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FRACTIONAL CURL OPERATOR AND FRACTIONAL WAVEGUIDES

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Abstract—Fractional curl operator has been utilized to study the fractional waveguide. The fractional waveguide may be regarded as intermediate step between the two given waveguides. The two given waveguides are related through the principle of duality. Behavior of field lines in fractional waveguides are studied with respect to fractional parameter α.

1. INTRODUCTION

Fractional calculus is a branch of mathematics that deals with operators having non-integer and/or complex order, e.g., fractional derivative and fractional integral [1]. Tools of fractional calculus have various applications in different disciplines of science and engineering, e.g., Optics, Control and Mechanics etc. Discussion on recent applications of tools of fractional calculus in science and engineering is available in [2]. Mathematical recipe to fractionalize a linear operator is available in [3, 4]. Recently, while exploring the roles and applications of fractional calculus in electromagnetics, a new fractional operator has been introduced [3]. The new fractional operator is termed as fractional curl operator.

Fractional curl operator has been utilized to find the new set of solutions to Maxwell's equations by fractionalizing the principle of duality [3]. New set of solutions is named as fractional dual solutions to the Maxwell equations. In electromagnetics, principle of duality states that if $(\underline{E}, \eta\underline{H})$ is one set of solutions (original solutions) to Maxwell equations, then other set of solutions (dual to the original solutions) is $(\eta\underline{H}, -\underline{E})$, where η is the impedance of the medium. The solutions which may be regarded as intermediate step between the original and dual to the original solutions may be obtained using the following relations [3]

$$\mathbf{E}_{\mathrm{fd}} = \frac{1}{(jk)^{\alpha}}(\nabla\times)^{\alpha}\mathbf{E}$$
$$\eta\mathbf{H}_{\mathrm{fd}} = \frac{1}{(jk)^{\alpha}}(\nabla\times)^{\alpha}\eta\mathbf{H}$$

where $(\nabla\times)^{\alpha}$ means fractional curl operator and $k = \omega\sqrt{\mu\epsilon}$ is the wavenumber of the medium. It may be noted that fd means fractional dual solutions. It is obvious from above set of equations that for $\alpha = 0$,

$$\mathbf{E}_{\mathrm{fd}} = \mathbf{E}, \qquad \eta\mathbf{H}_{\mathrm{fd}} = \eta\mathbf{H}$$

and for $\alpha = 1$

$$\mathbf{E}_{\mathrm{fd}} = \eta\mathbf{H}, \qquad \eta\mathbf{H}_{\mathrm{fd}} = -\mathbf{E}$$

Which are two sets of solutions to Maxwell's equations. The solutions which may be regarded intermediate step between the above two sets of solutions may be obtained by varying parameter α between zero and one. Naqvi et al. [5] afterward extended the work [3] and discussed the behavior of fractional dual solutions in an unbounded chiral medium. Lakhtakia [6] derived theorem which shows that a dyadic operator which commutes with curl operator can be used to find new solutions of the Faraday and Ampere-Maxwell equations. Veliev and Engheta [7] utilized the fractional curl operator to a fixed solution and obtained the fractional fields that represent the solution of reflection problem from an anisotropic surface. Naqvi and Abbas studied the behavior of fractional curl operator for complex and higher orders [8] and fractional dual solutions in metamaterial having negative permittivity and permeability [9]. Naqvi and Rizvi determined the sources corresponding to fractional dual solutions [10]. Recently Hussain and Naqvi [11], introduced the idea of fractional transmission lines. Naqvi et al. modelled the transmission through chiral layer using fractional curl operator [12].

Our interest is to connect two given waveguides through a fractional operator, and to determine solutions corresponding to the new waveguide which may be regarded as intermediate step between the two given waveguides. In present work, we have considered the fractional curl operator, $\frac{1}{(jk)^{\alpha}}(\nabla\times)^{\alpha}$, as an operator which connects the two given waveguides. Parallel plate waveguides have been considered as given waveguides. It is the requirement of fractional curl operator that two given problems should be related through the principle of duality. Therefore, it is considered that one waveguide is composed of perfect electric conductors (PEC) and it supports TM-mode while other waveguide is composed of perfect magnetic conductors (PMC) and it supports TE-mode. In this way we are proposing the solutions for fractional waveguides which may be regarded as intermediate step between the two given dual waveguides.

2. GENERAL WAVE BEHAVIOR ALONG GUIDING STRUCTURE

Consider a straight guiding structure having uniform cross-section and lying along z-axis. Assume that a wave is propagating in z-direction. For harmonic time dependence with an angular frequency ω, the dependency on z and t for all field components can be described by the exponential factor $\exp(-j\beta z)\exp(j\omega t)$. Hence in using a phasor representation in equations relating field quantities we may replace partial derivatives with respect to t and z by $(j\omega)$ and $(-j\beta)$ respectively. As an example, for a cosine reference we may write the instantaneous expression for $\mathbf{E}$ field in Cartesian coordinates as

$$E(x,y,z,t) = \mathrm{Re}\left[\mathbf{E}(x,y)\exp(j\omega t - j\beta z)\right] \tag{1}$$

where $\mathbf{E}(x,y)$ is a two dimensional vector phasor that depends only on cross sectional coordinates. The instantaneous expression for the $\mathbf{H}$ field can be written in a similar way. The electric and magnetic field intensities in source free dielectric region satisfy the following homogeneous vector Helmholtz's equations

$$\begin{aligned} \nabla^2\mathbf{E} + k^2\mathbf{E} &= 0 \\ \nabla^2\mathbf{H} + k^2\mathbf{H} &= 0 \end{aligned}$$

where $\mathbf{E}$ and $\mathbf{H}$ are three dimensional vector phasors and $k = \omega\sqrt{\mu\epsilon}$ is the wavenumber.

