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# ANALYSIS OF COUPLED OR SINGLE NONUNIFORM TRANSMISSION LINES USING STEP-BY-STEP NUMERICAL INTEGRATION

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**Abstract**—A method is proposed for analysis of arbitrarily loaded lossy and dispersive nonuniform single or coupled transmission lines. In this method, the transmission lines are subdivided to several uniform sections, at first. Then the voltage and current distributions are obtained using second order step-by-step numerical integration (second order finite difference method). The accuracy of the method is studied using analysis of some special types of single and coupled transmission lines.

## 1. INTRODUCTION

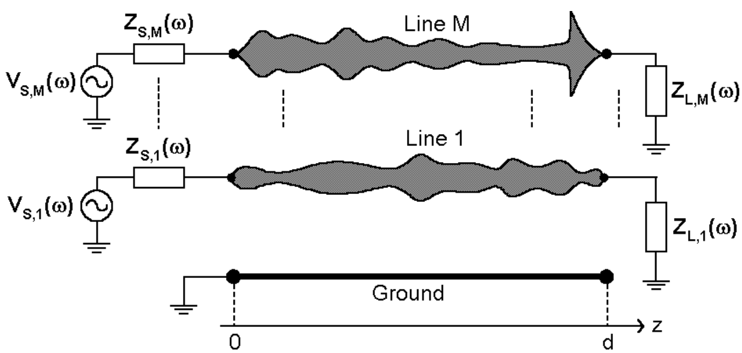
Single and coupled nonuniform transmission lines (NTLs) are widely used in RF and microwave circuits as resonators [1], impedance matching [1, 2], delay equalizers [3], filters [4], wave shaping [5], analog signal processing [6], VLSI interconnect [7] and etc. The differential equations describing these structures have non-constant coefficients, because of their length. The most used method is subdividing the nonuniform lines into many short sections [8–13]. In each section a uniform [8–11], linear [12], exponential [13] or other types of lines may be inserted. In some efforts, an exact or approximate closed form solution has been derived from the structures containing many cascaded short sections [8, 9]. These nonlinear equations have been solved analytically and without approximation only for a few special types of NTLs such as linear [12], exponential [13], power-law [14, 15], binomial [16], exponential power law [17] and hermite [18] types.

The subject of this paper is using second order step-by-step numerical integration to analyze coupled or single NTLs. In this proposed method, the transmission lines are subdivided to several uniform sections, at first. Then the voltage and current distributions are obtained using step-by-step numerical integration. Some closed relations, for which second derivative of the voltage has been considered, are obtained for this purpose. This method is applicable to all arbitrarily loaded lossy and dispersive coupled and single NTLs. The accuracy of the method is studied using analysis of some special kinds of single and coupled NTLs.

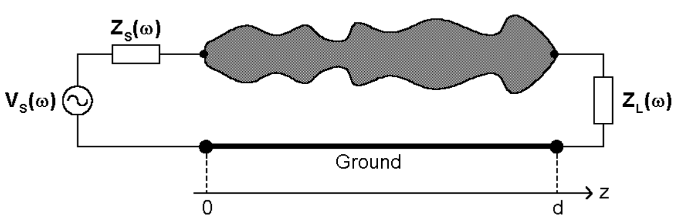
## 2. THE EQUATIONS OF SINGLE AND COUPLED NTLS

In this section, the frequency domain equations of loaded coupled or single NTLs are reviewed. It is assumed that the principal propagation mode of the lines is TEM or quasi-TEM. This assumption is valid when the widths in the cross section are small enough compared to the wavelength. Figure 1 shows typical coupled and single NTLs consisting of $M$ ($M = 1$ for single NTLs) lines with length of $d$ and with arbitrary terminal loads of $Z_{S,m}(\omega)$ and $Z_{L,m}(\omega)$, in which $m = 1, 2, \ldots, M$.

The differential equations describing lossy and dispersive coupled



(a)



(b)

**Figure 1.** A typical nonuniform transmission line terminated by arbitrary loads a) Coupled NTL with M lines b) Single NTL.

