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A COMBINED METHOD OF AUXILIARY SOURCES-REACTION MATCHING APPROACH FOR ANALYZING MODERATELY LARGE-SCALE ARRAYS OF CYLINDRICAL DIPOLES

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Abstract—In the present paper, a combined method of auxiliary sources (MAS)-reaction matching (RM) approach is presented for the analysis of arrays of arbitrarily located cylindrical dipoles. It is shown that the addition of auxiliary monopole terminal sources to each array element results in a superior solution with regard to the numerical stability of the computed quantities, the behavior of the current distributions of the array elements and the resulting errors of the electric field boundary condition. Numerical results are presented for various representative array configurations, in order to illustrate the features of the proposed method and exhibit its advantages over conventional Method of Moments (MoM) schemes, especially in cases of moderately large-scale arrays. Finally, a few concluding remarks are discussed.

1. INTRODUCTION

Arrays of parallel cylindrical dipoles are probably the most widely spread and the most exhaustively studied antenna arrays during the past decades. The reasons behind this fact are the simplicity in their realization, the existence of approximate analytical and numerical methods for the accurate prediction of their behavior, as well as their suitability to a large variety of practical applications. It is well known that the current distributions along the elements of any dipole array are governed by coupled integro-differential equations or equivalent integral equations, which are very difficult to cope with, especially when the array configuration is not characterized by symmetries.

Approximate analytical methods have been derived for the analysis of various regular arrays (such as linear, collinear, circular, etc), which have been summarized in a recent textbook [1]. However, in practice, wire antenna arrays are usually analyzed, designed and optimized by applying the method of moments (MoM), which has been extensively developed by numerous researchers in several variants. Nevertheless, significant discrepancies are usually encountered in the solutions obtained from different numerical methods or variants of the same method (e.g., MoM with different basis/testing functions) [2–5]. These are related to modeling inadequacies and differences in the degree to which each method has reached numerically stable results. Moreover, the numerical methods are usually accompanied by innate difficulties in the convergence of their solutions. This fact is an outcome of the non-solvability of the coupled equations discussed above when the approximate kernel is used. In particular, the failure of any method based on the approximate kernel to reach the exact solution in the case of an isolated tubular dipole has been elucidated in [6, 7] from a mixed mathematical and practical point of view. Apart from this fact, the matrix equations obtained from numerical methods when applied to such problems are highly susceptible to ill-conditioning, which restricts severely the number of expansion terms. It is worth mentioning that the latter is caused by round-off errors, and is not related to solvability issues.

In this paper, a method that is based on the combination of the method of auxiliary sources (MAS) with reaction matching (RM) is presented for the analysis of arrays of parallel cylindrical dipoles. The proposed method is an extension of the method presented in [8] for a single dipole. The MAS is a well-established numerical method, which has been recently recognized to be a branch of the generalized multipole techniques (GMTs) [9, 10]. Herein, as in [8], the RM [11] is utilized in conjunction with the MAS, which leads to the adoption of the initials MAS-RM. Although the resulting matrix equation is similar to that obtained from the MoM with piecewise sinusoidal (PWS) basis and testing functions, there are critical differences between the two methods, which are discussed hereinafter. As it will be shown, the proposed method is capable of analyzing large-scale arrays using very low discreteness densities, without significant degradation of the solution quality, which is in contrast to the well-known trade-off between the solution accuracy and the computational cost that accompanies conventional matrix methods.