Manipulating following two Maxwell curl equations,

$$\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H} \\ \nabla \times \mathbf{H} &= j\omega\epsilon\mathbf{E} \end{aligned}$$

the transverse components can be expressed in terms of longitudinal components (E_z, H_z), that is

$$E_x = -\frac{1}{h^2}\left(j\beta\frac{\partial E_z}{\partial x} + jk\frac{\partial \eta H_z}{\partial y}\right) \tag{2a}$$

$$E_y = -\frac{1}{h^2}\left(j\beta\frac{\partial E_z}{\partial y} - jk\frac{\partial \eta H_z}{\partial x}\right) \tag{2b}$$

$$H_x = -\frac{1}{h^2}\left(j\beta\frac{\partial H_z}{\partial x} - \frac{jk}{\eta}\frac{\partial E_z}{\partial y}\right) \tag{2c}$$

$$H_y = -\frac{1}{h^2}\left(j\beta\frac{\partial H_z}{\partial y} + \frac{jk}{\eta}\frac{\partial E_z}{\partial x}\right) \tag{2d}$$

It may be noted that $\beta = \sqrt{k^2 - h^2}$.

Using fractional operator $\frac{1}{(jk)^\alpha}(\nabla\times)^\alpha$, our objective is to study the new guiding structures which may be described by the following set of expressions

$$E_x^{\mathrm{fd}} = -\frac{1}{h^2}\left(j\beta\frac{\partial E_z^{\mathrm{fd}}}{\partial x} + jk\frac{\partial \eta H_z^{\mathrm{fd}}}{\partial y}\right) \tag{3a}$$

$$E_y^{\mathrm{fd}} = -\frac{1}{h^2}\left(j\beta\frac{\partial E_z^{\mathrm{fd}}}{\partial y} - jk\frac{\partial \eta H_z^{\mathrm{fd}}}{\partial x}\right) \tag{3b}$$

$$\eta H_x^{\mathrm{fd}} = -\frac{1}{h^2}\left(j\beta\frac{\partial \eta H_z^{\mathrm{fd}}}{\partial x} - jk\frac{\partial E_z^{\mathrm{fd}}}{\partial y}\right) \tag{3c}$$

$$\eta H_y^{\mathrm{fd}} = -\frac{1}{h^2}\left(j\beta\frac{\partial \eta H_z^{\mathrm{fd}}}{\partial y} + jk\frac{\partial E_z^{\mathrm{fd}}}{\partial x}\right) \tag{3d}$$

where $(E_z^{\mathrm{fd}}, \eta H_z^{\mathrm{fd}})$ is the new pair of longitudinal components in the new guiding structure. The guiding structures which contain $(E_z^{\mathrm{fd}}, \eta H_z^{\mathrm{fd}})$ as longitudinal components has been termed as fractional guiding structures in the next sections.

In studies related to fractional calculus, it is often assumed that two problems are given and the problems are related through an operator. This means that without operator we have quantities related to one given problem. Application of operator yields quantities related to another given problem. By fractionalizing the operator, our interest is to explore how quantities related to one problem are changing into the quantities related to the other problem. In present work, we are interested to consider two guiding structures which are related to each other through principle of duality. For $\alpha = 0$

$$\mathbf{E}^{\mathrm{fd}} = \mathbf{E} \qquad \eta\mathbf{H}^{\mathrm{fd}} = \eta\mathbf{H}$$

set of equations given in (3) reduce to set of equations given in (2) and it is assumed that it corresponds to one given guiding structure. For $\alpha = 1$

$$\mathbf{E}^{\mathrm{fd}} = \eta\mathbf{H} \qquad \eta\mathbf{H}^{\mathrm{fd}} = -\mathbf{E}$$

as a result equations (3a) and (3b) reduce to (2c) and (2d) while equations (3c) and (3d) reduce to negative of equations (2a) and

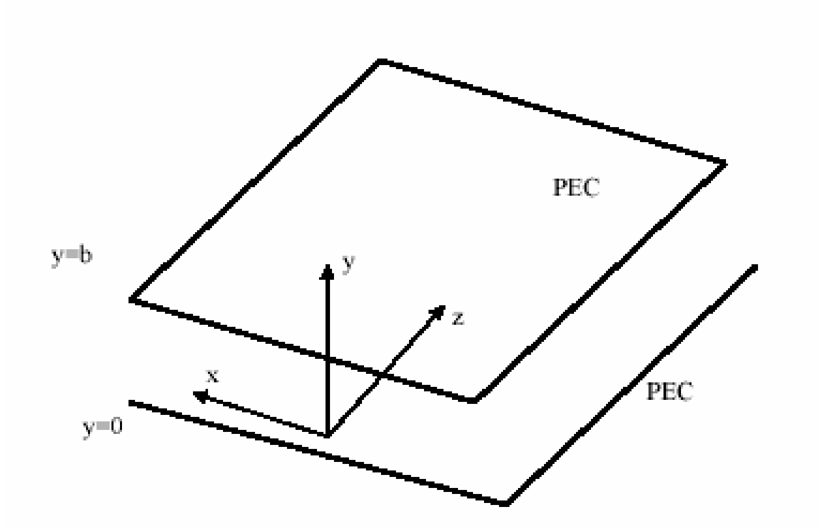

Figure 1. PEC walls parallel plate waveguide structure.

