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ON INDEPENDENCE, COMPLETENESS OF MAXWELL'S EQUATIONS AND UNIQUENESS THEOREMS IN ELECTROMAGNETICS

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Abstract—In this paper, the independence, completeness of Maxwell's equations and uniqueness theorems in electromagnetics are reviewed. It is shown that the four Maxwell's equations are independent and complete. A complete uniqueness theorem is proposed and proven for the first time by pointing out logic mistakes in the existing proof and presenting a truth table. Therefore, electrostatics and magnetostatics can be reduced from dynamical electromagnetics in all aspects including not only the equations as subsets of Maxwell's equations but also the corresponding uniqueness theorems. It is concluded that the axiomatic system of electromagnetic theory must consist of all four Maxwell's equations.

1. INTRODUCTION

In each discipline, we always try to identify the smallest, most compact set of laws or equations that could define the subject completely. This is the axiomatic system of the matter. The axiomatic laws are general physics laws that are not directly related to any particular cases such as specific material properties etc. The laws are independent when none of them can be deduced from others. The system is complete when no other laws are needed to describe the subject in any case other than the problem-related conditions. All other observations can be mathematically deduced, explained and solved based on those laws consistently and systematically. Although the laws in the axiomatic system must be abstracted from many observations (experiments in physics), they mean much more than any individual observation. They must be also compatible, not contradictory to each other. In mechanics, we have the great Newton's three laws. Correspondingly

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James Clerk Maxwell established Maxwell's equations in his famous treatise [1, Vols.1, 2] in electromagnetics. Unfortunately, it is widely accepted that only two of the Maxwell's equations are independent in electromagnetics. This issue will be reviewed in details from different angles in this paper. Several typical reasonings are commented. It is shown that all four of Maxwell's equations are actually independent. Without any of them, the system is incomplete.

It is noticed that the uniqueness theorems in electrostatics and magnetostatics are not consistent. The uniqueness theorem in electrostatics requires the normal components of electric fields on the boundary, and the uniqueness theorem in magnetostatics requires the tangential components of magnetic fields [2]. However, they satisfy the same mathematical equations with special sources according to the existing uniqueness theorem of a vector function [3]. Notice that the above two uniqueness theorems can not be reduced from that of electrodynamics (time-varying case). On the other hand, the equations of electrostatics and magnetostatics are considered as special cases of electrodynamics when $\frac{\partial}{\partial t} = 0$. Actually, the unification of electrostatics, magnetostatics and electrodynamics is a great contribution of J. C. Maxwell.

The existing uniqueness theorems of static and dynamic theories are also inconsistent. For example, in electrostatics normal components of electric field on the boundary are required; in magnetostatics tangential components of magnetic field are needed; however, in time-varying case, only tangential component of electric field or magnetic field are necessary [2, Sects. 3.20, 4.17 and 9.2]. Obviously, this is illogical.

In order to resolve the above theoretical difficulties consistently, let us review some existing conclusions that are widely accepted. In this paper, we will investigate some fundamental problems by analyzing the independence, uniqueness theorem and completeness of Maxwell's equations. A complete uniqueness theorem will be proven and all the above concerns can then be consistently explained.

2. INDEPENDENCE OF MAXWELL'S EQUATIONS

In modern notations due to Heinrich Hertz and Oliver Heaviside, Maxwell's equations are written as the following differential equations

[4, 5]:

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t) \tag{1a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t) \tag{1b}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \tag{1c}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \tag{1d}$$

where $\mathbf{E}$ is the electric (field) intensity, $\mathbf{H}$ the magnetic (field) intensity, $\mathbf{D}$ the electric displacement (or electric flux intensity), $\mathbf{B}$ the magnetic induction (or magnetic flux intensity). (1a) is based on Faraday's experiment (1825), (1b) is Ampère's law which is based on Biot-Savart experiment (1826) and Maxwell's great contribution — displacement current (1890) after 64 years. The last two equations are Gauss's laws based on Coulumb's type experiments (1785).

In terms of electric and magnetic fields, the above four field equations are the complete Maxwell's equations [6]. They are based on carefully selected independent experiments. We can not derive or substitute one by others. Readers who are interested in the history can consult [7, 8] or other physics textbooks. Some existing laws become natural consequences. For example, the empirical law of the conservation of electric charge.