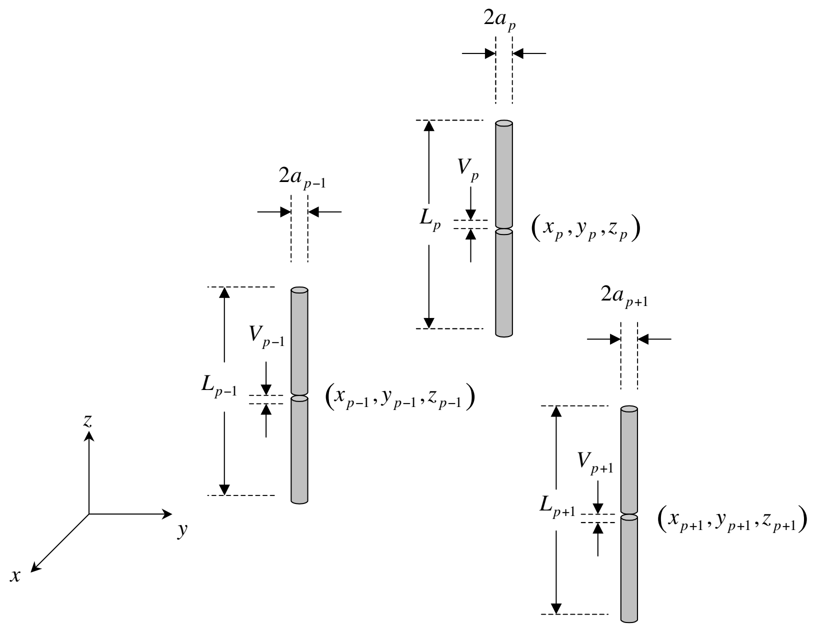


Figure 1. General arrangement of arbitrarily located parallel dipoles.

2. PROBLEM STATEMENT

Consider an arrangement of N_d parallel cylindrical dipoles with their centers located at (x_p, y_p, z_p), where $p = 1, 2, \ldots, N_d$, as shown in Fig. 1. The length and wire radius of each dipole are denoted by L_p and a_p, respectively. For the sake of convenience, the dipoles are assumed to be oriented along the z-axis of a coordinates system, as also depicted in Fig. 1. Moreover, an exponential $\exp(j\omega t)$ time dependent factor is assumed and suppressed throughout the analysis. Each dipole is driven either symmetrically or asymmetrically by a voltage generator of strength V_p, which imposes a driving field that is given by

$$E_p^g(z) = -V_p \delta(z - z_p^g), \tag{1}$$

where z_p^g is the location of the voltage generator. Apparently, parasitic elements can be modeled by setting $V_p = 0$.

3. FORMULATION

Although the fundamental ideas behind the MAS are quite simple, there are many important aspects, such as the type of the auxiliary sources, the locations of the auxiliary sources and matching points, as well as the correlation between them and the convergence behavior of the solution, that render the application of the method nontrivial [10].

With regard to the case under consideration, each element is modeled with the aid of an axially distributed set of auxiliary sources. Such a distribution of the auxiliary sources is proper only when the elements are sufficiently thin (compared to the operating wavelength). For reasons that have been reported elsewhere [8, 12], the auxiliary sources are selected to be sinusoidal dipoles, instead of elementary dipoles that are usually employed with the MAS [10]. The auxiliary dipoles are spatially overlapped in a manner that the center of each one is placed exactly at the ends of the two adjacent ones. Moreover, as in [8], two auxiliary monopoles are also introduced in each element, which are located in such a way that their inner ends coincide with the centers of the outer auxiliary dipoles and their outer ends are situated exactly at $z = z_p \pm L_p/2$.

In the following, the analysis is focused on the comparison between the solutions resulting in the presence and absence of the terminal monopoles. In accordance with the conventional model, $2N_p + 1$ auxiliary dipoles are located inside the element denoted by p, which are centered at positions $z_p + n\delta_p$, $n = 0, \pm 1, \ldots, \pm N_p$, with δ_p designating the distance between the centers of two adjacent auxiliary dipoles of length $2\delta_p = L_p/(N_p + 1)$. With regard to the improved model discussed above, two properly oriented auxiliary monopoles of length δ_p are added. All the auxiliary sources carry sinusoidal currents of unknown amplitudes $w_{(p,n)}$, which can be expressed as $w_{(p,n)} f_{(p,n)}(z) s_{(p,n)}(z)$, where $f_{(p,n)}(z)$ are properly shaped PWS functions and $s_{(p,n)}(z)$ are corrective functions that form the currents of the terminal monopoles, given by

$$f_{(p,n)}(z) = \begin{cases} \sin[k_0(\delta_p - |z - z_p - n\delta_p|)], & |z - z_p - n\delta_p| \leq \delta_p, \\ 0, \text{ elsewhere} \end{cases} \tag{2}$$

$$s_{(p,n)}(z) = \begin{cases} 1, & |n| \leq N_p \\ u(L_p/2 \mp z \pm z_p), & n = \pm(N_p + 1), \end{cases}$$

$$\text{where } u(z) = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0 \end{cases}, \tag{3}$$

where $k_0 = 2\pi/\lambda$ is the wavenumber of the vacuum, while $u(z)$ is the well known step function.

