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DYADIC GREEN'S FUNCTIONS FOR AN ELECTRICALLY GYROTROPIC MEDIUM

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Abstract—The complete set of dyadic Green's functions (DGFs) for an electrically gyrotropic medium is obtained using a new formulation technique, which consists of a matrix method with dyadic decomposition in the k-domain. The analytic expressions for DGFs are represented in a unique form in terms of characteristic field vectors that exist in an electrically gyrotropic medium. It is shown that the dyadic decomposition greatly facilitates the calculation of an inverse operation, which is crucial in derivation of Green's functions. The DGFs found here can be used to solve electromagnetic problems involving the ionosphere and new types of anisotropic materials such as ceramics and advanced composites.

1. INTRODUCTION

In numerous electromagnetic applications such as remote sensing, wave propagation and scattering, monolithic integrated circuits and optics, it is necessary to compute the electromagnetic field inside the medium. When the dyadic Green's function (DGF) of the medium is known, it is relatively easy to find the electromagnetic field in that environment.

The DGFs for isotropic media [1–3] and anisotropic media [4–7] have been formulated by numerous researchers for over a few decades. Because of the Hermitian structure of the permittivity or permeability tensors, the calculation of the dyadic Green's function for an electrically gyrotropic medium, such as cold plasma or a magnetically gyrotropic medium, such as ferrite in the presence of an external dc magnetic field $\overline{B}_0$ is more involved than other media. W. S. Weiglhofer [8, 9] represented DGF for an electrically and magnetically gyrotropic media in terms of a single scalar Green's function which is a solution of a fourth order partial differential equation. Electromagnetic DGF for multilayered symmetric electrically gyrotropic media was derived by S. Barkleshli [10] using the plane wave spectral, vector wave function expansion which was first introduced by L. B. Felsen and N. Marcuvitz [11]. L. W. Li *et al.*, [12] obtained DGFs in gyrotropic media using cylindrical vector wave functions.

In this paper, the complete set of DGFs for an electrically gyrotropic medium is derived using a new formulation technique, which consists of a matrix method with dyadic decomposition in the k-domain. First, the vector wave equation for dyadic Green's function is Fourier transformed and the problem is transformed into the k-domain. Since the equation in this domain becomes algebraic, representation of the DGF is reduced to finding the inverse of an electric wave matrix. Inverse operation is accomplished by decomposing the electric wave matrix into its dyadics. Once the inverse operation is completed, the DGF is constructed by expressing the adjoint of the wave matrix in terms of its eigenvectors or the characteristic field vectors using the matrix method. It is shown that the method that we introduce greatly simplifies the derivation of the DGFs for an electrically gyrotropic medium as compared to the existing methods.

2. FORMULATION AND SOLUTIONS

We would like to find the complete set of the DGFs for an unbounded electrically gyrotropic or a gyroelectric medium, which can be used to find the electromagnetic fields in the presence of current source distributions $\overline{J}(\overline{r})$ and $\overline{M}(\overline{r})$ as illustrated in Figure 1.

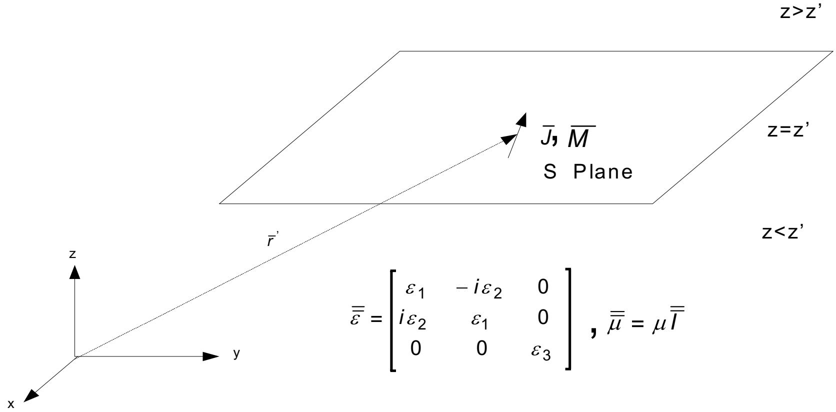

Figure 1. Imaginary S plane, parallel to xy plane, passing through the source at $z = z'$ in an unbounded electrically gyrotropic medium.