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0 \tag{2}$$

is a consequence of (1b) by taking divergence of (1b) and substituting (1c) into it. In fact, J. C. Maxwell was motivated by (2) when he introduced the great term $\frac{\partial \mathbf{D}}{\partial t}$ in (1b) [8] etc. Kirchhoff's laws become consequences of (1a) and (1b), etc. [9]. Unfortunately, in most existing (advanced) textbooks and literature about electromagnetic theory, it is widely accepted that Maxwell's equations (1a)–(1d) are not independent of one another [2, 4, 5, 10–19]. [20, p. 5] considers (1c) as the definition of $\rho(\mathbf{r}, t)$. This is incorrect. We already have the definition of electric charges, then $\rho(\mathbf{r}, t)$ can be defined directly based on charges. (1c) is essentially an experimental discovery.

The most popular statement is proposed in [2] and [20] etc. The authors referred "Maxwell's equations" to the curl equations (1a) and (1b) only, then "derived" the divergence (1c) and (1d) equations. Similar descriptions can be found in some books of mathematical physics as well [21, p. 603]. J. A. Stratton was probably the first person who tried to start from (1a) and (1b) to establish axiomatic electromagnetic theory. His famous book [2] has become a standard reference. Typical reasoning is given in section 1.3 of [2] or in section 1.5 of [19], it says taking divergence of (1a)

$$\nabla \cdot \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = \frac{\partial}{\partial t} \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \tag{3}$$

from which

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = c_1(\mathbf{r}), \quad \textit{independent of } t \textit{ only}! \tag{4}$$

Since whatever EM field is identically zero before an initial instant, we can conclude that the constant in (4) is zero, which results in (1d). Note that Maxwell's equations govern electromagnetic behaviors at everywhere in the universe. We can not conclude or prove mathematically that the EM field is identically zero at an initial instant at every point in the universe.

Another reasoning is described in books such as [17, p. 2] which says, "Such a constant, if not zero, then implies the existence of magnetic monopoles similar to free electric charges. Since magnetic monopoles have not been found to exist, this constant must be zero." Obviously, a basic assumption that "magnetic monopoles do not exist" is employed. This assumption can not be proven mathematically from (3), it can only be verified by experiments. And also this assumption itself is (1d) in words, which means we do not need to derive it from (1a) at all. In fact, some physicists conjecture the existence of magnetic monopoles based on the beauty of symmetry. In this case, we need to modify Maxwell's equations. But the number of equations stays the same. Furthermore, the $c_1(\mathbf{r})$ in (4) still cannot be mathematically proven to be necessarily the density of magnetic monopoles. This is similar to the charge divergence equation as described later. None of them can be derived from others [9]. Therefore, (1d) is independent of (1a). Of course, (1d) is compatible with (1a) since zero is one of many possibilities.

Similarly (1c) is the mathematical expression of the experimental fact that flux of $\mathbf{D}$ through a closed surface equals the electric source. The popular "derivation" is the following.

Taking divergence of (1b) yields

$$\frac{\partial \nabla \cdot \mathbf{D}(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0 \tag{5}$$

Then using the charge continuity equation (2), we arrive at

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{D} - \rho) = 0 \tag{6}$$

Since the field is zero at the initial instant [2, 17], (1c) holds.

But from (6), mathematically we can have only

$$(\nabla \cdot \mathbf{D} - \rho) = c_2(\mathbf{r}) \tag{7}$$

It is independent of time only, not really a constant. $c_2(\mathbf{r})$ can be determined to be zero by experiments only. The experiment is Coulumb's experiment (1785) or Cavendish's experiment (1773) which leads to just (1c), Gauss's law itself [7].

The above "derivation/reasoning" is an improper logical circle. For example, (2) can be derived from (1b) and (1c) by substituting (1c) into (5), but (1c) cannot be derived from (2) and (1c) without using(1c) (implicitly by assuming initial conditions) to conclude $c_2(\mathbf{r}) = 0$. Obviously, it is not good to choose (2) to be one of the fundamental equations in axiomatic electromagnetic theory. Of course, charge conservation is recognized as a fundamental belief in physics, Maxwell's equations must be compatible with it. This is the way how Maxwell introduced the milestone concept — displacement current. Maxwell's equations include much more information than (2). (2) becomes a natural conclusion in Maxwell's system after the introduction of displacement current. A comparable example can be found in Newton's mechanism systems. Once the first, second and third laws are established (based on many discoveries), the conservation of momentum is one of many important deductions,etc.

One more point we should point out is that if only (1a) and (1b) are independent, static problems cannot be included in the theoretical frame properly. W. C. Chew wrote in [4, p. 2] "For static problems where $\frac{\partial}{\partial t} = 0$, the electric field and magnetic field are decoupled. In this case, Equations (3) and (4) (i.e., (1c) and (1d)) cannot be derived from Equations (1) and (2) (i.e,(1a) and (1b)). Then, the electric field equations (1) and (4) are to be solved independently from the magnetic field equations (2) and (3)." This explanation indicates a logic drawback if only (1a) and (1b) are independent. It is expected that static theory must be reduced cases (subsets) of dynamic theory when $\frac{\partial}{\partial t} \to 0$ (practically).