NTLs in the frequency domain are given by

$$\frac{d\boldsymbol{V}(z,\omega)}{dz} = -\hat{\boldsymbol{Z}}(z,\omega)\boldsymbol{I}(z,\omega) \tag{1}$$

$$\frac{d\boldsymbol{I}(z,\omega)}{dz} = -\hat{\boldsymbol{Y}}(z,\omega)\boldsymbol{V}(z,\omega) \tag{2}$$

in which $\boldsymbol{V}$ and $\boldsymbol{I}$ are $M \times 1$ voltage and current vectors, respectively. Also we have

$$\hat{\boldsymbol{Z}}(\omega) = \boldsymbol{R}(z,\omega) + j\omega\boldsymbol{L}(z,\omega) \tag{3}$$

$$\hat{\boldsymbol{Y}}(\omega) = \boldsymbol{G}(z,\omega) + j\omega\boldsymbol{C}(z,\omega) \tag{4}$$

In (3)–(4), $\boldsymbol{R}, \boldsymbol{L}, \boldsymbol{G}$ and $\boldsymbol{C}$ are the per-unit-length matrices of the coupled transmission lines, whose dimensions are $M \times M$. These matrices are reduced to the distributed primary parameters $R$, $L$, $G$ and $C$, for the single transmission lines. Also, the characteristic impedance and the propagation coefficient of the single lines will be as follows, respectively

$$Z_0(z,\omega) = \sqrt{\frac{\hat{Z}(z,\omega)}{\hat{Y}(z,\omega)}} = \sqrt{\frac{R(z,\omega) + j\omega L(z,\omega)}{G(z,\omega) + j\omega C(z,\omega)}} \tag{5}$$

$$\begin{aligned}\gamma(z,\omega) &= \alpha(z,\omega) + j\beta(z,\omega) = \sqrt{\hat{Z}(z,\omega)\hat{Y}(z,\omega)} \\ &= \sqrt{[R(z,\omega) + j\omega L(z,\omega)][G(z,\omega) + j\omega C(z,\omega)]}\end{aligned} \tag{6}$$

Combining (1) and (2) with each other, gives the following differential equations for the voltage and current vectors of coupled NTLs.

$$\frac{d^2\boldsymbol{V}(z,\omega)}{dz^2} - \boldsymbol{f}(z,\omega)\frac{d\boldsymbol{V}(z,\omega)}{dz} - \boldsymbol{g}(z,\omega)\boldsymbol{V}(z,\omega) = \boldsymbol{0} \tag{7}$$

$$\boldsymbol{I}(z,\omega) = -\hat{\boldsymbol{Z}}^{-1}(z,\omega)\frac{d\boldsymbol{V}(z,\omega)}{dz} \tag{8}$$

Where

$$\boldsymbol{g}(z,\omega) = \hat{\boldsymbol{Z}}(z,\omega)\hat{\boldsymbol{Y}}(z,\omega) \tag{9}$$

$$\boldsymbol{f}(z,\omega) = \frac{d\hat{\boldsymbol{Z}}(z,\omega)}{dz}\hat{\boldsymbol{Z}}^{-1}(z,\omega) \tag{10}$$

Furthermore, the terminal conditions for loaded coupled NTLs are as follows

$$\boldsymbol{V}(0,\omega) + \boldsymbol{Z}_S(\omega)\boldsymbol{I}(0,\omega) = \boldsymbol{V}_S(\omega) \tag{11}$$

$$\boldsymbol{V}(d,\omega) - \boldsymbol{Z}_L(\omega)\boldsymbol{I}(d,\omega) = \boldsymbol{0} \tag{12}$$

Where $\boldsymbol{Z}_S$ and $\boldsymbol{Z}_L$ are diagonal source and load matrices, respectively. One sees from (7)–(12) that, analytically solving the equations of general type coupled or single NTLs is a very hard problem.