(2b) respectively and it is assumed that it corresponds to other given guiding structure. For $0 < \alpha < 1$, set of equations (3) may provide us solutions for guiding structure which may be regarded as intermediate step between the two given guiding structures. In next section we shall apply the fractional curl operator on parallel plate waveguides.

3. FRACTIONAL PARALLEL PLATE WAVEGUIDE

Consider a parallel plate waveguide consisting of two perfect electric conducting plates separated by a dielectric medium with constitutive parameters ϵ and μ. The separation between the two parallel plates is b as shown in Figure 1. One plate is located at $y = 0$, while other plate is located at $y = b$. The plates are assumed to be of infinite extent. Let us suppose that a TM wave ($H_z = 0$) is propagating in z-direction. The axial component of electric field is given by

$$\begin{aligned}\hat{z}E_z(y,z) &= \hat{z}A_n \sin(hy)\exp(-j\beta z)\\ &= \hat{z}\frac{A_n}{2j}\left[\exp(jhy - j\beta z) - \exp(-jhy - j\beta z)\right] \qquad (4a)\end{aligned}$$

where $h = \frac{n\pi}{b}$. The transverse components of the fields are

$$\hat{y}E_y(y,z) = -\hat{y}\frac{j\beta}{h}A_n \cos(hy)\exp(-j\beta z)$$

$$= -\hat{y}\frac{j\beta}{h}\frac{A_n}{2}\left[\exp(jhy - j\beta z) + \exp(-jhy - j\beta z)\right] \quad (4b)$$

$$\hat{x}\eta H_x(y,z) = \hat{x}\frac{jk}{h}A_n\cos(hy)\exp(-j\beta z)$$
$$= \hat{x}\frac{jk}{h}\frac{A_n}{2}\left[\exp(jhy - j\beta z) + \exp(-jhy - j\beta z)\right] \quad (4c)$$

where $\beta = \sqrt{k^2 - h^2}$ and $k = \omega\sqrt{\mu\epsilon}$. Fields inside the waveguide may be considered as combination of two TEM plane waves bouncing back and forth obliquely between the two conducting plates as shown in Figure 2. That is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \quad (5a)$$
$$\eta\mathbf{H} = \eta\mathbf{H}_1 + \eta\mathbf{H}_2 \quad (5b)$$

where $(\mathbf{E}_1, \mathbf{H}_1)$ are the electric and magnetic fields associated with one plane wave and are given below

$$\mathbf{E}_1 = \frac{A_n}{2}\left(-j\hat{z} - \frac{j\beta}{h}\hat{y}\right)\exp(jhy - j\beta z) \quad (6a)$$
$$\eta\mathbf{H}_1 = \hat{x}\frac{jk}{h}\frac{A_n}{2}\exp(jhy - j\beta z) \quad (6b)$$

while electric and magnetic fields $(\mathbf{E}_2, \mathbf{H}_2)$ associated with second plane wave and are given below

$$\mathbf{E}_2 = \frac{A_n}{2}\left(j\hat{z} - \frac{j\beta}{h}\hat{y}\right)\exp(-jhy - j\beta z) \quad (7a)$$
$$\eta\mathbf{H}_2 = \hat{x}\frac{jk}{h}\frac{A_n}{2}\exp(-jhy - j\beta z) \quad (7b)$$

Fields $\mathbf{E}_1$ and $\mathbf{H}_1$ given by equations (6) are related through the Maxwell equations as

$$\nabla \times \mathbf{E}_1 = -j\omega\mu\mathbf{H}_1$$
$$j(h\hat{y} - \beta\hat{z}) \times \mathbf{E}_1 = -j\omega\mu\mathbf{H}_1$$
$$\frac{1}{(jk)}(-jh\hat{y} + j\beta\hat{z}) \times \mathbf{E}_1 = \eta\mathbf{H}_1$$
$$\mathbf{k}_1 \times \mathbf{E}_1 = \eta\mathbf{H}_1 \quad (8a)$$

Similarly

$$\frac{1}{(jk)}(-jh\hat{y} + j\beta\hat{z}) \times \eta\mathbf{H}_1 = -\mathbf{E}_1$$
$$\mathbf{k}_1 \times \eta\mathbf{H}_1 = -\mathbf{E}_1 \quad (8b)$$

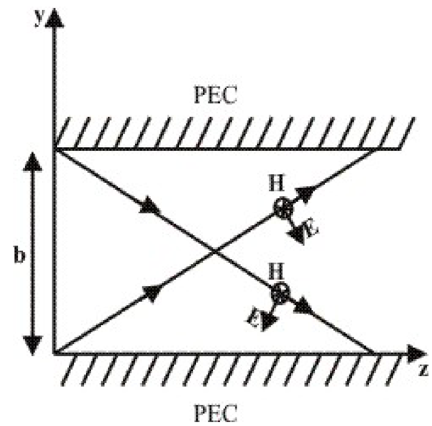

Figure 2. TM mode propagating in a PEC parallel plates waveguide, $\alpha = 0$, $\alpha = 4$.

where $\mathbf{k}_1 = \frac{1}{(jk)}(-jh\hat{y} + j\beta\hat{z})$.