Consequently, the four Maxwell's equations (1a)–(1d) are independent and compatible (not contradictory one another), the divergence equation of $\mathbf{B}$ (1d) is compatible with the curl equation of $\mathbf{E}$ (1a), and the divergence equation of $\mathbf{D}$ (1c) is compatible with the curl equation of $\mathbf{H}$ (1b). We can not derive mathematically any one of them from others without introducing additional assumption that is usually one of Maxwell's equations, or without falling into improper logical circles. This conclusion will be used in the next section.

3. COMPLETENESS AND UNIQUENESS THEOREM OF MAXWELL'S EQUATIONS

It is expected that if a set of equations is complete, the solutions are uniquely determined by all the equations with the particular properties and conditions of the case under study. About electromagnetic fields, physicists have been believing that J. A. Maxwell already established the complete electrodynamic theory — Maxwell's equations. Maxwell's equations completely determine the electromagnetic fields, and are the fundamental equations of the theory of such as electromagnetics [6, 8, 22]. In theoretical aspects, if Maxwell's equations are complete, all important electromagnetic theorems and principles like Poynting theorem, duality theorem, reciprocity theorem etc. [4, 23] are important corollaries of Maxwell's equations, which means they can be derived from Maxwell's equations without additional equations or assumptions. None of them can substitute Maxwell's equations themselves as postulates of electrodynamics.

As we discussed in the previous section, Maxwell's equations (1a)–(1d) are independent and compatible. Now we may ask: are they complete? The answer is, as general physical laws, they are complete; however, they are not complete to uniquely determine fields in particular cases. Constitutive equations are needed. Although constitutive equations are not general physics laws, they are needed for two reasons. The physical reason is that constitutive relations reflect medium polarizations that are related to the case under study. This is an important topic in solid physics [8]. They are independent of Maxwell's equations.

Let us discuss the mathematical reasons. The existing reasons are questionable. For example, in [24, p. 3], the author claims "The necessity of using constitutive relations to supplement the Maxwell's equations is clear from the following mathematical observations. There are a total of $12 (= 3 \times 4)$ scalar unknowns for the four field vectors **E**, **H**, **B**, and **D**. As we have learned (3) and (4) are not independent equations; they can be derived from (1), (2) and (5). The independent equations are (1) and (2), which constitute six (3×2) scalar equations. Thus we need six more scalar equations. These are the constitutive relations." Although J. D. Jackson indicated that the four Maxwell's equations provide a complete description of the classical electromagnetics [8, p. 239], he gave a similar reason for the necessity of constitutive equations. In his argument, there are 14 (8 from Maxwell's equations +6 from two constitutive equations) scalar equations, but only 12 variables [8, p. 14]. The numbers do not match. Similar statements can be found in [2, Sec. 1.5] [4, p. 5] and many other references. Obviously, the above mathematical observations are based on our knowledge about solutions to a set of linear algebra equations, the Cramer's rule: N linear algebra equations of N unknowns can determine these N unknowns **uniquely** if the determinant is nonzero [26, Sec. 1.9–2] [27]. Although this is true for linear algebra equations, it has not been shown that it is applicable to vector differential equation systems. However we strongly believe it is true in any case. A contradictory example is the uniqueness theorem of a vector function discussed in [3, p. 92–97], which reads: "A vector is **uniquely** specified by giving its divergence and its curl within a region and its normal component over the boundary." Mathematically

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = s \tag{8a}$$

$$\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{c} \tag{8b}$$

and the boundary conditions determine $\mathbf{F}$ uniquely. There are 4 scalar equations, but only 3 scalar unknowns. We should not be confused by some techniques used in electrostatics that are related to the above theorem. The above equations are converted into Possion's equation by introducing a scalar potential. In many cases, we solve the field by solving potentials. However, the introduction of potential relies on the curl equation. Implicitly, the same number (4 in this case) of scalar equations is still used in term of $\mathbf{F}(\mathbf{r})$.

The essential idea behind the above discussions is the corresponding uniqueness theorem, rather than the number of equations. In this sense, the expression about N linear algebra equations should be probably considered as the first uniqueness theorem in science. Various uniqueness theorems in mathematics are discussed in [28] for scalar differential equations and in [29] for vector differential equations in electricity.