For an electrically gyrotropic medium such as cold plasma, the relative permittivity and relative permeability tensors with the presence of an external dc magnetic field $\overline{B}_0 = \hat{b}_0 B_0$ are defined in dyadic form as

$$\overline{\overline{\varepsilon}} = \varepsilon_1 \left(\overline{\overline{I}} - \hat{b}_0 \hat{b}_0 \right) + i\varepsilon_2 \left(\hat{b}_0 \times \overline{\overline{I}} \right) + \varepsilon_3 \hat{b}_0 \hat{b}_0 \tag{1}$$

$$\overline{\overline{\mu}} = \mu \overline{\overline{I}} \tag{2}$$

Also note that $\mu = 1$ for a cold plasma. When $\hat{b}_0 = \hat{z}$, the permittivity tensor given in Eq. (1) can be represented in matrix form as

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \tag{3}$$

Maxwell's equations for the problem illustrated in Figure 1 in the presence of impressed magnetic current density $\overline{M}(\overline{r})$ and the electric current density $\overline{J}(\overline{r})$ can be written as

$$\nabla \times \overline{E} = i\omega\mu_0 \overline{\overline{\mu}} \cdot \overline{H} - \overline{M} \tag{4}$$

$$\nabla \times \overline{H} = -i\omega\varepsilon_0 \overline{\overline{\varepsilon}} \cdot \overline{E} + \overline{J} \tag{5}$$

The linearity of Maxwell's equations implies linear dependence of $\overline{E}$ and $\overline{H}$ on the excitations $\overline{J}$ and $\overline{M}$. Then in Figure 1 at any point, $\overline{E}$ and $\overline{H}$ can be represented as

$$\overline{E}(\overline{r}) = \int_{V'} \overline{\overline{G}}{}^{e}_{ee} \left(\overline{r}, \overline{r}'\right) \cdot \overline{J}\left(\overline{r}'\right) d^3\overline{r}' + \int_{V'} \overline{\overline{G}}{}^{e}_{em} \left(\overline{r}, \overline{r}'\right) \cdot \overline{M}\left(\overline{r}'\right) d^3\overline{r}' \tag{6}$$

$$\overline{H}(\overline{r}) = \int_{V'} \overline{\overline{G}}{}^{e}_{me} \left(\overline{r}, \overline{r}'\right) \cdot \overline{J}\left(\overline{r}'\right) d^3\overline{r}' + \int_{V'} \overline{\overline{G}}{}^{e}_{mm} \left(\overline{r}, \overline{r}'\right) \cdot \overline{M}\left(\overline{r}'\right) d^3\overline{r}' \tag{7}$$

$\overline{J}$ and $\overline{M}$ can be written as

$$\overline{J}(\overline{r}) = \int_{V'} \delta\left(\overline{r} - \overline{r}'\right) \overline{\overline{I}} \cdot \overline{J}\left(\overline{r}'\right) d^3\overline{r}' \tag{8}$$

$$\overline{M}(\overline{r}) = \int_{V'} \delta\left(\overline{r} - \overline{r}'\right) \overline{\overline{I}} \cdot \overline{M}\left(\overline{r}'\right) d^3\overline{r}' \tag{9}$$

where $\overline{\overline{I}}$ is a unit dyad.

