from (31)

$$\overline{\overline{\mathsf{A}}}_o = -\frac{1}{B^3}(\overline{\overline{\mathsf{B}}}_o^2 - \frac{3B^2}{4}\overline{\overline{\mathsf{I}}}) = \frac{1}{2B}(\boldsymbol{e}_1\boldsymbol{\varepsilon}_1 + \boldsymbol{e}_2\boldsymbol{\varepsilon}_2 + \boldsymbol{e}_3\boldsymbol{\varepsilon}_3 - 3\boldsymbol{e}_4\boldsymbol{\varepsilon}_4) = \frac{\overline{\overline{\mathsf{B}}}_o}{B^2}. \tag{41}$$

To verify (29) we expand

$$(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)|(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}_o) = \frac{1}{B^2}(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^2$$

$$= \frac{1}{4}((\boldsymbol{e}_1\boldsymbol{\varepsilon}_1 + \boldsymbol{e}_2\boldsymbol{\varepsilon}_2 + \boldsymbol{e}_3\boldsymbol{\varepsilon}_3 - 3\boldsymbol{e}_4\boldsymbol{\varepsilon}_4){}^\wedge_\wedge\sum\boldsymbol{e}_i\boldsymbol{\varepsilon}_i)^2 = \sum_{i<j}\boldsymbol{e}_{ij}\boldsymbol{\varepsilon}_{ij} = \overline{\overline{\mathsf{I}}}^{(2)}. \tag{42}$$

5. VERIFYING IN EIGENEXPANSIONS

In principle, the inverse expression could also be derived by expanding the medium dyadic applying suitable vector bases. However, to compile the result in coordinate-independent form is not a straightforward task. It is instructive to verify the derived result (29) by using eigenexpansions. Let us assume that the right eigenvectors of the dyadic $\overline{\overline{\mathsf{B}}}_o$ are $\boldsymbol{e}_i$ and that they form a four-dimensional basis. The corresponding reciprocal basis of one-forms is denoted by $\boldsymbol{\varepsilon}_i$. Denoting the eigenvalues by B_i we can write the eigenexpansions [15]

$$\overline{\overline{\mathsf{B}}}_o = \sum_{i=1}^{4} B_i\boldsymbol{e}_i\boldsymbol{\varepsilon}_i, \tag{43}$$

$$\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o = \sum_{i<j}(B_i + B_j)\boldsymbol{e}_{ij}\boldsymbol{\varepsilon}_{ij}. \tag{44}$$

Because of the trace-free property we have

$$B_4 = -(B_1 + B_2 + B_3) \tag{45}$$

whence

$$\bar{\bar{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o = (B_1+B_2)(\boldsymbol{e}_{12}\boldsymbol{\varepsilon}_{12} - \boldsymbol{e}_{34}\boldsymbol{\varepsilon}_{34})$$

$$+(B_2+B_3)(\boldsymbol{e}_{23}\boldsymbol{\varepsilon}_{23} - \boldsymbol{e}_{14}\boldsymbol{\varepsilon}_{14}) + (B_3+B_1)(\boldsymbol{e}_{31}\boldsymbol{\varepsilon}_{31} - \boldsymbol{e}_{24}\boldsymbol{\varepsilon}_{24}). \quad (46)$$

The inverse of this is quite simply

$$(\bar{\bar{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o)^{-1} = (B_1+B_2)^{-1}(\boldsymbol{e}_{12}\boldsymbol{\varepsilon}_{12} - \boldsymbol{e}_{34}\boldsymbol{\varepsilon}_{34})$$

$$+(B_2+B_3)^{-1}(\boldsymbol{e}_{23}\boldsymbol{\varepsilon}_{23} - \boldsymbol{e}_{14}\boldsymbol{\varepsilon}_{14}) + (B_3+B_1)^{-1}(\boldsymbol{e}_{31}\boldsymbol{\varepsilon}_{31} - \boldsymbol{e}_{24}\boldsymbol{\varepsilon}_{24}). \quad (47)$$