In electromagnetics, the uniqueness theorem of a vector function implies four special cases [11, p. 63]. They are also special cases of Maxwell's equations: static or decoupled. Most textbooks discussed the third case in [11, p. 63]) as electrostatics and the second case as magnetostatics. In the proof of the uniqueness theorem of a vector function both divergence and curl equations are used explicitly. In the above cases, constitutive relations are needed if they are considered as reduced cases of Maxwell's equations. Based on the uniqueness theorem, we can solve the equations via any expedient means.

For time-varying electromagnetic fields, the uniqueness theorem is given in many textbooks [2, Sec. 9.1.2] in time domain, [4, Sec. 1.5.2] [23, Sec. 7.3] [10, Sec. 372] [17] in frequency domain, [19, Sec. 3.3] in both domains. The proofs are essentially based on the Poynting theorem of the difference fields, or more generally, the so-called method of energy integrals [21]. Note that in the proofs, only the two curl equations of Maxwell's equations are used, which means fields can be solved uniquely only by solving the two curl equations. If this is true, the two divergence equations (even the charge continuity equation) are not fundamental equations. At least they should be derivable from the two curl equations, only the two curl equations are independent. As some authors claimed "... we need only work with the first two ..." [4, p. 5]. But this is even contradictory to the existing statement "only three of (1a)–(2) are independent, since (2) is not used in the proof" [2]. However, we already show that (1a)–(1d) are independent from the physical and mathematical point of views. What is the problem?

Let us review the proof of the uniqueness theorem for time-varying electromagnetic fields. According to [21], the method of energy integrals was first introduced into electromagnetic theory by A. Rubinowicz [25], then cited by others.

The original method of energy integrals was developed for wave equations in mechanics/acoustics or the so-called hyperbolic type differential equations [21]. For example, for the two-dimensional wave equation

$$L(u) = u_{tt} - u_{xx} - u_{yy} = 0 \tag{9}$$

we can transfer (9) into the following identity

$$2u_t L(u) = -2(u_t u_x)_x - 2(u_t u_y)_y + (u_x^2)_t + (u_y^2)_t + (u_t^2)_t = 0 \tag{10}$$

then prove the uniqueness theorem by integrating it. See the details in [21, p. 644–646]. The above transformation is identical if u_t is nonzero or nonsingular. Note that the starting equation (9) is inherently a second order differential equation based on the physical equation, Newton's second law [30, p. 791 and p. 827]. However, the physical laws in electromagnetics, Maxwell's equation (1a)–(1d) are inherently first order differential equations. There are two ways to apply the method of energy integral to electromagnetics.

First, we can recognize Maxwell's equations as wave equations [4, Sec. 1.5.2]. R. Courant claimed in [21, p. 178] that "Maxwell's system of differential equations is essentially hyperbolic" based on the characteristic form of the two curl equations. In this method, we take the risk of extending solution sets since we derived the wave equations (Helmholtz's equation) by taking divergence of the original curl equations (1a) and (1b). It is easy to show that differentiating equations may extend solution space. As a consequence, we will need to impose all the original Maxwell (four) equations on the solutions. Maxwell's equations are not pure hyperbolic.

The existing uniqueness theorem of electrodynamics is typically stated as [2, p. 487] "An electromagnetic field is uniquely determined within a bounded region V at all time $t > 0$ by the initial values of electric and magnetic vectors throughout V, and the values of the tangential component of the electric vector (or of the magnetic vector) over the boundaries for $t \geq 0$." The most popular proof of the uniqueness theorem in electromagnetics is based on Poynting identity of the difference fields. For simplicity, we discuss the time domain case in which the constitutive equations are

$$\mathbf{D}(\mathbf{r}, t) = \overline{\epsilon}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}, t) \tag{11a}$$

$$\mathbf{B}(\mathbf{r}, t) = \overline{\mu}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r}, t) \tag{11b}$$

where $\overline{\epsilon}(\mathbf{r})$ and $\overline{\mu}(\mathbf{r})$ are 3×3 tensors (rank of 2), independent of time (stationary). Substituting (11) into (1a) and (1b), we have two curl equations about $\mathbf{E}$ and $\mathbf{H}$. In order to comment on the existing proof, let us summarize it. According to the reduction to absurdity (proof by contradiction), usually we assume the current source $\mathbf{J}$ is given, there are two different solutions, then the differences

$$\mathbf{e}(\mathbf{r}, t) = \mathbf{E}_2(\mathbf{r}, t) - \mathbf{E}_1(\mathbf{r}, t) \not\equiv 0 \tag{12a}$$

$$\mathbf{h}(\mathbf{r}, t) = \mathbf{H}_2(\mathbf{r}, t) - \mathbf{H}_1(\mathbf{r}, t) \not\equiv 0 \tag{12b}$$

satisfy

$$\nabla \times \mathbf{e}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\left[\overline{\mu}(\mathbf{r}) \cdot \mathbf{h}(\mathbf{r}, t)\right] \tag{13a}$$

$$\nabla \times \mathbf{h}(\mathbf{r}, t) = \frac{\partial}{\partial t}\left[\overline{\epsilon}(\mathbf{r}) \cdot \mathbf{e}(\mathbf{r}, t)\right] \tag{13b}$$