The dyadic Green's functions $\overline{\overline{G}}{}^{e}_{ee}(\overline{r}, \overline{r}')$, $\overline{\overline{G}}{}^{e}_{mm}(\overline{r}, \overline{r}')$ are called electric type and magnetic type, respectively, and $\overline{\overline{G}}{}^{e}_{me}(\overline{r}, \overline{r}')$, $\overline{\overline{G}}{}^{e}_{em}(\overline{r}, \overline{r}')$ are called magnetic-electric type and electric-magnetic type

DGFs, respectively, for an electrically gyrotropic medium. The superscript of the DGF refers to the type of the gyrotropic medium and the first and the second subscripts show the type of the dyadic Green's function. The subscript 'e' refers to an electric type and 'm' refers to a magnetic type DGF. The superscript 'e' stands for an electrically gyrotropic medium. When Eqs. (6)–(9) are substituted into Eqs. (4)–(5), we obtain following vector wave equations for an electrically gyrotropic medium.

$$\left[\nabla \times \nabla \times \overline{\overline{I}} - k_0^2 \mu \overline{\overline{\varepsilon}}\right] \cdot \overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}') = i\omega\mu_0\mu\overline{\overline{I}}\delta(\overline{r} - \overline{r}') \quad (10a)$$

$$\left[\nabla \times \overline{\overline{\varepsilon}}^{-1} \cdot \nabla \times \overline{\overline{I}} - k_0^2 \mu \overline{\overline{I}}\right] \cdot \overline{\overline{G}}_{mm}^{e}(\overline{r}, \overline{r}') = i\omega\varepsilon_0\overline{\overline{I}}\delta(\overline{r} - \overline{r}') \quad (10b)$$

$$\left[\nabla \times \nabla \times \overline{\overline{I}} - k_0^2 \mu \overline{\overline{\varepsilon}}\right] \cdot \overline{\overline{G}}_{em}^{e}(\overline{r}, \overline{r}') = -\nabla \times \overline{\overline{I}}\delta(\overline{r} - \overline{r}') \quad (10c)$$

$$\left[\nabla \times \overline{\overline{\varepsilon}}^{-1} \cdot \nabla \times \overline{\overline{I}} - k_0^2 \mu \overline{\overline{I}}\right] \cdot \overline{\overline{G}}_{me}^{e}(\overline{r}, \overline{r}') = \nabla \times \overline{\overline{\varepsilon}}^{-1}\delta(\overline{r} - \overline{r}') \quad (10d)$$

2.1. Electric Type DGF $\overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}')$ for an Electrically Gyrotropic Medium

The electric type DGF for an electrically gyrotropic medium with $\mu = 1$ satisfies the second order dyadic differential equation given by (10a)

$$\left[\nabla \times \nabla \times \overline{\overline{I}} - k_0^2 \overline{\overline{\varepsilon}}\right] \cdot \overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}') = i\omega\mu_0\overline{\overline{I}}\delta(\overline{r} - \overline{r}') \quad (11)$$

where $k_0^2 = \omega^2\mu_0\varepsilon_0$ and ω is the angular frequency. To facilitate the construction of the DGF $\overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}')$, which is a solution of the second order differential equation given in Eq. (11), we transform the problem to one in the *k-domain*. This is accomplished by introducing the Fourier transform pair of DGF $\overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}')$ as follows.

$$\overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}') = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \overline{\overline{G}}_{ee}^{e}(\overline{k}, \overline{r}') e^{i\overline{k}\cdot\overline{r}} d^3\overline{k} \quad (12a)$$

$$\overline{\overline{G}}_{ee}^{e}(\overline{k}, \overline{r}') = \int_{-\infty}^{\infty} \overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}') e^{i\overline{k}\cdot\overline{r}} d^3\overline{r} \quad (12b)$$