The unknown EM field at any observation point is expressed as the superimposition of the EM fields produced by the auxiliary sources, thus

$$\vec{F}(\vec{r}) = \sum_{p=1}^{N_d} \sum_{n=-(N_p+1)}^{N_p+1} w_{(p,\,n)} \vec{F}_{(p,\,n)}(\vec{r}). \tag{4}$$

The field vector $\vec{F}$ stands for the total radiated electric $\vec{E}$ or magnetic field $\vec{H}$, while $\vec{F}_{(p,\,n)}$ is the corresponding field radiated by the auxiliary source denoted by the subscript pair $(p,\, n)$. The latter is obtained by properly shifting the EM field of a sinusoidal dipole or monopole, as follows

$$\vec{F}_{(p,\,n)}(x,\, y,\, z) = \begin{cases} \vec{F}_D(x - x_p,\, y - y_p,\, z - z_p - n\delta_p), & |n| \leq N_p \\ \vec{F}_M^{\mp}(x - x_p,\, y - y_p,\, z - z_p - n\delta_p), & n = \pm(N_p+1) \end{cases}. \tag{5}$$

In (5), the vector $\vec{F}_D$ stands for the EM field generated by a sinusoidal dipole, while the vectors $\vec{F}_M^+$ and $\vec{F}_M^-$ stand for the EM fields generated by the upper and lower half of a sinusoidal dipole, respectively, which correspond to the positive and negative monopoles described in [8].

The unknown weighting coefficients entering into (4) are derived by enforcing the electric field boundary condition associated with each one of the elements composing the array under consideration; therefore

$$E_z(x_q + a_q \cos\gamma_q,\, y_q + a_q \sin\gamma_q,\, z) = E_q^g(z), \quad |z - z_q| \leq \frac{L_q}{2}, \tag{6}$$

where γ_q is the angle of azimuth as measured with respect to the element axis denoted by q. Following a procedure similar to that presented in [8], the boundary condition of (6) is enforced in the RM sense. The latter consists in the multiplication of both sides of (6) with a projection of the sinusoidal current distribution of each auxiliary source onto the surface of each element and integration of the product over the corresponding interval [8, 11]. Repetition of this procedure for all the auxiliary sources leads to an algebraic system of equations given by

$$\sum_{p=1}^{N_d} \sum_{n=-(N_p+1)}^{N_p+1} Z_{(p,\,n),\,(q,\,m)} w_{(p,\,n)} = -V_{(q,\,m)}, \begin{cases} q = 1,\, 2,\, \ldots,\, N_d \\ m = 0,\, \pm 1,\, \ldots,\, \pm N_q,\, \pm(N_q + 1) \end{cases}. \tag{7}$$

If the terminal monopoles are not taken into account, the algebraic system of (7) is expressed as

$$\sum_{p=1}^{N_d} \sum_{n=-N_p}^{N_p} Z_{(p,\,n),\,(q,\,m)} w_{(p,\,n)} = -V_{(q,\,m)}, \left\{ \begin{array}{c} q = 1,\, 2,\, \ldots,\, N_d \\ m = 0,\, \pm 1,\, \ldots,\, \pm N_q \end{array} \right. . \tag{8}$$

In both cases, the symbol $Z_{(p,\,n),\,(q,\,m)}$ designates the reaction integral, which is expressed by means of the mutual impedance $Z(s,\,h)$ between two parallel in echelon sinusoidal sources that are separated horizontally by s and vertically by h, as

$$Z_{(p,\,n),\,(q,\,m)} = \left\{ \begin{array}{c} Z\left(\sqrt{(x_q - x_p)^2 + (y_q - y_p)^2},\; |z_q - z_p + m\delta_q - n\delta_p|\right),\; q \neq p \\ Z(a_q,\; |m - n|\delta_q),\; q = p \end{array} \right. . \tag{9}$$

When $q \neq p$, the reaction interval lies on the element axis rather than on its surface. The computation of the reaction terms involves the mutual impedance between two sinusoidal sources, which may be two dipoles, a monopole and a dipole, or two monopoles. These terms are computable using either numerical integration algorithms or available closed form expressions [13, 14]. The voltages $V_{(q,\,m)}$ in (7) and (8) denote the reaction integrals of the driving fields, which are given by

$$V_{(q,\,m)} = \left\{ \begin{array}{cc} -V_q \sin[k_0(\delta_p - |z_q^g - z_q - m\delta_q|)], & |z_q^g - z_q - m\delta_q| \leq \delta_q \\ 0,\; \text{elsewhere} & \end{array} \right. . \tag{10}$$