## 3. ANALYSIS OF NTLS USING STEP-BY-STEP METHOD

In this section, the analysis of arbitrary coupled or single NTLs using step-by-step numerical integration is proposed. First, the transmission lines are subdivided to $N$ uniform sections with length of $\Delta z = d/N$. Then, two differential equations (7) and (8) are discretized to obtain the following difference equations, respectively

$$\begin{aligned}\boldsymbol{V}(d-(n+1)\Delta z,\omega) &\cong 2\boldsymbol{V}(d-n\Delta z,\omega)-\boldsymbol{V}(d-(n-1)\Delta z,\omega)\\&\quad+\frac{d^2\boldsymbol{V}(z,\omega)}{dz^2}\Big|_{z=d-n\Delta z}\Delta z^2\\&=\Big(2\boldsymbol{I}_d+\boldsymbol{g}(d-n\Delta z,\omega)\Delta z^2-\boldsymbol{f}(d-n\Delta z,\omega)\Delta z\Big)\boldsymbol{V}(d-n\Delta z,\omega)\\&\quad+(\boldsymbol{f}(d-n\Delta z,\omega)\Delta z-\boldsymbol{I}_d)\boldsymbol{V}(d-(n-1)\Delta z,\omega);\\&\quad n=1,2,\ldots,N-1\end{aligned}\tag{13}$$

$$\begin{aligned}\boldsymbol{I}(d-n\Delta z,\omega)&=-\hat{\boldsymbol{Z}}^{-1}(d-n\Delta z,\omega)\frac{d\boldsymbol{V}(z,\omega)}{dz}\Big|_{z=d-n\Delta z}\\&\cong-\hat{\boldsymbol{Z}}^{-1}(d-n\Delta z,\omega)(\boldsymbol{V}(d-(n-1)\Delta z,\omega)-\boldsymbol{V}(d-n\Delta z,\omega))/\Delta z;\\&\quad n=1,2,\ldots,N\end{aligned}\tag{14}$$

To obtain (13)–(14), the forward difference and three points approximations has been used for the first and second derivatives of the voltage function, respectively. To use (13), the voltage at $z = d - \Delta z$ is required. The voltage at this point can be found using (7)–(8) and (12) in the power series expansion of the voltage function, as follows

$$\begin{aligned}\boldsymbol{V}(d-\Delta z,\omega)&\cong\boldsymbol{V}(d,\omega)-\frac{d\boldsymbol{V}(z,\omega)}{dz}\Big|_{z=d}\Delta z+\frac{d^2\boldsymbol{V}(z,\omega)}{dz^2}\Big|_{z=d}\Delta z^2/2\\&=\boldsymbol{V}(d,\omega)+\hat{\boldsymbol{Z}}(d,\omega)\boldsymbol{I}(d,\omega)\Delta z+(-\boldsymbol{f}(d,\omega)\hat{\boldsymbol{Z}}(d,\omega)\boldsymbol{I}(d,\omega)\\&\quad+\boldsymbol{g}(d,\omega)\boldsymbol{V}(d,\omega))\Delta z^2/2\\&=\Big(\boldsymbol{I}_d+\hat{\boldsymbol{Z}}(d,\omega)\boldsymbol{Z}_L^{-1}\Delta z+0.5\boldsymbol{g}(d,\omega)\Delta z^2\\&\quad-0.5\boldsymbol{f}(d,\omega)\hat{\boldsymbol{Z}}(d,\omega)\boldsymbol{Z}_L^{-1}\Delta z^2\Big)\boldsymbol{V}(d,\omega)\end{aligned}\tag{15}$$

Using (13), the voltages of all sections are obtained step-by-step from $z = d$ to $z = 0$. However, the vector $\boldsymbol{V}(d, \omega)$ is required to be known in this process. To find this vector, one may use an optimization approach. In the optimization approach, the following defined error, which is obtained from the terminal condition at $z = 0$, as in (11), has to become minimum and near to zero.

$$Error = \sqrt{\frac{1}{M}\sum_{m=1}^{M} |\boldsymbol{E}(m)|^2} \tag{16}$$

in which

$$\boldsymbol{E} = \boldsymbol{V}_S(\omega) - (\boldsymbol{V}(0,\omega) + \boldsymbol{Z}_S(\omega)\boldsymbol{I}(0,\omega)) \tag{17}$$

Of course, for single lines, i.e., $M = 1$, the unknown voltage $V(d,\omega)$, which behaves like a scale factor, can be assumed one volt, at first. Then its correct value is obtained so that the defined error in (16) becomes zero. In this way, we have

$$V(d,\omega) = \frac{V_S(\omega)}{V(0,\omega) + Z_S I(0,\omega)} \tag{18}$$

## 4. EXAMPLES AND RESULTS

In this section, two special types of single and coupled transmission lines (uniform and linearly nonuniform) are analyzed both using analytical formulas and using the proposed method. The time consumed for the examples was less than 1.0 sec. using a Pentium-4 PC and MATLAB program.