Fields $\mathbf{E}_2$ and $\mathbf{H}_2$ given by equations (7) are also related through Maxwell equation as given below

$$\begin{aligned}\nabla \times \mathbf{E}_2 &= -j\omega\mu\mathbf{H}_2\\ -j(h\hat{y} + \beta\hat{z}) \times \mathbf{E}_2 &= -j\omega\mu\mathbf{H}_2\\ \frac{1}{(jk)}(jh\hat{y} + j\beta\hat{z}) \times \mathbf{E}_2 &= \eta\mathbf{H}_2\\ \mathbf{k}_2 \times \mathbf{E}_2 &= \eta\mathbf{H}_2 \end{aligned} \tag{9a}$$

Similarly

$$\begin{aligned}\frac{1}{(jk)}(jh\hat{y} + j\beta\hat{z}) \times \eta\mathbf{H}_2 &= -\mathbf{E}_2\\ \mathbf{k}_2 \times \eta\mathbf{H}_2 &= -\mathbf{E}_2 \end{aligned} \tag{9b}$$

where $\mathbf{k}_2 = \frac{1}{(jk)}(jh\hat{y} + j\beta\hat{z})$. It may be noted that $|\mathbf{k}_1| = |\mathbf{k}_2|$. It may also be deduced from above expressions that for set of fields $(\mathbf{E}_1, \mathbf{H}_1)$, the operator $\left(\frac{1}{jk_1}\nabla\times\right)$ is equivalent to cross product operator $(\mathbf{k}_1\times)$ while for set of fields $(\mathbf{E}_2, \mathbf{H}_2)$, the operator $\left(\frac{1}{jk_2}\nabla\times\right)$ is equivalent to cross product operator given by $(\mathbf{k}_2\times)$. It is also obvious that if $(\mathbf{E}_1, \eta\mathbf{H}_1)$ is one set of solutions to Maxwell's equation then other set of solutions to Maxwell's equations is $(\eta\mathbf{H}_1, -\mathbf{E}_1)$. Similarly if $(\mathbf{E}_2, \eta\mathbf{H}_2)$ is one set of solutions to Maxwell's equation then other set of solutions to Maxwell's equations is $(\eta\mathbf{H}_2, -\mathbf{E}_2)$. Our interest is to determine the fields which may be regarded as intermediate step of the field $(\mathbf{E}, \eta\mathbf{H})$ and $(\eta\mathbf{H}, -\mathbf{E})$, that is, new set of solutions $(\mathbf{E}_{\rm fd}, \eta\mathbf{H}_{\rm fd})$. For this purpose solutions sets $(\mathbf{E}_{ifd}, \eta\mathbf{H}_{ifd})$ with $i = 1, 2$ are required. $(\mathbf{E}_{ifd}, \eta\mathbf{H}_{ifd})$ may be obtained by using the following relations

$$\mathbf{E}_{ifd} = \frac{1}{(jk)^\alpha}\left[(\nabla\times)^\alpha \mathbf{E}_i\right] \tag{10a}$$

$$\eta\mathbf{H}_{ifd} = \frac{1}{(jk)^\alpha}\left[(\nabla\times)^\alpha \eta\mathbf{H}_i\right], \qquad i = 1, 2 \tag{10b}$$

Solutions $(\mathbf{E}_{\rm fd}, \eta\mathbf{H}_{\rm fd})$ may be obtained by linear combination of $(\mathbf{E}_{\rm 1fd}, \eta\mathbf{H}_{\rm 1fd})$ and $(\mathbf{E}_{\rm 2fd}, \eta\mathbf{H}_{\rm 2fd})$, that is

$$\mathbf{E}_{\rm fd} = \mathbf{E}_{\rm 1fd} + \mathbf{E}_{\rm 2fd} \tag{11a}$$

$$\eta\mathbf{H}_{\rm fd} = \eta\mathbf{H}_{\rm 1fd} + \eta\mathbf{H}_{\rm 2fd} \tag{11b}$$

In order to determine the fractional dual solutions $(\mathbf{E}_{ifd}, \eta\mathbf{H}_{ifd})$, the eigenvalues and eigenvectors of the two cross product operators $(\mathbf{k}_1\times, \mathbf{k}_2\times)$ are required. Eigenvectors and eigenvalues of the operator $(\mathbf{k}_1\times)$ are

$$\mathbf{A}_1 = \frac{1}{\sqrt{2}}\left[\hat{x} - j\frac{\beta}{k}\hat{y} - j\frac{h}{k}\hat{z}\right], \qquad a_1 = j$$

$$\mathbf{A}_2 = \frac{1}{\sqrt{2}}\left[\hat{x} + j\frac{\beta}{k}\hat{y} + j\frac{h}{k}\hat{z}\right], \qquad a_2 = -j$$

$$\mathbf{A}_3 = -j\frac{h}{k}\hat{y} + j\frac{\beta}{k}\hat{z}, \qquad a_3 = 0$$

Fields $(\mathbf{E}_1, \mathbf{H}_1)$ may be expressed in terms of the eigenvectors of the operator, that is

$$\mathbf{E}_1 = [P\mathbf{A}_1 + Q\mathbf{A}_2 + R\mathbf{A}_3]\exp(jhy - j\beta z) \tag{12}$$

The coefficients are given below

$$P = \frac{A_n}{2\sqrt{2}}\frac{k}{h}$$

$$Q = -\frac{A_n}{2\sqrt{2}}\frac{k}{h}$$

$$R = 0$$

The expression for $\mathbf{E}_{\rm fd}$ is obtained by applying fractional curl operator on vector $\mathbf{E}$. Fractionalization of curl operator means fractionalization of the equivalent cross product operator. Fractionalization of cross product operator means fractionalization of eigenvalues of the operator. Fractionalizing the eigenvalues of the operator yields

$$\mathbf{E}_{1\mathrm{fd}} = [(a_1)^\alpha P\mathbf{A}_1 + (a_2)^\alpha Q\mathbf{A}_2 + (a_3)^\alpha R\mathbf{A}_3] \exp(jhy - j\beta z) \qquad (13)$$