After solving the linear system of (7) or (8), the EM field is readily computable from the closed form expressions of (4) and (5). The electric current density on the surface of each element is derived from the magnetic field boundary condition. Extensive checks have shown that the contribution of the distant auxiliary sources to the tangential magnetic field on each element is negligible when compared to the contribution of the inner auxiliary sources. Therefore, the current distribution along each element is calculated by

$$I_p(z) \approx 2\pi a_p \left\{ \widehat{z} \cdot \left[\widehat{n}_{\,p} \times \sum_{n=-(N_p+1)}^{N_p+1} w_{(p,\,n)} \vec{H}_{(p,\,n)}\left(x_p + a_p,\, y_p,\, z\right) \right] \right\} \tag{11}$$

where $\widehat{n}_p$ it the unit vector that is normal to the surface of the element denoted by p.

In accordance with the MoM, the current distribution on each element is computed as the weighted superimposition of the corresponding basis functions. It is worth mentioning that the algebraic systems of (7) and (8) are identical to those obtained from the MoM with PWS basis and testing functions (corresponding to the currents of the auxiliary sources discussed above). Thus, for comparison purposes, the current distribution on each element is obtained from

$$I_p(z) = \sum_{n=-(N_p+1)}^{N_p+1} w_{(p,n)} f_{(p,n)}(z) s_{(p,n)}(z). \tag{12}$$

It is noted that there is an important difference between the MAS and the MoM, which is related to the behavior of the current distribution near $z = z_p \pm L_p/2$. Namely, for tubular elements, the current distribution must vanish at $z = z_p \pm L_p/2$, in order to satisfy the current continuity, which, on the other hand, does not hold exactly in cases of solid or capped wires. Nonetheless, in the majority of the MoM codes, the basis functions are selected so that the current distribution vanishes at $z = z_p \pm L_P/2$. Although this is not far from the exact situation in most cases, the drawback of this approach is that the slopes of the current near these points are determined exclusively from the first derivative of the outer basis functions. In contrast, the current distribution resulting from the MAS-RM is not accompanied by any restriction related to its value or slope, either when the auxiliary monopoles are employed or not. Thus, it is believed that (11) represents the true current distribution more accurately, subject to the model limitations and the influence of the end caps. In cases of moderately thicker elements, the accurate modeling of each element necessitates the addition of properly selected auxiliary sources near the ends. This would make it possible to model the radiation from the end caps and enforce the associated boundary conditions, which is, however, beyond the scopes of the present paper.

4. NUMERICAL RESULTS

All the examined cases were treated using auxiliary dipoles and monopoles, as well as auxiliary dipoles only, in order to exhibit the differences between the resulting solutions. In general, the solution stability can be examined by checking the resulting EM field (or any related parameter of interest) for an increasing number of auxiliary

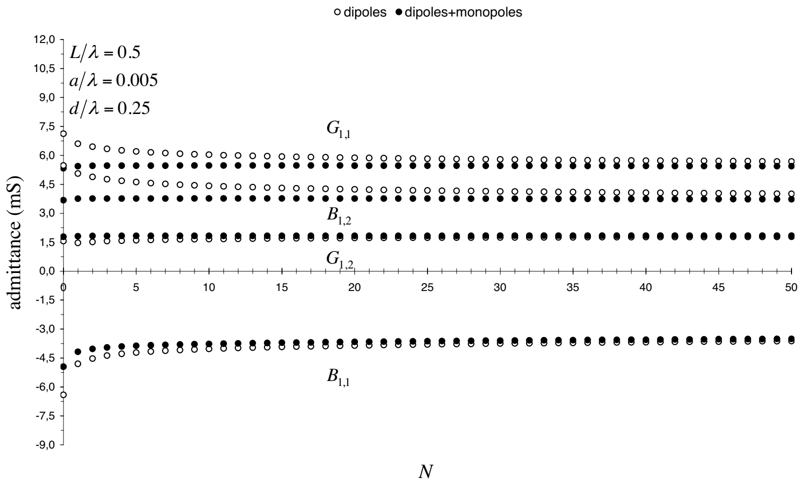
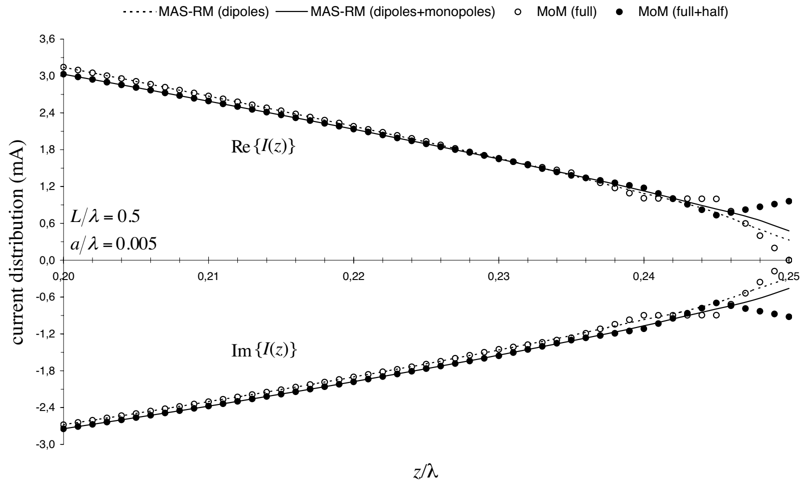