*Example 1: (Uniform Single Transmission Line)*

Consider a lossless and uniform single transmission line ($M = 1$, $R = G = 0$). Assume $Z_0 = \sqrt{L/C} = 50\,\Omega$, $\gamma = j\beta = j\omega\sqrt{LC} = j\omega/c$ ($c$ is the velocity of the light), $d = 20\,\text{cm}$, $Z_S = 50\,\Omega$, $Z_L = 100\,\Omega$ and $V_S = 1\,\text{V}$. Figures 2–3, shows the amplitude of voltage distribution obtained from the proposed method considering $N = 20$ and $N = 50$ sections and assuming the excited frequency to be $f = 1.0$ or $2.0\,\text{GHz}$, respectively. One sees a good agreement between the exact solutions and the solutions obtained from the proposed method. It is seen and also evident that, as the number of sections, $N$, increases the accuracy of the obtained solution increases. Also, the error has been spread along the whole length of the line. Furthermore, as the source frequency (or equivalently the length of the line) increases, the accuracy of the method decreases.

*Example 2: (Uniform Coupled Transmission Lines)*

Consider a lossless uniform coupled microstrip structure with $M = 2$ strips. The substrate permittivity is $\varepsilon_r = 10$, the width of

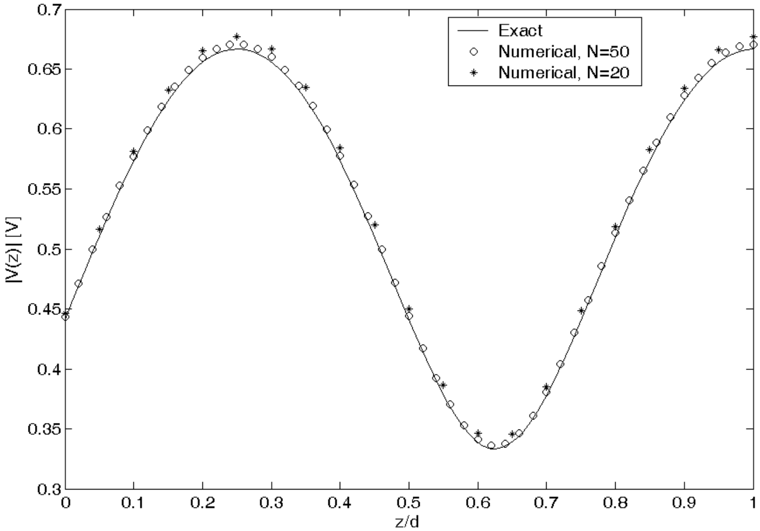



**Figure 2.** The amplitude of the voltage of uniform single line at frequency of $f = 1.0\,\mathrm{GHz}$, obtained from exact formulas and from the proposed method with $N = 20$ and $N = 50$ sections.

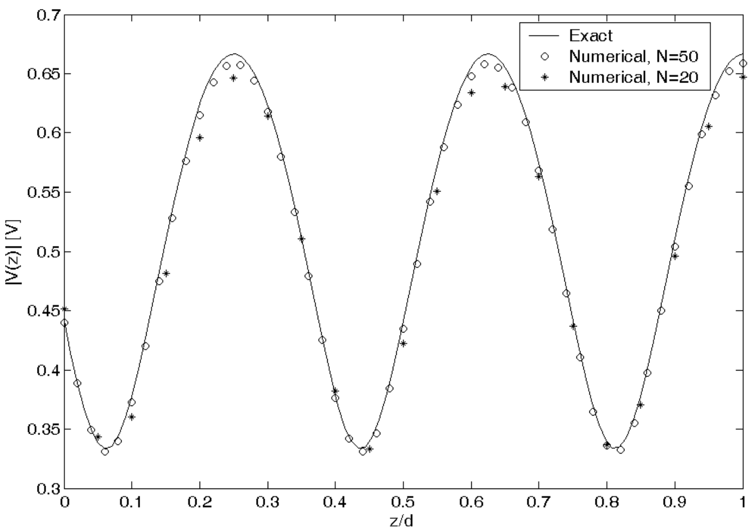



**Figure 3.** The amplitude of the voltage of uniform single line at frequency of $f = 2.0\,\mathrm{GHz}$, obtained from exact formulas and from the proposed method with $N = 20$ and $N = 50$ sections.