Now let us try to verify the right-hand side of (29) by finding its eigenexpansion. Applying (45) the two scalar invariants have the expansions

$$\begin{aligned}
\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(3)} &= B_1B_2B_3 + B_1B_2B_4 + B_1B_3B_4 + B_2B_3B_4 \\
&= -(B_1+B_2)(B_2+B_3)(B_3+B_1), \qquad (48)\\
\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)} &= B_1B_2 + B_2B_3 + B_3B_1 + B_1B_4 + B_2B_4 + B_3B_4 \\
&= -(B_1^2 + B_2^2 + B_3^2 + B_1B_2 + B_2B_3 + B_3B_1) \\
&= -(B_3+B_1)(B_1+B_2) - B_2^2 - B_3^2. \qquad (49)
\end{aligned}$$

The last expression can be rewritten as

$$B_2^2 + B_3^2 = -\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)} - (B_3+B_1)(B_1+B_2) \qquad (50)$$

and it is equally valid for all permutations of the indices. Starting now from

$$\overline{\overline{\mathsf{B}}}_o{}^2 = \sum_{i=1}^4 B_i^2 \boldsymbol{e}_i\boldsymbol{\varepsilon}_i, \qquad (51)$$

we can write in analogy with (44)

$$\bar{\bar{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o{}^2 = \sum_{i<j}(B_i^2 + B_j^2)\boldsymbol{e}_{ij}\boldsymbol{\varepsilon}_{ij}. \qquad (52)$$

This can be expanded as

$$\bar{\bar{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o{}^2 = (B_1^2+B_2^2)\boldsymbol{e}_{12}\boldsymbol{\varepsilon}_{12} + (B_2^2+B_3^2)\boldsymbol{e}_{23}\boldsymbol{\varepsilon}_{23} + (B_3^2+B_1^2)\boldsymbol{e}_{31}\boldsymbol{\varepsilon}_{31}$$

$$+\sum_{i=1}^3 (B_i^2 + (B_1+B_2+B_3)^2)\boldsymbol{e}_{i4}\boldsymbol{\varepsilon}_{i4}. \qquad (53)$$

Inserting

$$B_1^2 + (B_1+B_2+B_3)^2 = 2(B_1+B_2)(B_1+B_3) + B_2^2 + B_3^3 \qquad (54)$$

and similarly permutating the indices we have

$$\begin{aligned}\overline{\overline{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o{}^2 = & (B_1^2 + B_2^2)(\boldsymbol{e}_{12}\boldsymbol{\varepsilon}_{12} + \boldsymbol{e}_{34}\boldsymbol{\varepsilon}_{34}) + (B_2^2 + B_3^2)(\boldsymbol{e}_{23}\boldsymbol{\varepsilon}_{23} + \boldsymbol{e}_{14}\boldsymbol{\varepsilon}_{14}) \\ & + (B_3^2 + B_1^2)(\boldsymbol{e}_{31}\boldsymbol{\varepsilon}_{31} + \boldsymbol{e}_{24}\boldsymbol{\varepsilon}_{24}) + 2(B_3 + B_1)(B_1 + B_2)\boldsymbol{e}_{14}\boldsymbol{\varepsilon}_{14} \\ & + 2(B_1 + B_2)(B_2 + B_3)\boldsymbol{e}_{24}\boldsymbol{\varepsilon}_{24} + 2(B_2 + B_3)(B_3 + B_1)\boldsymbol{e}_{34}\boldsymbol{\varepsilon}_{34}.\end{aligned} \tag{55}$$

Inserting (50) we can finally construct the expansion

$$\begin{aligned}\overline{\overline{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o{}^2 = & -\mathrm{tr}\overline{\overline{\mathsf{B}}}_o^{(2)}\overline{\overline{\mathsf{I}}}^{(2)} - (B_3 + B_1)(B_2 + B_3)(\boldsymbol{e}_{12}\boldsymbol{\varepsilon}_{12} - \boldsymbol{e}_{34}\boldsymbol{\varepsilon}_{34}) \\ & -(B_1 + B_2)(B_3 + B_1)(\boldsymbol{e}_{23}\boldsymbol{\varepsilon}_{23} - \boldsymbol{e}_{14}\boldsymbol{\varepsilon}_{14}) \\ & -(B_1 + B_2)(B_2 + B_3)(\boldsymbol{e}_{31}\boldsymbol{\varepsilon}_{31} - \boldsymbol{e}_{24}\boldsymbol{\varepsilon}_{24}),\end{aligned} \tag{56}$$

which together with (48) will finally verify (47). It is obvious that arriving at (29) through this route would require some knowledge of the form of the final result.