Solutions to the Maxwell equations, which may be regarded as intermediate step between the solutions set $(\mathbf{E}_1, \eta\mathbf{H}_1)$ and solutions set $(\eta\mathbf{H}_1, -\mathbf{E}_1)$ are given by

$$\begin{aligned}\mathbf{E}_{1\mathrm{fd}} &= (\mathbf{k}_1\times)^\alpha\, \mathbf{E}_1\\ &= \frac{A_n}{2}\frac{k}{h}\left[j\sin\left(\frac{\alpha\pi}{2}\right)\hat{x} - j\frac{\beta}{k}\cos\left(\frac{\alpha\pi}{2}\right)\hat{y}\right.\\ &\qquad \left.-j\frac{h}{k}\cos\left(\frac{\alpha\pi}{2}\right)\hat{z}\right]\exp(jhy - j\beta z) \qquad (14\mathrm{a})\\ \eta\mathbf{H}_{1\mathrm{fd}} &= (\mathbf{k}_1\times)^\alpha\, \eta\mathbf{H}_1\\ &= \frac{A_n}{2}\frac{k}{h}\left[j\cos\left(\frac{\alpha\pi}{2}\right)\hat{x} + j\frac{\beta}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{y} + j\frac{h}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{z}\right]\\ &\qquad \times\exp(jhy - j\beta z) \qquad (14\mathrm{b})\end{aligned}$$

Which shows that electric field vector and magnetic field vectors are rotated counterclockwise by an angle $\alpha\pi/2$.

Eigenvectors and eigenvalues of the operator $(\mathbf{k}_2\times)$ are

$$\begin{aligned}\mathbf{A}_1 &= \frac{1}{\sqrt{2}}\left[\hat{x} - j\frac{\beta}{k}\hat{y} + j\frac{h}{k}\hat{z}\right], && a_1 = j\\ \mathbf{A}_2 &= \frac{1}{\sqrt{2}}\left[\hat{x} + j\frac{\beta}{k}\hat{y} - j\frac{h}{k}\hat{z}\right], && a_2 = -j\\ \mathbf{A}_3 &= j\frac{h}{k}\hat{y} + j\frac{\beta}{k}\hat{z}, && a_3 = 0\end{aligned}$$

Fields may be expressed in terms of eigenvectors of the operator $\mathbf{k}_2\times$

$$\mathbf{E}_2 = [P\mathbf{A}_1 + Q\mathbf{A}_2 + R\mathbf{A}_3]\exp(-jhy - j\beta z) \qquad (15)$$

where coefficients are

$$\begin{aligned}P &= \frac{A_n}{2\sqrt{2}}\frac{k}{h}\\ Q &= -\frac{A_n}{2\sqrt{2}}\frac{k}{h}\\ R &= 0\end{aligned}$$

Solutions to the Maxwell equations, which may be regarded as intermediate step between the solutions set $(\mathbf{E}_2, \eta\mathbf{H}_2)$ and solutions set $(\eta\mathbf{H}_2, -\mathbf{E}_2)$ are given by

$$\begin{aligned}
\mathbf{E}_{2\mathrm{fd}} &= (\mathbf{k}_2\times)^{\alpha}\,\mathbf{E}_2 \\
&= \frac{A_n}{2}\frac{k}{h}\exp\left(-j\alpha\pi\right)\left[-j\sin\left(\frac{\alpha\pi}{2}\right)\hat{x}-j\frac{\beta}{k}\cos\left(\frac{\alpha\pi}{2}\right)\hat{y}\right. \\
&\quad \left.+j\frac{h}{k}\cos\left(\frac{\alpha\pi}{2}\right)\hat{z}\right]\exp(-jhy-j\beta z) \qquad (16a) \\
\eta\mathbf{H}_{2\mathrm{fd}} &= (\mathbf{k}_2\times)^{\alpha}\,\eta\mathbf{H}_2 \\
&= \frac{A_n}{2}\frac{k}{h}\exp\left(-j\alpha\pi\right)\left[j\cos\left(\frac{\alpha\pi}{2}\right)\hat{x}-j\frac{\beta}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{y}\right. \\
&\quad \left.+j\frac{h}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{z}\right]\exp(-jhy-j\beta z) \qquad (16b)
\end{aligned}$$

Which shows that electric field vector and magnetic field vectors are rotated counter-clockwise by an angle $\alpha\pi/2$.

Solutions to the Maxwell equations, which may be regarded as intermediate step between the solutions set $(\mathbf{E},\eta\mathbf{H})$ and solutions set $(\eta\mathbf{H},-\mathbf{E})$ may be obtained by substituting results by (14) and (16) in (11) and are given below

$$\begin{aligned}
\mathbf{E}_{\mathrm{fd}} &= A_n\frac{k}{h}\exp\left(-j\frac{\alpha\pi}{2}\right)\left[-\sin\left(\frac{\alpha\pi}{2}\right)\sin\left(hy+\frac{\alpha\pi}{2}\right)\hat{x}\right. \\
&\quad \left.-j\frac{\beta}{k}\cos\left(\frac{\alpha\pi}{2}\right)\cos\left(hy+\frac{\alpha\pi}{2}\right)\hat{y}+\frac{h}{k}\cos\left(\frac{\alpha\pi}{2}\right)\sin\left(hy+\frac{\alpha\pi}{2}\right)\hat{z}\right] \\
&\quad \times\exp(-j\beta z) \qquad (17a) \\
\eta\mathbf{H}_{\mathrm{fd}} &= A_n\frac{k}{h}\exp\left(-j\frac{\alpha\pi}{2}\right)\left[j\cos\left(\frac{\alpha\pi}{2}\right)\cos\left(hy+\frac{\alpha\pi}{2}\right)\hat{x}\right. \\
&\quad \left.-\frac{\beta}{k}\sin\left(\frac{\alpha\pi}{2}\right)\sin\left(hy+\frac{\alpha\pi}{2}\right)\hat{y}+j\frac{h}{k}\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(hy+\frac{\alpha\pi}{2}\right)\hat{z}\right] \\
&\quad \times\exp(-j\beta z) \qquad (17b)
\end{aligned}$$