strips and the gap between them are equal to the thickness of the substrate. This inhomogeneous structure will has the following the per-unit-length matrices.

$$\boldsymbol{L}(z) = \boldsymbol{L}_0 = \begin{bmatrix} 425.6 & 74.83 \\ 74.83 & 425.6 \end{bmatrix} \text{nH/m} \tag{19}$$

$$\boldsymbol{C}(z) = \boldsymbol{C}_0 = \begin{bmatrix} 174.9 & -14.25 \\ -14.25 & 174.9 \end{bmatrix} \text{pF/m} \tag{20}$$

$$\boldsymbol{R}(z) = \boldsymbol{G}(z) = \boldsymbol{0} \tag{21}$$

Assume that $d = 20\,\text{cm}$, $f = 1.0\,\text{GHz}$, $Z_{S,1} = Z_{S,2} = 50\,\Omega$, $Z_{L,1} = Z_{L,2} = 50\,\Omega$, $V_{S,1} = 1\,\text{V}$ and $V_{S,2} = 0$. The exact voltages of this structure can be determined using the modal decomposing method [10]. Figures 4–5, compare the amplitude and the angle of voltages of two lines, obtained from the modal decomposing method and from the proposed method considering $N = 20$ and $N = 50$ sections. One sees a good agreement between the exact solutions and the solutions obtained from the proposed method.

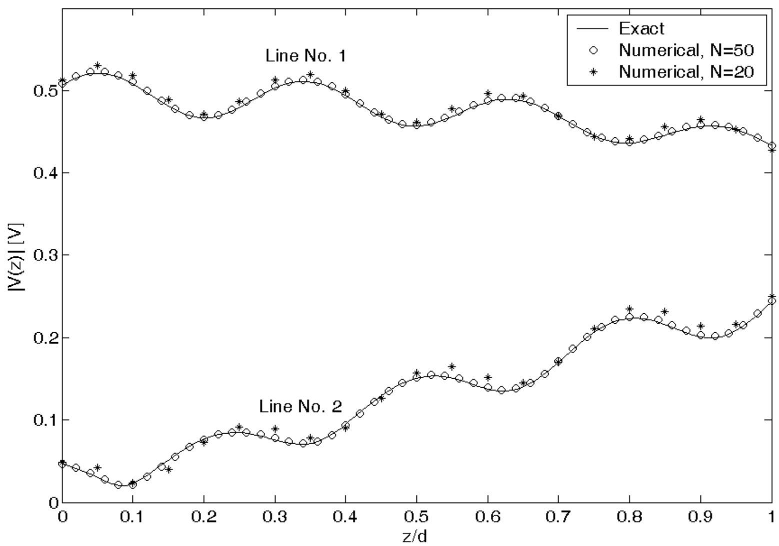



**Figure 4.** The amplitude of the voltage of uniform coupled transmission lines at frequency of $f = 1.0\,\text{GHz}$, obtained from exact formulas and from the proposed method with $N = 20$ and $N = 50$ sections.