6. DISCUSSION

Applying dyadic algebra to electromagnetic analysis requires a toolbox of operational rules or dyadic identities, an example of which is given in the Appendix A of [6] (its upgraded version is available through http://www.ismolindell.com/publications/monographs/pdf/iden.pdf). One of the main shortcomings has been the lack of an analytic coordinate-free expression for the inverse of a dyadic mapping bivectors to bivectors or two-forms to two-forms or their modified counterparts. There exist formulas for some special cases like medium dyadics of the co-called Q-media [8], i.e., dyadics of the form $\overline{\overline{\mathsf{Q}}}^{(2)}$, or medium dyadics of self-dual media [16] which satisfy an algebraic equation of the second order.

In the present paper a step forward in this direction is taken by introducing the inverse for dyadics of the form $\overline{\overline{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o$ with $\mathrm{tr}\overline{\overline{\mathsf{B}}}_o = 0$, which when transposed corresponds to the medium dyadic for the class of skewon media. It does not appear very straightforward to extend the result (29) to the class of general IB-media, i.e., for dyadics $\overline{\overline{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}$ with $\mathrm{tr}\overline{\overline{\mathsf{B}}} \neq 0$. However, (29) can be generalized to another extended class of dyadics $\overline{\overline{\mathsf{D}}} \in \mathbb{E}_2\mathbb{F}_2$ defined by

$$\overline{\overline{\mathsf{D}}} = \overline{\overline{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o + \boldsymbol{A}\boldsymbol{\Pi}, \qquad \mathrm{tr}\overline{\overline{\mathsf{B}}}_o = 0, \tag{57}$$

where $\boldsymbol{A} \in \mathbb{E}_2$ is a bivector and $\boldsymbol{\Pi} \in \mathbb{F}_2$ a two-form. In fact, the expression for the inverse can be directly written by applying the rule

[6]

$$\overline{\overline{\mathsf{D}}}^{-1} = (\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^{-1} - \frac{(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^{-1}|\boldsymbol{A}\boldsymbol{\Pi}|(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^{-1}}{1-\mathbf{\Pi}|(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o)^{-1}|\boldsymbol{A}}. \tag{58}$$

Medium dyadics defined by $\overline{\overline{\mathsf{M}}} = \overline{\overline{\mathsf{D}}}^T$ have nonzero principal and axion parts in the general case.

APPENDIX A. IDENTITIES

In this section we list some dyadic identities required in the analysis without derivation. Rules for their derivation can be found in [6]. The dyadics $\overline{\overline{\mathsf{A}}}, \overline{\overline{\mathsf{B}}}, \cdots$ are elements of the space $\mathbb{E}_1\mathbb{F}_1$, i.e., dyadics mapping vectors to vectors, unless otherwise specified. The dimension of the basic vector space is 4 so that $\mathrm{tr}\overline{\overline{\mathsf{I}}} = 4$ and $\mathrm{tr}\overline{\overline{\mathsf{I}}}^{(2)} = 6$.

$$(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})|(\overline{\overline{\mathsf{C}}}{}^\wedge_\wedge\overline{\overline{\mathsf{D}}}) = (\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{C}}}){}^\wedge_\wedge(\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{D}}}) + (\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{D}}}){}^\wedge_\wedge(\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{C}}}) \tag{A1}$$

$$(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})\lfloor\lfloor\overline{\overline{\mathsf{C}}}^T = (\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{C}}}^T)\overline{\overline{\mathsf{B}}} + (\overline{\overline{\mathsf{B}}}||\overline{\overline{\mathsf{C}}}^T)\overline{\overline{\mathsf{A}}} - \overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{C}}}|\overline{\overline{\mathsf{B}}} - \overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{C}}}|\overline{\overline{\mathsf{A}}} \tag{A2}$$

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})^T = \mathrm{tr}(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})\,\overline{\overline{\mathsf{I}}}^{(2)} - ((\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T){}^\wedge_\wedge\overline{\overline{\mathsf{I}}} + \overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}} \tag{A3}$$