It may be noted that for $\alpha=0$

$$\begin{aligned}
\mathbf{E}_{\mathrm{fd}} &= A_n\left[-j\frac{\beta}{h}\cos(hy)\hat{y}+\sin(hy)\hat{z}\right]\exp(-j\beta z)=\mathbf{E} \\
\eta\mathbf{H}_{\mathrm{fd}} &= A_n\left[j\frac{k}{h}\cos(hy)\hat{x}\right]\exp(-j\beta z)=\eta\mathbf{H}
\end{aligned}$$

Which represents a TM mode propagating along a PEC parallel plate waveguide as given by equation (4). For $\alpha=1$

$$\mathbf{E}_{\mathrm{fd}} = A_n\left[j\frac{k}{h}\cos(hy)\hat{x}\right]\exp(-j\beta z)=\eta\mathbf{H}$$

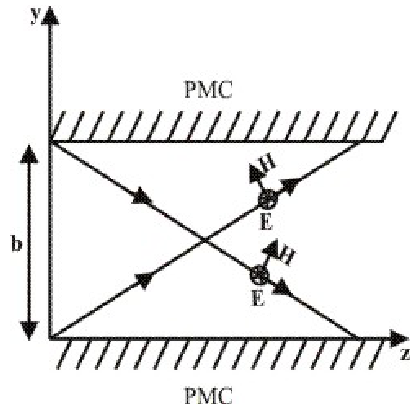

Figure 3. TE mode propagating in a PMC parallel plates waveguide, $\alpha = 1$.

$$\eta \mathbf{H}_{\text{fd}} \;=\; A_n \left[j\frac{\beta}{h}\cos(hy)\hat{y} - \sin(hy)\hat{z} \right] \exp(-j\beta z) = -\mathbf{E}$$

Which shows that electric field vector and magnetic field vectors are rotated counterclockwise by an angle $\pi/2$ and tangential components of magnetic field are zero at $y = 0$ and $y = b$ as shown in Figure 3. Which represents a TE mode propagating along a PMC parallel plate waveguide and new situation is dual to the equations given in (4). For $0 < \alpha < 1$, it may be considered as parallel plate waveguide which is of intermediate step of PEC and PMC plates. The propagating mode contains both electric and magnetic fields in the axial direction. This means that fractional dual solutions connect two waveguides which are related through principle of duality and give solutions to waveguides which are intermediate step between the two waveguides. The waveguides corresponding to these solutions may be termed as fractional dual waveguide.

For $1 < \alpha < 2$, $2 < \alpha < 3$ and $3 < \alpha < 4$, there is further rotation of electric and magnetic field vectors as for range of $0 < \alpha < 1$. The solutions for $\alpha = 2$ are

$$\mathbf{E}_{\text{fd}} \;=\; A_n \left[j\frac{\beta}{h}\cos(hy)\hat{y} - \sin(hy)\hat{z} \right] \exp(-j\beta z) = -\mathbf{E}$$

$$\eta \mathbf{H}_{\text{fd}} \;=\; -A_n \left[j\frac{k}{h}\cos(hy)\hat{x} \right] \exp(-j\beta z) = -\eta \mathbf{H}$$

and the walls of the waveguide are PEC. The solutions for $\alpha = 3$ are

$$\mathbf{E}_{\text{fd}} \;=\; -A_n \left[j\frac{k}{h}\cos(hy)\hat{x} \right] \exp(-j\beta z) = -\eta \mathbf{H}$$

$$\eta\mathbf{H}_{\text{fd}} = A_n\left[-j\frac{\beta}{h}\cos(hy)\hat{y}+\sin(hy)\hat{z}\right]\exp(-j\beta z)=\mathbf{E}$$

and the walls of the waveguide are PMC. The solutions for $\alpha = 4$ are

$$\mathbf{E}_{\text{fd}} = A_n\left[-j\frac{\beta}{h}\cos(hy)\hat{y}+\sin(hy)\hat{z}\right]\exp(-j\beta z)=\mathbf{E}$$

$$\eta\mathbf{H}_{\text{fd}} = A_n\left[j\frac{k}{h}\cos(hy)\hat{x}\right]\exp(-j\beta z)=\eta\mathbf{H}$$

with walls are PEC. The behaviors are shown in Figures 3 to Figure 5.

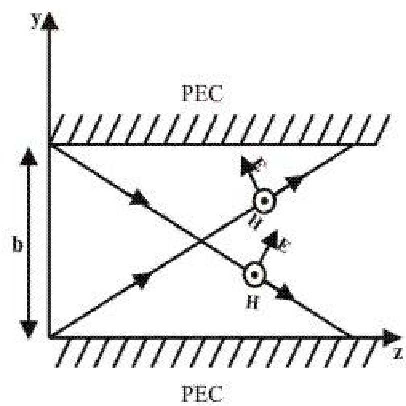

Figure 4. TM mode propagating in a PEC parallel plates waveguide, $\alpha = 2$.