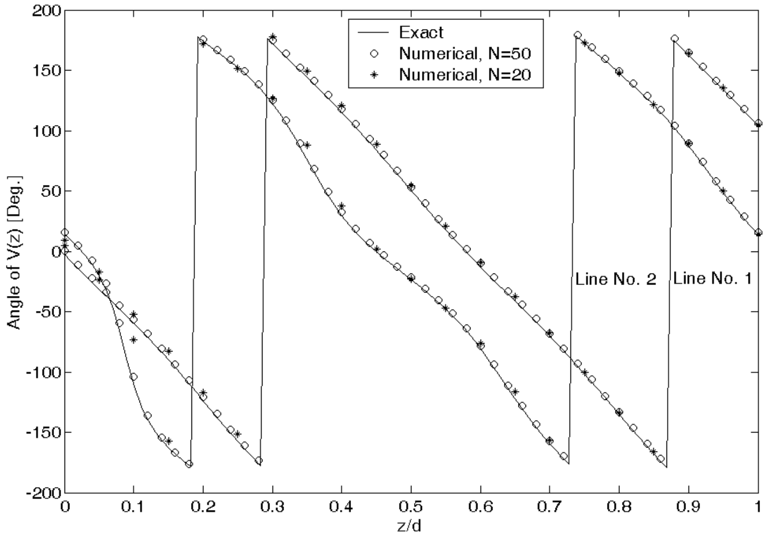



**Figure 5.** The angle of the voltage of uniform coupled transmission lines at frequency of $f = 1.0\,\mathrm{GHz}$, obtained from exact formulas and from the proposed method with $N = 20$ and $N = 50$ sections.

*Example 3: (Linearly Nonuniform Single Transmission Line)*

Consider a lossless and linearly varied single NTL with the following distributed primary parameters

$$L(z) = L_0(1 + k(z/d)) \tag{22}$$

$$C(z) = \frac{C_0}{1 + k(z/d)} \tag{23}$$

$$R(z) = G(z) = 0 \tag{24}$$

This type of transmission line will have the following secondary parameters defined in (5)–(6)

$$Z_0(z) = \sqrt{L_0/C_0}(1 + k(z/d)) \tag{25}$$

$$\gamma = j\beta = j\omega\sqrt{L_0C_0} \tag{26}$$

Assume that $Z_0(0) = \sqrt{L_0/C_0} = 50\,\Omega$, $\beta = \omega\sqrt{L_0C_0} = \omega/c$, $d = 20\,\mathrm{cm}$, $f = 1.0\,\mathrm{GHz}$, $Z_S = 50\,\Omega$, $Z_L = 100\,\Omega$, $V_S = 1\,\mathrm{V}$ and $k = 0.5$, 1.0 or 1.5. Figure 6, compares the amplitude of voltage of the line, obtained from (A1)–(A7) (in the Appendix) and from the proposed method considering $N = 100$ sections. Again, a good agreement is

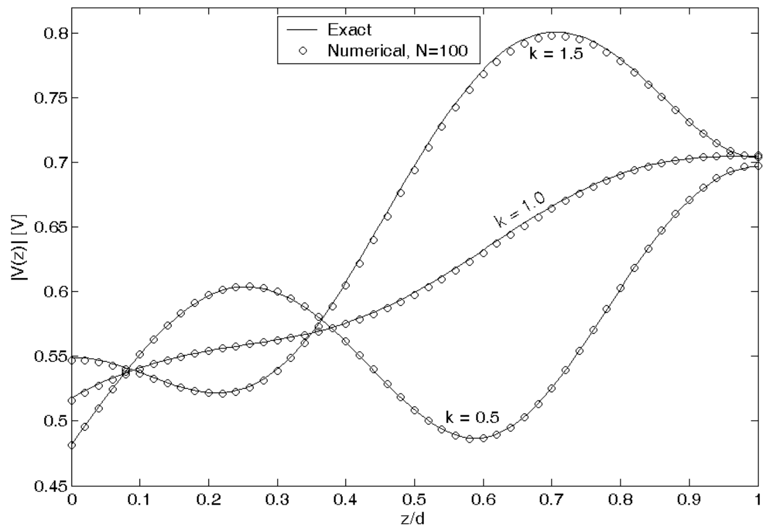



**Figure 6.** The amplitude of the voltage of linearly nonuniform single line at frequency of $f = 1.0\,\text{GHz}$, obtained from exact formulas and from the proposed method with $N = 100$ sections.

seen between the results from analytical solution and the results from the proposed method. Of course, to have the same errors as in the two previous examples, we had to use more sections. This may be due to more variations of the linearly nonuniform lines with respect to uniform lines.