After substituting Eq. (12) into Eq. (11) and using the identity

$$\delta(\overline{r} - \overline{r}') = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')} d^3\overline{k} \quad (13)$$

we obtain the Fourier-transformed DGF

$$\overline{\overline{G}}_{ee}^{e}(\overline{k}, \overline{r}') = -i\omega\mu_0 \left[\overline{k}\,\overline{k} + k_0^2\overline{\overline{\varepsilon}}\right]^{-1} e^{-i\overline{k}\cdot\overline{r}'} \quad (14)$$

where $\overline{k}$ and $\overline{\overline{k}}$ are defined as

$$\overline{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z \tag{15a}$$

$$\overline{\overline{k}} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \tag{15b}$$

The DGF $\overline{\overline{G}}^{e}_{ee}(\overline{k}, \overline{r}')$ given in Eq. (14) can be expressed in terms of the electric wave matrix $\overline{\overline{W}}_E$ as

$$\overline{\overline{G}}^{e}_{ee}(\overline{k}, \overline{r}') = -i\omega\mu_0 \overline{\overline{W}}_E^{-1} e^{-i\overline{k}\cdot\overline{r}'} \tag{16}$$

where we introduce

$$\overline{\overline{W}}_E = \left[\overline{\overline{k}}\,\overline{\overline{k}} + k_0^2\overline{\overline{\varepsilon}}\right] \tag{17}$$

as an *electric wave matrix* for an electrically gyrotropic medium. When we substitute Eq. (16) into Eq. (12a), we obtain

$$\overline{\overline{G}}^{e}_{ee}(\overline{r}, \overline{r}') = \frac{-i\omega\mu_0}{(2\pi)^3} \int_{-\infty}^{\infty} \overline{\overline{W}}_E^{-1} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')} d^3\overline{k}$$

or

$$\overline{\overline{G}}^{e}_{ee}(\overline{r}, \overline{r}') = \frac{-i\omega\mu_0}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{adj\left(\overline{\overline{W}}_E\right)}{\left|\overline{\overline{W}}_E\right|} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')} d^3\overline{k} \tag{18}$$

As a result, the problem of finding the DGF $\overline{\overline{G}}^{e}_{ee}(\overline{r}, \overline{r}')$ is simplified to finding the inverse of an electric wave matrix $\overline{\overline{W}}_E$ which is equal to

$$\overline{\overline{W}}_E^{-1} = \frac{adj\left(\overline{\overline{W}}_E\right)}{\left|\overline{\overline{W}}_E\right|} \tag{19}$$

Using Maxwell's Eqs. (4)–(5) in the source free region, i.e., $\overline{J} = 0$ and $\overline{M} = 0$, it can be shown that the dispersion equation for a gyroelectric medium is given by

$$\left|\overline{\overline{k}}\,\overline{\overline{k}} + k_0^2\overline{\overline{\varepsilon}}\right| = 0 \quad \text{or} \quad \left|\overline{\overline{W}}_E\right| = 0.$$

The determinant of the electric wave matrix for a gyroelectric medium can be written as

$$\left|\overline{\overline{W}}_E\right| = k_o^2\varepsilon_3\left(k_z^2 - k_{zI}^2\right)\left(k_z^2 - k_{zII}^2\right) \tag{20}$$

Note that $\left|\overline{\overline{W}}_E\right| = 0$ when $k_z^2 = k_{zI}^2$ or $k_z^2 = k_{zII}^2$. k_{zI}^2 and k_{zII}^2 are the wavenumbers squared and defined as

$$\frac{k_{zI}^2}{k_0^2} = \frac{\left[2\varepsilon_1\varepsilon_3 - \frac{k_\rho^2}{k_0^2}\left(\varepsilon_1+\varepsilon_3\right)\right] + \left[\frac{k_\rho^4}{k_0^4}\left(\varepsilon_1-\varepsilon_3\right)^2 + 4\varepsilon_2^2\varepsilon_3\left(\varepsilon_3 - \frac{k_\rho^2}{k_0^2}\right)\right]^{\frac{1}{2}}}{2\varepsilon_3} \tag{21a}$$