Dot multiplying (13a) with $\mathbf{h}$, and (13b) with $\mathbf{e}$, then subtracting the first result from the second, we have

$$\nabla \cdot (\mathbf{e} \times \mathbf{h}) = -\frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{h} \cdot \overline{\mu} \cdot \mathbf{h} + \mathbf{e} \cdot \overline{\epsilon} \cdot \mathbf{e}\right) \tag{14}$$

Integrating in the volume V bounded by surface S yields

$$\int_S (\mathbf{e} \times \mathbf{h}) \cdot \hat{n} dS = -\frac{1}{2}\frac{\partial}{\partial t}\int_V \left(\mathbf{h} \cdot \overline{\mu} \cdot \mathbf{h} + \mathbf{e} \cdot \overline{\epsilon} \cdot \mathbf{e}\right) dV \tag{15}$$

Since at the initial time t_0, the fields are given, for $t \leq t_0$, $\mathbf{e} = 0$ and $\mathbf{h} = 0$, integrating again respect to time yields

$$\int_{t_0}^{t} dt \int_S (\mathbf{e} \times \mathbf{h}) \cdot \hat{n} dS + \frac{1}{2}\int_V \left(\mathbf{e} \cdot \overline{\epsilon} \cdot \mathbf{e}\right) dV + \frac{1}{2}\int_V \left(\mathbf{h} \cdot \overline{\mu} \cdot \mathbf{h}\right) dV = 0 \tag{16}$$

We are posing boundary conditions on S. Only the first integral is related to S. If at any time on S

$$(\mathbf{e} \times \mathbf{h}) \cdot \hat{n} \equiv 0 \tag{17}$$

In terms of boundary conditions, we have chosen

$$\hat{n} \times \mathbf{e} = 0, \quad on\ some\ parts\ of\ S \tag{18a}$$

$$\hat{n} \times \mathbf{h} = 0, \quad on\ the\ rest\ parts\ of\ S \tag{18b}$$

we have

$$\int_V (\mathbf{e} \cdot \overline{\epsilon} \cdot \mathbf{e})\, dV + \int_V (\mathbf{h} \cdot \overline{\mu} \cdot \mathbf{h})\, dV = 0 \tag{19}$$

They are the simplest conditions satisfying (17). For physical materials, $\overline{\epsilon}$ and $\overline{\mu}$ must be positive definite since the introduction of dielectrics increases the stored energy in the same region. The integrals in (19) are associated with the energy (even not exactly [2]), expressed in independent variables $\mathbf{e}$ and $\mathbf{h}$, they must be non-negative definite, thus they equal zeros, which lead to

$$\mathbf{e} = 0 \tag{20a}$$

$$\mathbf{h} = 0 \tag{20b}$$

(20a) and (20b) are contradictory to the assumptions (12a) and (12b). We concluded that the uniqueness theorem is proved.

From the discussions about independence in Section 2, the above proof and then the given uniqueness theorem is physically, mathematically and logically questionable. Now, we can question the proof at least in two aspects. First, only the curl equations are used. If the four Maxwell's equations are independent, this theorem must be incomplete. Second, in order to satisfy (17), there are other choices. The existing choices (18a) and (18b) are contradictory to the uniqueness because they imply that the normal components can be undetermined. Logically and most importantly, the method of proof by contradiction has been misused in this case. In many cases, the method of contradiction involves only one statement or variable. The basic structure is that if we want to prove "If $A, then\ B$", we work on it by assuming that A and $NOT\ B$ are true [31]. Here B is usually a single judgment that involves one variable in mathematics. The contradiction method is successfully applied to the proofs of many uniqueness theorems [3, 21, 29] where only a single scalar or vector unknown is involved. However, there are two vector unknowns $\mathbf{e}$ and $\mathbf{h}$ in (13). By logics [32], we have a truth table as follows.

Table 1. Truth table of **e** and **h**.