Finally, one may conclude that the proposed method is applicable to all arbitrary NTLs. Also, it is concluded that as the excitation frequency, the length of the line (with respect to the wavelength) and the variations of the primary parameters increase, the necessary number of sections increases. To obtain a crude relation for the amount of error, consider a lossless and uniform line. The relative error in (13) will be as follows

$$\begin{aligned} E &\cong \frac{1}{12V}\left|\frac{d^4V}{dz^4}\right|\Delta z^4 = \frac{1}{12}(\hat{Z}\hat{Y})^2\Delta z^4 = \frac{1}{12}\beta^4\Delta z^4 \\ &= \frac{1}{12}\left(2\pi\frac{\Delta z}{\lambda}\right)^4 = \frac{130}{N^4}\left(\frac{d}{\lambda}\right)^4 \end{aligned} \tag{27}$$

in which $\lambda$ is the wavelength. For example, to have the relative error less than $10^{-4}$ for two cases of $d/\lambda = 0.6$ and 1.2 (as in Example 1), $N$ must be greater than 21 and 41, respectively.

## 5. CONCLUSIONS

The second order step-by-step numerical integration was used to analyze coupled or single Nonuniform Transmission Lines (NTLs). In this proposed method, the transmission lines are subdivided to several uniform sections, at first. Then the voltage and current distributions are obtained using step-by-step numerical integration. Some closed relations, for which second derivative of the voltage has been considered, are obtained for this purpose. It was seen that, as the variations of the primary parameters, the excitation frequency and the length of the line (with respect to the wavelength) increases, the necessary number of sections increases. The method is evaluated using analysis of some special kinds of coupled and single lines. The method is very simple and fast and can be used for all arbitrarily loaded lossy and dispersive coupled or single NTLs.

## APPENDIX A.

The exact voltage and current of lossless linearly varied single nonuniform transmission lines are determined analytically. The exact voltage and current distributions of this type of transmission line have been determined as follows [12]

$$V(z,\omega)=(1+k(z/d))\left[K_1 J_1\left(\frac{\beta d}{k}(1+k(z/d))\right)+K_2 Y_1\left(\frac{\beta d}{k}(1+k(z/d))\right)\right] \tag{A1}$$

$$I(z,\omega)=-\frac{1}{j\beta Z_0(z)}\frac{dV(z,\omega)}{dz} \tag{A2}$$

In (A1), $J_1(\cdot)$ and $Y_1(\cdot)$ are respectively the first and second type of Bessel functions with degree of one. The coefficients $K_1$ and $K_2$ are determined using (A1)–(A2) in terminal conditions (11)–(12), as follows

$$K_1=\frac{a_4}{a_1 a_4-a_2 a_3}V_S=-\frac{a_4}{a_3}K_2 \tag{A3}$$

in which

$$a_1 = \left(1+j\frac{k}{\beta d}\frac{Z_S(\omega)}{Z_0(0)}\right) J_1\left(\frac{\beta d}{k}\right)+j\frac{Z_S(\omega)}{Z_0(0)}J_1'\left(\frac{\beta d}{k}\right) \tag{A4}$$

$$a_2 = \left(1+j\frac{k}{\beta d}\frac{Z_S(\omega)}{Z_0(0)}\right) Y_1\left(\frac{\beta d}{k}\right)+j\frac{Z_S(\omega)}{Z_0(0)}Y_1'\left(\frac{\beta d}{k}\right) \tag{A5}$$

$$a_3 = \left(1-j\frac{k}{(1+k)\beta d}\frac{Z_L(\omega)}{Z_0(d)}\right) J_1\left(\frac{(1+k)\beta d}{k}\right)$$

$$-j\frac{Z_L(\omega)}{Z_0(d)}J_1'\left(\frac{(1+k)\beta d}{k}\right) \quad \text{(A6)}$$

$$a_4 = \left(1 - j\frac{k}{(1+k)\beta d}\frac{Z_L(\omega)}{Z_0(d)}\right)Y_1\left(\frac{(1+k)\beta d}{k}\right) -j\frac{Z_L(\omega)}{Z_0(d)}Y_1'\left(\frac{(1+k)\beta d}{k}\right) \quad \text{(A7)}$$

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