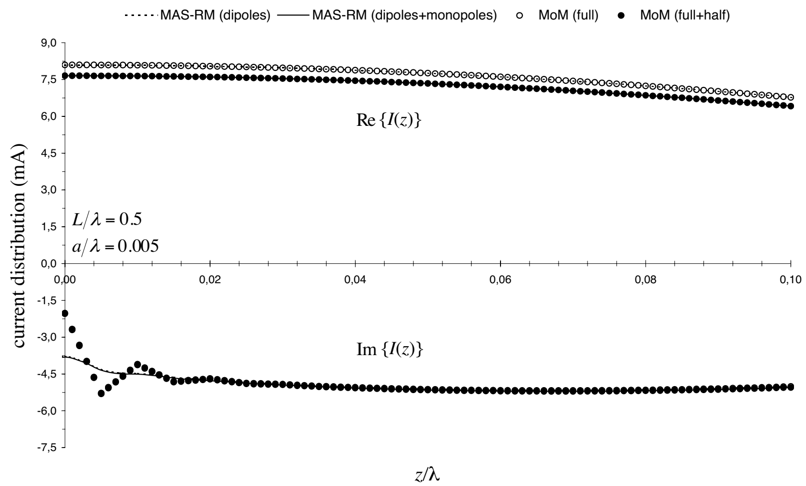
Figure 2. Computed self- and mutual admittance of two identical non-staggered dipoles versus $N(L/\lambda = 0.5,\ a/\lambda = 0.005,\ d/\lambda = 0.025)$.

sources. Finally, the solution is evaluated as satisfactory, though not necessarily convergent in a strict mathematical sense, when the calculated quantities are sufficiently stable.

At first, numerical results are presented for a two-element array of non-staggered identical elements with $L/\lambda = 0.5$ and $a/\lambda = 0.005$, having their centers distanced by $d/\lambda = 0.25$. The computed self- and mutual admittances are illustrated in Fig. 2 as a function of the parameter N, where $N_1 = N_2 = N$. Obviously, the incorporation of the terminal monopoles leads to a remarkably stable solution, since practically unchangeable results are obtained over a wide range of variation of the parameter N. On the other hand, when the terminal monopoles are not taken into account, a significantly larger N is needed to reach a solution of comparable stability. It is also noted that the same behavior is observed when the MoM is applied, since the base currents computed from (12) are roughly equal to those obtained from (11) for moderately small N. However, as N tends to and becomes larger than $L/(2a)$, the magnitudes of the auxiliary sources currents oscillate both near the driving point and the ends of each element. This phenomenon, which is not contingent upon round-off errors or the feed type, always occurs, either when the terminal monopoles are taken into account or not, and causes nonphysical oscillations in the current distribution resulting from (12). On the contrary, the current distribution of (11), which is consistent with the magnetic



(a)