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}})^T = \mathrm{tr}\overline{\overline{\mathsf{A}}}\,\overline{\overline{\mathsf{I}}}^{(2)} - \overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{I}}} \tag{A4}$$

$$\overline{\overline{\mathsf{A}}}^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)T} = \overline{\overline{\mathsf{A}}}^3 - \mathrm{tr}\overline{\overline{\mathsf{A}}}\,\overline{\overline{\mathsf{A}}}^2 + \mathrm{tr}\overline{\overline{\mathsf{A}}}^{(2)}\,\overline{\overline{\mathsf{A}}} \tag{A5}$$

$$\overline{\overline{\mathsf{A}}}^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T = \mathrm{tr}\overline{\overline{\mathsf{A}}}\,\overline{\overline{\mathsf{A}}}^{(2)} - \overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}^2 \tag{A6}$$

$$\mathrm{tr}\overline{\overline{\mathsf{A}}}^{(4)} = \frac{1}{24}\Big((\mathrm{tr}\overline{\overline{\mathsf{A}}})^4 - 6(\mathrm{tr}\overline{\overline{\mathsf{A}}})^2\mathrm{tr}\overline{\overline{\mathsf{A}}}^2 + 8\mathrm{tr}\overline{\overline{\mathsf{A}}}\,\mathrm{tr}\overline{\overline{\mathsf{A}}}^3 + 3(\mathrm{tr}\overline{\overline{\mathsf{A}}}^2)^2 - 6\mathrm{tr}\overline{\overline{\mathsf{A}}}^4\Big). \tag{A7}$$

$$\mathrm{tr}\overline{\overline{\mathsf{A}}}^{(3)} = \frac{1}{6}\Big((\mathrm{tr}\overline{\overline{\mathsf{A}}})^3 - 3\mathrm{tr}\overline{\overline{\mathsf{A}}}\,\mathrm{tr}\overline{\overline{\mathsf{A}}}^2 + 2\mathrm{tr}\overline{\overline{\mathsf{A}}}^3\Big) \tag{A8}$$

$$\mathrm{tr}\overline{\overline{\mathsf{A}}}^{(2)} = \frac{1}{2}\Big((\mathrm{tr}\overline{\overline{\mathsf{A}}})^2 - \mathrm{tr}\overline{\overline{\mathsf{A}}}^2\Big) \tag{A9}$$

$$\overline{\overline{\mathsf{I}}}^{(2)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T = 3\overline{\overline{\mathsf{I}}} \tag{A10}$$

$$\overline{\overline{\mathsf{I}}}^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T = 2\overline{\overline{\mathsf{I}}}^{(2)}, \qquad \overline{\overline{\mathsf{I}}}^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)T} = 3\overline{\overline{\mathsf{I}}} \tag{A11}$$

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T = \overline{\overline{\mathsf{I}}}^{(3)}, \quad \overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)T} = \overline{\overline{\mathsf{I}}}^{(2)}, \quad \overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(3)T} = \overline{\overline{\mathsf{I}}} \tag{A12}$$

The following operation maps a dyadic $\overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{F}_1$ to a dyadic $\overline{\overline{\mathsf{X}}} \in \mathbb{E}_3\mathbb{F}_3$:

$$\overline{\overline{\mathsf{X}}} = \overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^T = \mathrm{tr}\overline{\overline{\mathsf{A}}}\,\overline{\overline{\mathsf{I}}}^{(3)} - \overline{\overline{\mathsf{I}}}^{(2)}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}} \tag{A13}$$

The same mapping is inverted as

$$\overline{\overline{\mathsf{A}}} = \overline{\mathsf{I}}^{(4)} \lfloor\lfloor \overline{\overline{\mathsf{X}}}^T = \mathrm{tr}\overline{\overline{\mathsf{X}}}\,\overline{\mathsf{I}} - \overline{\overline{\mathsf{X}}} \lfloor\lfloor \overline{\mathsf{I}}^{(2)T} \tag{A14}$$

Cayley-Hamilton identity:

$$\overline{\overline{\mathsf{A}}}^4 - \mathrm{tr}\overline{\overline{\mathsf{A}}}\,\overline{\overline{\mathsf{A}}}^3 + \mathrm{tr}\overline{\overline{\mathsf{A}}}^{(2)}\,\overline{\overline{\mathsf{A}}}^2 - \mathrm{tr}\overline{\overline{\mathsf{A}}}^{(3)}\,\overline{\overline{\mathsf{A}}} + \mathrm{tr}\overline{\overline{\mathsf{A}}}^{(4)}\,\overline{\mathsf{I}} = 0. \tag{A15}$$

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