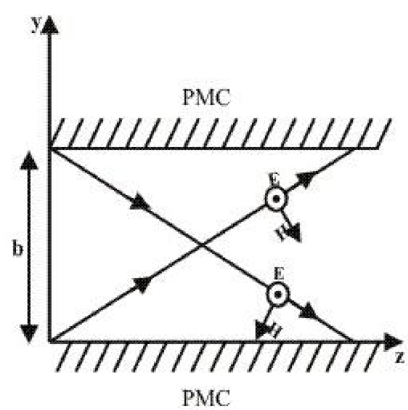

Figure 5. TE mode propagating in a PMC parallel plates waveguide, $\alpha = 3$.

It is obvious from the results that behavior of solutions with respect to fractional parameter is periodic with period 4. It may be noted that results are consistent with the published work [3, 10].

It is obvious from equation (17) that surface impedance of the walls of fractional waveguide is

$$Z_{\text{walls}} = -\frac{E_{xfd}}{H_{zfd}} = \frac{E_{zfd}}{H_{xfd}} = \eta j \tan\left(hy + \frac{\alpha\pi}{2}\right) \tag{18}$$

This means that for $\alpha = 0$, walls of the waveguide are of perfect electric conductors while for $\alpha = 1$, walls of the waveguide are of perfect magnetic conductor. For non integer values of the fractional parameter α, the walls of waveguide may be regarded as intermediate step between the perfect electric conductor and perfect magnetic conductor.

Fractional dual waveguide is a waveguide which carries field as a combination of two plane waves given by expressions (14) and (16). These plane waves are propagating in z-direction while bouncing back and forth obliquely between the two plates. The impedance of the plates is given by expression (18) and is a function of fractional parameter α. It may be noted that for integer values of the fractional parameter α, electric and magnetic fields propagate in the form of either TM or TE-modes and for non-integer values of α, they propagate in the form of hybrid modes. The behaviors are consistent with the previous work [3].

4. BEHAVIOR OF FIELD LINES INSIDE THE FRACTIONAL WAVEGUIDE

In this section our interest is to study the behavior of field lines with respect to fractional parameter α. We have selected yz-plane as observation plane. The instantaneous field expressions are obtained by multiplying the phasor vector expressions (17a) with $\exp(j\omega t)$ and taking the real part of the product. In the yz-plane of the waveguide, $\mathbf{E}_{\text{fd}}$ has both y- and z-components. That is

$$\begin{aligned} E_{yfd} &= -A_n \frac{\beta}{h} \cos\left(\frac{\alpha\pi}{2}\right) \cos\left(hy + \frac{\alpha\pi}{2}\right) \sin\left(\beta z + \frac{\alpha\pi}{2} - \omega t\right) \\ E_{zfd} &= A_n \cos\left(\frac{\alpha\pi}{2}\right) \sin\left(hy + \frac{\alpha\pi}{2}\right) \cos\left(\beta z + \frac{\alpha\pi}{2} - \omega t\right) \end{aligned}$$

Equation describing the behavior of electric field lines at a given t can be found from the following relation.

$$\frac{dy}{E_{yfd}} = \frac{dz}{E_{zfd}}$$

For example, at $t = 0$,

$$-\beta\cos\left(\frac{\alpha\pi}{2}\right)\frac{\sin\left(\beta z+\frac{\alpha\pi}{2}\right)}{\cos\left(\beta z+\frac{\alpha\pi}{2}\right)}dz = h\cos\left(\frac{\alpha\pi}{2}\right)\frac{\sin\left(hy+\frac{\alpha\pi}{2}\right)}{\cos\left(hy+\frac{\alpha\pi}{2}\right)}dy$$

Integration gives

$$-\ln\cos\left(\beta z+\frac{\alpha\pi}{2}\right) = \ln\cos\left(hy+\frac{\alpha\pi}{2}\right)+c_1$$

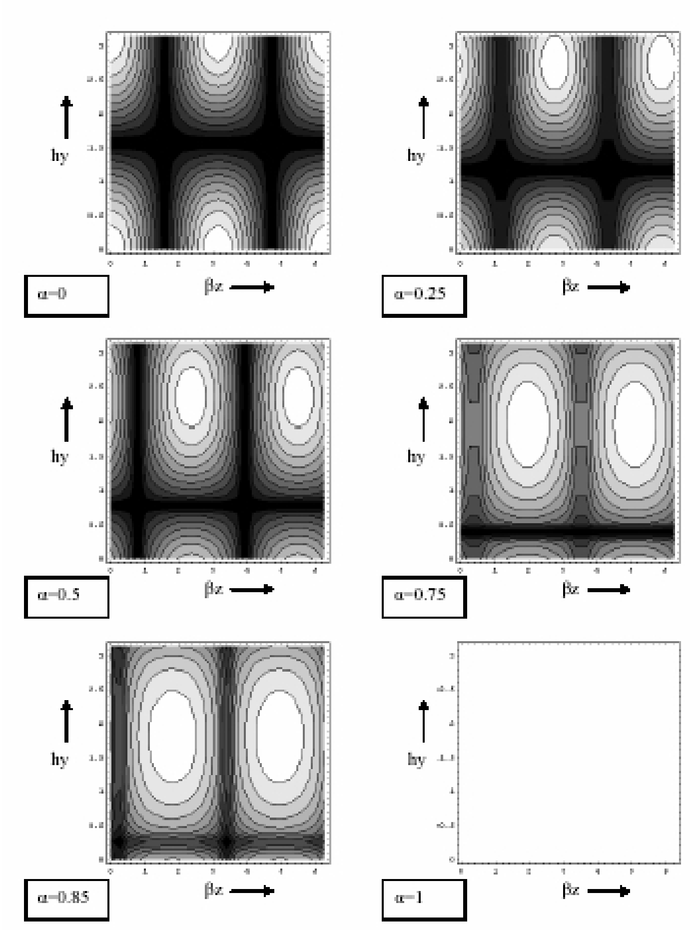

Figure 6. Electric field (E_{fd}) lines behavior at different values of α.