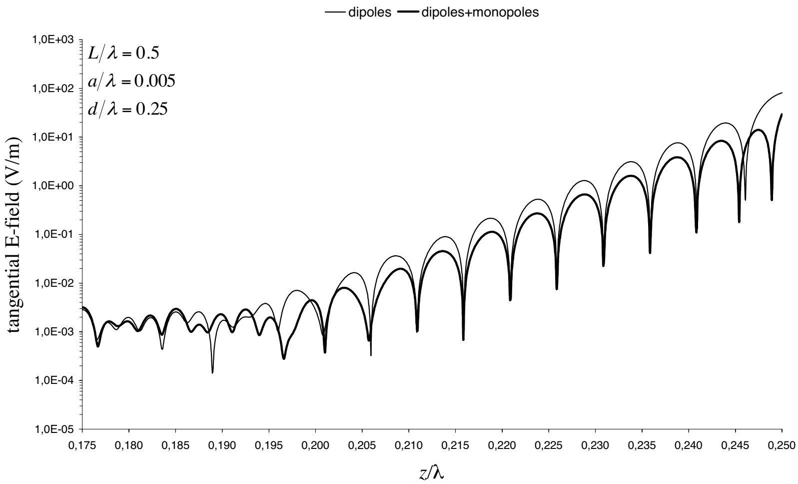
(b)

Figure 3. Computed current distribution on a symmetrically driven isolated dipole ($L/\lambda = 0.5$, $a/\lambda = 0.005$, $N + 1 = 50$). (a). Current distribution near the ends. (b). Current distribution near the driving point.

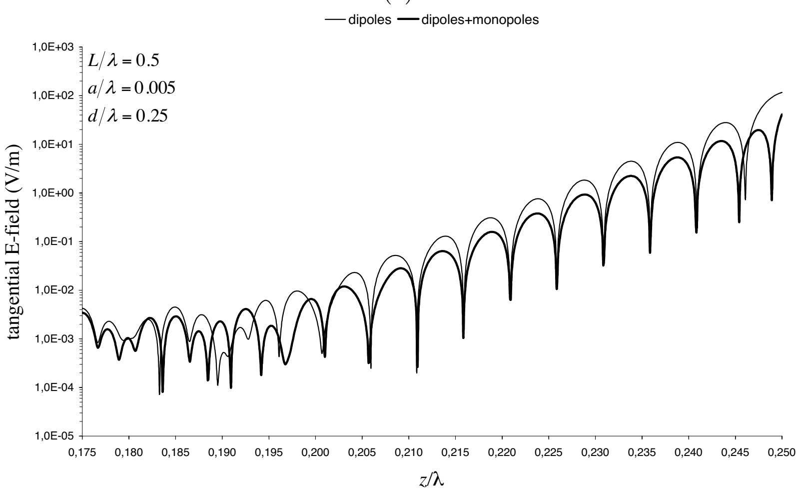
field boundary condition, is free from any nonphysical oscillations, regardless of the oscillations in the magnitudes of the auxiliary sources currents discussed above. This behavior is illustrated in Fig. 3, where the resulting current distributions on an isolated dipole (with $L/\lambda = 0.5$ and $a/\lambda = 0.005$) are shown. The presented curves correspond to both MAS-RM and MoM results for $N + 1 = 50$, in both the presence and absence of the terminal monopoles and the terminal half-PWS functions, respectively. From this sketch, it is apparent that the MoM solutions suffer from nonphysical behavior of the resulting current distributions near the ends and the driving point, due to the limitations regarding their values and slopes discussed above. For much larger N (in particular, when $N \gg L/(2a)$), numerical instabilities are typically encountered, which are not related to the oscillations observed in Fig. 3, but are caused by ill-conditioning of the matrix equation and render the solution physically meaningless in any case. This is the reason why all the above-mentioned issues can be assessed only with double precision arithmetic. Otherwise, severe ill-conditioning is encountered even for moderately small N, which forbids the examination of the solution.

In addition, many different two-element array configurations (with non-staggered and staggered elements) have been examined over a wide range of the geometrical parameters involved. In any case, for moderately small N, it was found that the results computed without considering the terminal monopoles depend strongly upon N and differ more significantly from those obtained by taking into consideration the terminal monopoles. For larger N, the differences are insignificant, at least from a practical point of view. Of course, after a certain limit, the solutions become unstable and, probably, completely meaningless, due to round-off errors. Thus, it is important to mention that all the comments made about the solution behavior regard the dynamic range in which the solution behavior is not affected dominantly by round-off errors.