$$\frac{k_{zII}^2}{k_0^2} = \frac{\left[2\varepsilon_1\varepsilon_3 - \frac{k_\rho^2}{k_0^2}\left(\varepsilon_1+\varepsilon_3\right)\right] - \left[\frac{k_\rho^4}{k_0^4}\left(\varepsilon_1-\varepsilon_3\right)^2 + 4\varepsilon_2^2\varepsilon_3\left(\varepsilon_3 - \frac{k_\rho^2}{k_0^2}\right)\right]^{\frac{1}{2}}}{2\varepsilon_3} \tag{21b}$$

where

$$k_\rho^2 = k_x^2 + k_y^2 \tag{22}$$

The adjoint of $\overline{\overline{W}}_E$ or $adj\left(\overline{\overline{W}}_E\right)$ can be written as

$$\begin{aligned} adj\overline{\overline{W}}_E \;=\; & \left(k_0^4 adj\overline{\overline{\varepsilon}} - k^2k_0^2\varepsilon_3\overline{\overline{I}}\right) + \hat{k}\hat{k}\left[k^2\left(k^2 - k_0^2\varepsilon_1\right)\right] \\ & + \hat{b}_0\hat{b}_0\left[k^2k_0^2\left(\varepsilon_3-\varepsilon_1\right)\right] \\ & + \left(\hat{k}\times\hat{b}_0\right)\left(\hat{k}\times\hat{b}_0\right)\left[k^2k_0^2\left(\varepsilon_3-\varepsilon_1\right)\right] \\ & + i\varepsilon_2k^2k_0^2\left[\hat{k}\left(\hat{k}\times\hat{b}_0\right) - \left(\hat{k}\times\hat{b}_0\right)\hat{k}\right] \end{aligned} \tag{23}$$

where

$$k^2 = k_\rho^2 + k_z^2 = k_x^2 + k_y^2 + k_z^2 \tag{24}$$

$$\hat{k} = \frac{\overline{k}}{\left|\overline{k}\right|} \tag{25}$$

We can now represent $adj\left(\overline{\overline{W}}_E\right)$ in matrix form as

$$adj\overline{\overline{W}}_E = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{26}$$

where the elements of the matrix in Eq. (26) are given in Appendix A by Eqs. (A1)–(A9).

We perform the integration over k_z after substituting Eqs. (20), (26) into (18). The poles of the integrand occur at the zeros of $\left|\overline{\overline{W}}_E\right|$ denoted by $k_z = \pm k_{zI}$ and $k_z = \pm k_{zII}$. Assuming the medium to be slightly lossy, i.e., $\mathrm{Im}k_z \ll \mathrm{Re}k_z$, $\mathrm{Im}k_z > 0$ and performing the contour integration over k_z, we obtain the following result for $z > z'$:

$$\overline{\overline{G}}{}^e_{ee}(\overline{r}, \overline{r}') = \frac{-\omega\mu_0}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \left\{\frac{1}{k_0^2\varepsilon_3\left(k_{zI}^2 - k_{zII}^2\right)}\left[\frac{adj\overline{\overline{W}}_E\left(k_{zI}\right)}{k_{zI}}e^{i\overline{k}_I\cdot(\overline{r}-\overline{r}')} - \frac{adj\overline{\overline{W}}_E\left(k_{zII}\right)}{k_{zII}}e^{i\overline{k}_{II}\cdot(\overline{r}-\overline{r}')}\right]\right\},\ z > z' \tag{27}$$