Cases	$\mathbf{e} \equiv 0$	$\mathbf{h} \equiv 0$	$\mathbf{e} \equiv 0 \wedge \mathbf{h} \equiv 0$	Explanations
1	T	T	T	Both **E** and **H** are unique
2	T	F	F	**E** is unique, but not **H**
3	F	T	F	**H** is unique, but not **E**
4	F	F	F	Both **E** and **H** are not unique

If we want to show that Case 1 in the truth table is true, we have to negate all other three cases. Unfortunately, in the existing proof of the uniqueness theorem of electromagnetics, only Case 4 (implied in the assumption (12)) is negated as the contradiction of Case 1. However, Case 2, Case 3 and Case 4 are all contradictory (fields are not unique) to Case 1 (fields are unique). Thus, the proof is logically incomplete. We will show in the next section that if (1a)–(1d) are considered together with constitutive equations (11a) and (11b), a complete uniqueness theorem can be proved without any physical, mathematical and logical doubts.

4. A COMPLETE UNIQUENESS THEOREM

Based on the above discussions about the independence, completeness and uniqueness theorem of Maxwell's equations, a complete uniqueness theorem is presented and proved in this section.

Theorem 1 *An electromagnetic field is uniquely determined within a bounded region V at all time $t > 0$ by the initial values of electric and magnetic vectors throughout V, and the values (both tangential and normal components) of the electric vector and of the magnetic vector over the boundaries for $t \geq 0$. In equations, considering the constitutive equations (11a) and (11b), the solution to*

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\left[\overline{\mu} \cdot \mathbf{H}(\mathbf{r}, t)\right] \tag{21a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial}{\partial t}\left[\overline{\epsilon} \cdot \mathbf{E}(\mathbf{r}, t)\right] + \mathbf{J}(\mathbf{r}, t) \tag{21b}$$

$$\nabla \cdot \left[\overline{\epsilon} \cdot \mathbf{E}(\mathbf{r}, t)\right] = \rho(\mathbf{r}, t) \tag{21c}$$

$$\nabla \cdot \left[\overline{\mu} \cdot \mathbf{H}(\mathbf{r}, t)\right] = 0 \tag{21d}$$

$$\mathbf{E}(\mathbf{r}, t)\big|_{\mathbf{r}\ on\ S;\ t\geq 0} = \mathbf{E}_0(\mathbf{r}, t)\big|_{\mathbf{r}\ on\ S;\ t\geq 0} \tag{21e}$$

$$\mathbf{H}(\mathbf{r}, t)\big|_{\mathbf{r}\ on\ S;\ t\geq 0} = \mathbf{H}_0(\mathbf{r}, t)\big|_{\mathbf{r}\ on\ S;\ t\geq 0} \tag{21f}$$

is unique, provided the existence [33] and the initial values in V are given.

Proof. In order to consider the truth Table 1, we need to combine two logic methods, proof by contradiction and exclusive case [32], instead of proof by contradiction only. All four cases have to be considered by using the method of exclusive cases. When we negate Cases 2, 3 and 4, we have to use the method of proof by contradiction.

Assuming the difference fields are $\mathbf{e}(\mathbf{r},t)$ and $\mathbf{h}(\mathbf{r},t)$, we have from (21)

$$\nabla\times\mathbf{e}(\mathbf{r},t)=-\frac{\partial}{\partial t}\left[\overline{\mu}\cdot\mathbf{h}(\mathbf{r},t)\right] \tag{22a}$$

$$\nabla\times\mathbf{h}(\mathbf{r},t)=\frac{\partial}{\partial t}\left[\overline{\epsilon}\cdot\mathbf{e}(\mathbf{r},t)\right] \tag{22b}$$

$$\nabla\cdot\left[\overline{\epsilon}\cdot\mathbf{e}(\mathbf{r},t)\right]=0 \tag{22c}$$

$$\nabla\cdot\left[\overline{\mu}\cdot\mathbf{h}(\mathbf{r},t)\right]=0 \tag{22d}$$

$$\mathbf{e}(\mathbf{r},t)|_{\mathbf{r}\ on\ S;\ t\geq 0}=0 \tag{22e}$$

$$\mathbf{h}(\mathbf{r},t)|_{\mathbf{r}\ on\ S;\ t\geq 0}=0 \tag{22f}$$

Let us negate Cases 2, 3 and 4 in the truth table one by one.