Another very important issue that is worth to be examined is the degree of satisfaction of the electric field boundary condition, especially due to the differences between the computed results for small N. In any case, the tangential electric field varies along each element and reaches its maximum values at $z_p \pm L_p/2$, with magnitude that strongly depends upon the specific array configuration and the number and type of the auxiliary sources used. Selected results regarding the magnitude of the tangential electric field are depicted in Fig. 4 for the non-staggered configuration of Fig. 2 and $N + 1 = 50$. The presented curves correspond to both symmetric ($V_1 = V_2$) and anti-symmetric ($V_1 = -V_2$) excitation of unit amplitude. Apparently, the



(a)



(b)

Figure 4. Plots of the magnitude of the tangential electric field computed along symmetrically and anti-symmetrically driven identical non-staggered dipoles ($L/\lambda = 0.5$, $a/\lambda = 0.005$, $d/\lambda = 0.25$, $N + 1 = 50$). (a). Symmetric excitation. (b). Anti-symmetric excitation.

addition of the terminal monopoles leads to a noteworthy reduction of the boundary condition error in both cases, as shown in Fig. 4. This fact reveals that the addition of the terminal monopoles improves the accuracy of the model and explains, at least in part, the notable differences between the numerical results obtained with and without utilizing the terminal monopoles for small N.

According to the preceding, it is apparent that the proposed method can be utilized for the analysis of moderately large-scale arrays of cylindrical dipoles, as long as it is feasible to keep the number of unknowns sufficiently low without degradation of the solution quality. It is noted that conventional matrix methods are typically characterized by N_T^3 order of complexity, where N_T is the number of unknowns. As an outcome, such methods become inefficient as the number of elements is increased and the number of expansion terms per element remains high enough to attain a reliable solution. This fact has led to the development of efficient iterative techniques for solving the resulting matrix equations (for example, refer to [15, 16]), which are characterized by lower complexity. However, the complexity reduction is not achieved without cost. For example, the algorithm presented in [15] necessitates the selection of a proper threshold percentage for the formation of a sparse matrix that retains only strong interactions between segments, whereas the technique proposed in [16] requires a proper grouping of the elements in a way that depends on the array configuration; otherwise, the matrix solver output may be inaccurate, unstable, or even divergent. The method presented herein can be considered to be an alternative to any technique aiming at the complexity reduction for solving moderately large problems. In particular, instead of utilizing an algorithm of lower complexity to analyze a moderately large-scale array, the proposed method can be applied for a smaller number of unknowns per element, inasmuch as this will not affect the solution quality. Even when the number of unknowns is not small enough to reduce the execution time comparably to the sophisticated iterative solvers, the proposed method has the advantage of direct applicability to a wide range of array configurations (e.g., arrays of unequal and/or unequally spaced elements) without any limitations such as those discussed above. As a "benchmark case", a 27-element Yagi-Uda array is considered, in order to check the validity of the computed results and exhibit the important savings in the execution time that can be potentially achieved in cases of moderately large-scale arrays. The parameters of the antenna under consideration are the same as those in [17], which are also contained in Table 1. The magnitudes of the computed base currents are depicted in Fig. 5 for the two test cases reported in Table 1. More specifically, the test cases "L"

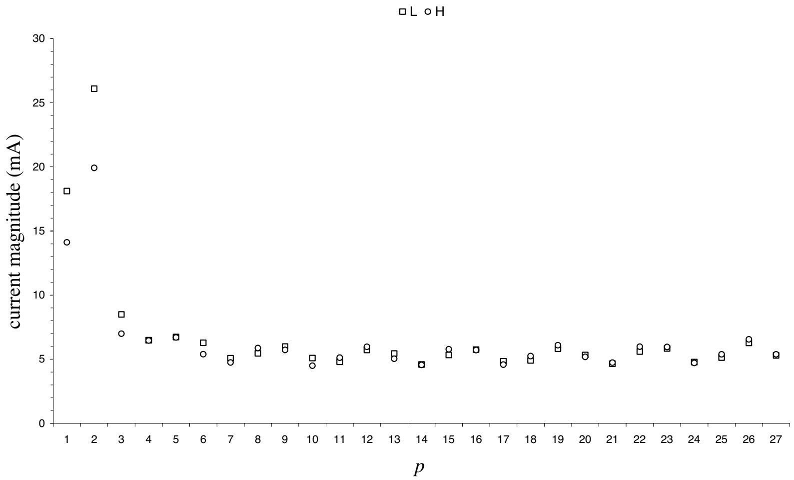
and "H" correspond to low and high discreteness densities associated with the numbers of the auxiliary sources used, respectively. Obviously, the addition of the terminal monopoles was found to lead to significant differences between the predicted base currents for small numbers of auxiliary sources, which become smaller as the number of the auxiliary sources is increased. The savings in memory allocation and execution time are obvious from the data contained in Table 2. It is apparent from Fig. 5 that the addition of the terminal monopoles leads to reliable results that are reached in their absence for a dramatically larger number of unknowns, which reinforces the above-mentioned statement.