or

$$\left[\cos\left(\beta z + \frac{\alpha\pi}{2}\right)\cos\left(hy + \frac{\alpha\pi}{2}\right)\right] = c_2, \qquad 0 < \alpha < 1 \qquad (19)$$

where c_1 and c_2 are constants. Similarly from $\mathbf{H}_{\rm fd}$, we have

$$H_{yfd} = -A_n \frac{\beta}{h}\sin\left(\frac{\alpha\pi}{2}\right)\sin\left(hy + \frac{\alpha\pi}{2}\right)\cos\left(\beta z + \frac{\alpha\pi}{2} - \omega t\right)$$
$$H_{zfd} = A_n \sin\left(\frac{\alpha\pi}{2}\right)\cos\left(hy + \frac{\alpha\pi}{2}\right)\sin\left(\beta z + \frac{\alpha\pi}{2} - \omega t\right)$$

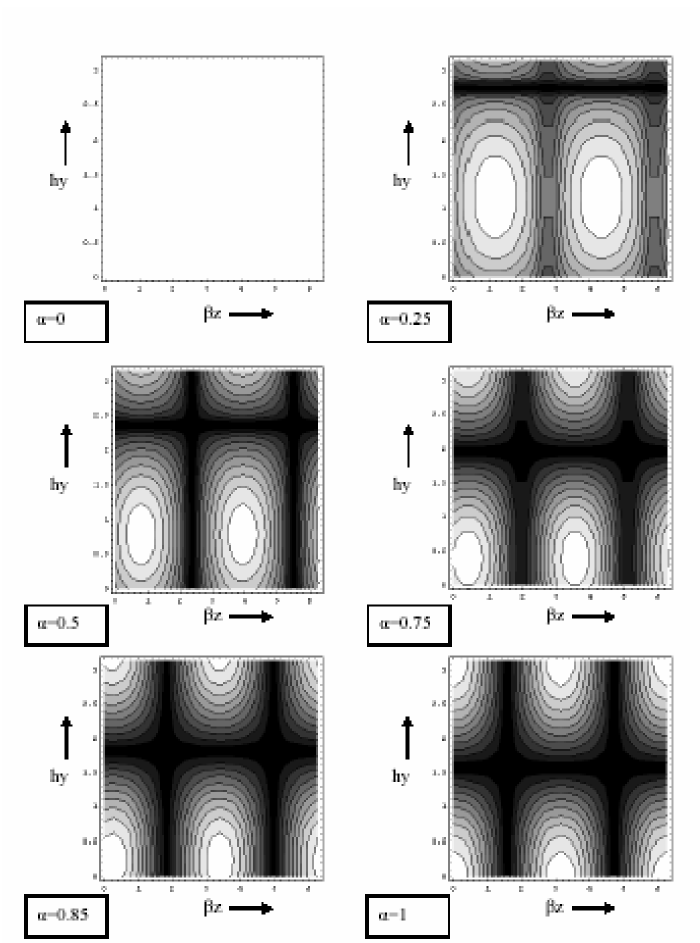

Figure 7. Magnetic field ($H_{\rm fd}$) lines behavior at different values of α.

Field lines for $\mathbf{H}_{\mathrm{fd}}$ in yz-plane may be obtained using the following relations

$$\left[\sin\left(\beta z+\frac{\alpha\pi}{2}\right)\sin\left(hy+\frac{\alpha\pi}{2}\right)\right]=c_3,\qquad 0<\alpha<1 \qquad (20)$$

where c_3 is constant. Using expressions (19) and (20) field lines are plotted as given in Figure 6 and Figure 7. It is observed from the plots that for $\alpha = 0$, there is no contribution of field lines due to H_{fd} while E_{fd} lines are contributing. It is also noted that for $\alpha = 0$, E_{fd} lines are normal to the interfaces. Which shows that walls of the waveguide may be considered of perfect electric conductor. For $\alpha = 1$, H_{fd} lines are contributing while E_{fd} lines disappear and H_{fd} lines are normal to the interfaces or tangential components of the field lines are zero at the boundaries of the parallel plates waveguide. Which shows that walls of the waveguide may be considered of perfect magnetic conductors. For $0 < \alpha < 1$, both H_{fd} and E_{fd} lines are contributing. For non-integer values of the fractional parameter α, both the field lines contain tangential as well as normal components at the boundaries. Which shows that walls of the waveguide are of intermediate step between the perfect electric conductor and perfect magnetic conductor.

5. CONCLUSIONS

It is noted that for $\alpha = 0$, we are dealing with a perfect electric conducting parallel plate waveguide carrying TM mode in z-direction. This may be interpreted as the superposition of two plane waves bouncing back and forth obliquely between the two conducting plates. As fractional parameter α takes values from zero towards unity, there are two activities happening. One is the electric and magnetic field vectors are being rotated by an angle $\alpha\pi/2$ in the counterclockwise direction. Other is that perfect electric conductor is changing to perfect magnetic conductor. That is parallel plates are intermediate step of PEC and PMC conductor. As α becomes equal to unity rotation angle becomes equal to $\pi/2$ and PEC parallel plates waveguide changes to PMC parallel plates waveguide. The behavior of solutions and properties of guiding structure has also been studied for values of fractional parameter greater than unity. It is found that solutions and properties of guiding structure are periodic with respect to fractional parameter with period 4.

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