Case 2: Assume

$$\mathbf{e}(\mathbf{r},t)\equiv 0\ and\ \mathbf{h}(\mathbf{r},t)\not\equiv 0 \tag{23}$$

In this case, the original proof (for case 4) is invalid since $\mathbf{e}(\mathbf{r},t)\equiv 0$. An equation cannot be multiplied by constant 0. (23) is interpreted as: the electric field is unique, but not the magnetic field. Then the electromagnetic field is not unique. Substituting (23) into (22) yields

$$\overline{\mu}\cdot\mathbf{h}(\mathbf{r},t)=\mathbf{b}(\mathbf{r}) \tag{24a}$$

$$\nabla\times\mathbf{h}(\mathbf{r},t)=0 \tag{24b}$$

$$\nabla\cdot\left[\overline{\mu}\cdot\mathbf{h}(\mathbf{r},t)\right]=0 \tag{24c}$$

$$\mathbf{h}(\mathbf{r},t)|_{\mathbf{r}\ on\ S;\ t\geq 0}=0 \tag{24d}$$

(22c) and (22e) are identities. (24a) shows that under the assumption of (23), the difference (it is important to notice that it is not necessarily the original) magnetic field must be time independent, i.e., static. Unfortunately $\mathbf{b}(\mathbf{r})$ cannot be determined by (24a) itself. Then,

$$\nabla\times\mathbf{h}(\mathbf{r})=0 \tag{25a}$$

$$\nabla\cdot\left[\overline{\mu}\cdot\mathbf{h}(\mathbf{r})\right]=0 \tag{25b}$$

$$\mathbf{h}(\mathbf{r})|_{\mathbf{r}\ on\ S}=0 \tag{25c}$$

Rewrite (25c) as

$$\hat{n} \times \mathbf{h}(\mathbf{r})|_{\mathbf{r}\ on\ S} = 0 \tag{26a}$$
$$\hat{n} \cdot \mathbf{h}(\mathbf{r})|_{\mathbf{r}\ on\ S} = 0 \tag{26b}$$

since the tangential and normal parts of $\mathbf{h}(\mathbf{r})$ are independent and unique in a given coordinate system. (25a) and (26a) imply that one can introduce

$$\mathbf{h}(\mathbf{r}) = -\nabla\phi_h(\mathbf{r}) \tag{27}$$

Substituting (27) into (25b) and (26b) yields

$$\nabla \cdot [\overline{\mu} \cdot \nabla\phi_h(\mathbf{r})] = 0 \tag{28a}$$
$$\hat{n} \cdot [\overline{\mu} \cdot \nabla\phi_h(\mathbf{r})]|_{\mathbf{r}\ on\ S} = 0 \tag{28b}$$

Considering the identity [34, p. 487]

$$\nabla \cdot [\phi_h(\overline{\mu} \cdot \nabla\phi_h)] = \phi_h\nabla \cdot (\overline{\mu} \cdot \nabla\phi_h) + (\nabla\phi_h) \cdot (\overline{\mu} \cdot \nabla\phi_h) \tag{29}$$

and (28a), one obtains

$$\nabla \cdot [\phi_h(\overline{\mu} \cdot \nabla\phi_h)] = (\nabla\phi_h) \cdot \overline{\mu} \cdot (\nabla\phi_h) \tag{30}$$

Integrating over the volume, (30) becomes

$$\int_S [\phi_h(\overline{\mu} \cdot \nabla\phi_h)] \cdot \hat{n}dS = \int_V (\nabla\phi_h) \cdot \overline{\mu} \cdot (\nabla\phi_h)dV \tag{31}$$

The left volume integral is converted to a surface S by using the divergence theorem. Note that discontinuities (of $\overline{\mu}$) are allowed [34, p. 488]. Substituting (28b) into (31) yields

$$\int_V (\nabla\phi_h) \cdot \overline{\mu} \cdot (\nabla\phi_h)dV = 0 \tag{32}$$

Since physical $\overline{\mu}$ makes the integral non-negative definite, (32) can only be satisfied in the case

$$\nabla\phi_h \equiv 0 \tag{33}$$

from which

$$\mathbf{h}(\mathbf{r}) \equiv 0 \tag{34}$$

This is contradictory to the assumption (23). Then Case 2 is negated.

Case 3: Assume

$$\mathbf{e}(\mathbf{r}, t) \not\equiv 0 \; and \; \mathbf{h}(\mathbf{r}, t) \equiv 0 \tag{35}$$

i.e., the electric field is not unique, but the magnetic field is unique. Similarly to the Case 2, we have from (35) and (22) the following homogeneous static problem about the difference electric field,

$$\nabla \times \mathbf{e}(\mathbf{r}) = 0 \tag{36a}$$

$$\nabla \cdot [\bar{\epsilon} \cdot \mathbf{e}(\mathbf{r})] = 0 \tag{36b}$$

$$\mathbf{e}(\mathbf{r})|_{\mathbf{r} \; on \; S} = 0 \tag{36c}$$

(36) is mathematically the same as (25). Then the assumption (35) is negated exactly in the same way.