Table 1. Parameters of the 27-element Yagi-Uda array and the associated parameters N_p for the two test cases considered (L and H).

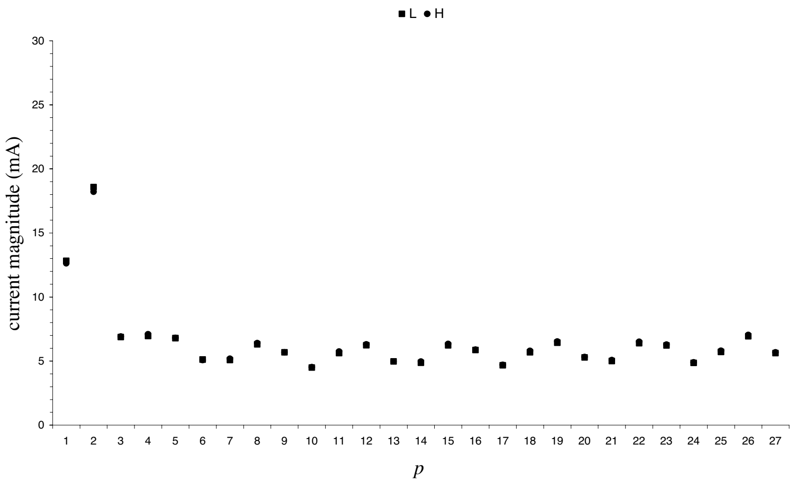
p	L_p/λ	x_p/λ	Test L $N_p+1=\left\lceil \frac{L_p}{20a_p} \right\rceil$	Test H $N_p+1=\left\lceil \frac{L_p}{2a_p} \right\rceil$
1	0.500	-0.125	9	84
2	0.470	0	8	79
3-27	0.406	$(p-2)0.34$	7	68
Elements with $a/\lambda = 0.003$ and centers at $(x_p,0,0)$. The symbol $\lceil A \rceil$ stands for the nearest integer that is greater than or equal to A.				

Table 2. Numbers of unknowns and execution times for the two test cases considered (L and H).

Test	N_T, Execution Time (sec) (*)	
	Dipoles	Dipoles and Monopoles
L	357, 104	411, 134
H	3699, 28043	3753, 28869
(*) Execution times on an Intel P4-2.6GHz/HT based PC with 512MB RAM		



(a)



(b)

Figure 5. Comparative plots of the computed current magnitudes that correspond to the test cases L and H (Tables 1 and 2). (a). Auxiliary Sources: sinusoidal dipoles. (b). Auxiliary Sources: sinusoidal dipoles and monopoles.

5. CONCLUSION

In this paper, a novel combination of the MAS with the RM technique is proposed for the treatment of arbitrary arrays of parallel cylindrical dipoles. From the extended checks performed, it was concluded that the addition of auxiliary terminal monopoles leads to an essential improvement on the numerical stability of the computed results and a reduction of the error associated with the fulfillment of the electric field boundary condition on each element. Furthermore, it was found that the calculation of the current distributions by invoking the magnetic field boundary condition results in smooth curves, which are insensitive to oscillations that usually accompany MoM solutions as the number of expansion terms is increased. Thereby, from a practical point of view, the number of expansion terms is only limited by ill-conditioning and not by the appearance of oscillations in the magnitudes of the auxiliary sources currents. Conclusively, the remarks of this paper are summarized as follows:

- The solutions obtained from the MAS-RM, in the presence of the terminal monopoles, tend to remarkably stable results that are reached by the conventional MoM (without utilizing terminal half-PWS functions) for a significantly larger number of expansion terms. Therefore, the addition of the terminal auxiliary sources in the MAS-RM model (half-PWS functions in the MoM) benefits the numerical stability of the solution.
- The current distributions resulting from the MAS-RM are smooth and stable over a very wide dynamic range of variation for the number of the expansion terms, regardless of any oscillations in the magnitudes of the auxiliary sources currents. The latter restricts the number of expansion terms in the MoM even before ill-conditioning begins to occur, either when the terminal half-PWS functions are added or not.
- The boundary condition errors occurring are significantly smaller in the presence of the terminal monopoles (terminal half-PWS functions in the MoM).

Finally, the method proposed in this paper provides an efficient alternative to the widely spread MoM codes, especially for treating moderately large-scale arrays.

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