Case 4: Assume

$$\mathbf{e}(\mathbf{r}, t) \not\equiv 0 \; and \; \mathbf{h}(\mathbf{r}, t) \not\equiv 0 \tag{37}$$

It is the case in which both the electric and magnetic fields are not unique. This case is negated in the most popular proof.

Consequently, only Case 1 could be true if the solution exists. Therefore, the proof is completed.

Note that all four Maxwell's equations are used in the proof, and also no more laws are needed. In fact, the physical explanation of the difference field equation is very clear: An electromagnetic system (here it is the difference fields) is null if and only if there is no static and time-varying sources anywhere and isolated (no energy exchange).

5. UNIQUENESS THEOREMS IN ELECTROSTATICS AND MAGNETOSTATICS

As we pointed out in Section 1 and Section 4, the existing uniqueness theorems in electrostatics and magnetostatics are not consistent with each other and with the existing uniqueness theorem of electrodynamics. This is a paradox. On one hand, the field equations in electrostatics and magnetostatics can be reduced from Maxwell's equations if the four Maxwell's equations are independent as concluded in early sections. On the other hand, their uniqueness theorems can not be deduced from the same Maxwell's equations . We can now resolve the paradox based on the present uniqueness theorem of Maxwell's equations.

5.1. Uniqueness Theorem in Electrostatics

The existing uniqueness theorem requires only the normal components of electric field on the boundary S [2, 35]. From Maxwell's equations (1a) and (1c), the electrostatic equations are

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \tag{38a}$$

$$\nabla \cdot (\overline{\epsilon} \cdot \mathbf{E})(\mathbf{r}) = \rho(\mathbf{r}) \tag{38b}$$

As a result of the uniqueness theorem 1, the corresponding uniqueness theorem in electrostatics is proposed as follows.

Theorem 2 *An electrostatic boundary value problem in a region V bounded by S is uniquely determined by*

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \tag{39a}$$

$$\nabla \cdot (\overline{\epsilon} \cdot \mathbf{E}(\mathbf{r})) = \rho(\mathbf{r}) \tag{39b}$$

$$\mathbf{E}(\mathbf{r})|_{\mathbf{r}\ on\ S} = \mathbf{E}_0(\mathbf{r})|_{\mathbf{r}\ on\ S} \ \text{ with } n \times E_0|_{\mathbf{r}\ on\ S} \equiv 0 \tag{39c}$$

The proof is exactly the same as the Case 3 in Theorem 1 although the original fields are not necessarily static in Case 3.

5.2. Uniqueness Theorem in Magnetostatics

The same discussions are suitable for magnetostatics. We have the following uniqueness theorem.

Theorem 3 *A magnetostatics boundary value problem in a region V bounded by S is uniquely determined by*

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) \tag{40a}$$

$$\nabla \cdot (\overline{\mu} \cdot \mathbf{H}(\mathbf{r})) = 0 \tag{40b}$$

$$\mathbf{H}(\mathbf{r})|_{\mathbf{r}\ on\ S} = \mathbf{H}_0(\mathbf{r})|_{\mathbf{r}\ on\ S} \ \text{ with } n \cdot H_0|_{\mathbf{r}\ on\ S} \equiv 0 \tag{40c}$$

The proof is the same as the Case 2 in Theorem 1 since only the difference field is involved.

The traditional proof of the uniqueness theorem is based on the introduction of a vector potential from (40b) [2] [36, p. 258–259]. It is applicable to Theorem 3 by using the identity

$$\begin{aligned} &\int \left[(\nabla \times \mathbf{A}) \cdot \overline{\mu}^{-1} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \overline{\mu}^{-1} \cdot \nabla \times \mathbf{A}) \right] dV \\ &= \int_S (\mathbf{A} \times \overline{\mu}^{-1} \cdot \nabla \times \mathbf{A}) dS \end{aligned} \tag{41}$$

if we define the difference magnetic flux intensity

$$\mathbf{b}(\mathbf{r}) = \nabla \times \mathbf{A} \tag{42}$$

However, if we consider that **B** is a function of **H**, the proof in the Case 2 of Theorem 1 is more natural and general. Consequently, Theorem 2 and Theorem 3 are now derivable from Theorem 1. The uniqueness theorems of electrodynamics, electrostatics and magnetostatics are compatible. Therefore, the axiomatic system of electromagnetic theory must consist of all four Maxwell's equations.

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