Similarly, when $z < z'$, $\overline{\overline{G}}{}^e_{ee}(\overline{r}, \overline{r}')$ can be obtained by assuming $\mathrm{Im}(-k_{zI}) < 0$ and $\mathrm{Im}(-k_{zII}) < 0$ as

$$\overline{\overline{G}}{}^e_{ee}(\overline{r}, \overline{r}') = \frac{-\omega\mu_0}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \left\{\frac{1}{k_0^2\varepsilon_3\left(k_{zI}^2 - k_{zII}^2\right)}\left[\frac{adj\overline{\overline{W}}_E\left(-k_{zI}\right)}{k_{zI}}e^{i\overline{\kappa}_I\cdot(\overline{r}-\overline{r}')} - \frac{adj\overline{\overline{W}}_E\left(-k_{zII}\right)}{k_{zII}}e^{i\overline{\kappa}_{II}\cdot(\overline{r}-\overline{r}')}\right]\right\},\ z < z' \tag{28}$$

where

$$\overline{k}_I = \overline{k}_\rho + \hat{z}k_{zI} \tag{29a}$$

$$\overline{k}_{II} = \overline{k}_\rho + \hat{z}k_{zII} \tag{29b}$$

$$\overline{\kappa}_I = \overline{k}_\rho - \hat{z}k_{zI} \tag{29c}$$

$$\overline{\kappa}_{II} = \overline{k}_\rho - \hat{z}k_{zII} \tag{29d}$$

$\overline{k}_I$, $\overline{k}_{II}$ represent the wave vectors for the upward $(+z)$ traveling waves of type I and type II. $\overline{\kappa}_I$, $\overline{\kappa}_{II}$ represent those for the downward $(-z)$ traveling waves.

Since the matrix $adj\left(\overline{\overline{W}}_E\right)$ is a *Hermitian* matrix, accordingly it satisfies the following relation

$$adj\left(\overline{\overline{W}}_E\right) = \left(adj\left(\overline{\overline{W}}_E\right)\right)^\dagger \tag{30}$$

It can be shown that $adj\left(\overline{\overline{W}}_E\right)$ can be written as a single dyad in terms of its eigenvectors by solving the eigenvalue problem $adj\overline{\overline{W}}_E.\hat{u} = \lambda\hat{u}$.

The details of the derivation to represent $adj\left(\overline{\overline{W}}_E\right)$ as a single dyad is given in Appendix B.

We can then represent the DGF $\overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}')$ given by Eq. (27) in dyadic form when $z > z'$ as

$$\begin{aligned}\overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}') &= \frac{-\omega\mu_0}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \left\{\frac{1}{k_0^2\varepsilon_3\left(k_{zI}^2 - k_{zII}^2\right)}\right. \\ &\quad \left[\frac{\alpha_I}{k_{zI}}\hat{e}_{nI}\left(k_{zI}\right)\hat{e}_{nI}^{*}\left(k_{zI}\right)e^{i\overline{k}_I\cdot(\overline{r}-\overline{r}')}\right. \\ &\quad \left.\left. - \frac{\alpha_{II}}{k_{zII}}\hat{e}_{nII}\left(k_{zII}\right)\hat{e}_{nII}^{*}\left(k_{zII}\right)e^{i\overline{k}_{II}\cdot(\overline{r}-\overline{r}')}\right]\right\}, \quad z > z' \end{aligned} \tag{31a}$$

Similarly, the DGF $\overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}')$ given by Eq. (28) is written in dyadic form when $z < z'$ as

$$\begin{aligned}\overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}') &= \frac{-\omega\mu_0}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \left\{\frac{1}{k_0^2\varepsilon_3\left(k_{zI}^2 - k_{zII}^2\right)}\right. \\ &\quad \left[\frac{\alpha_I}{k_{zI}}\hat{e}_{nI}\left(-k_{zI}\right)\hat{e}_{nI}^{*}\left(-k_{zI}\right)e^{i\overline{\kappa}_I\cdot(\overline{r}-\overline{r}')}\right. \\ &\quad \left.\left. - \frac{\alpha_{II}}{k_{zII}}\hat{e}_{nII}\left(-k_{zII}\right)\hat{e}_{nII}^{*}\left(-k_{zII}\right)e^{i\overline{\kappa}_{II}\cdot(\overline{r}-\overline{r}')}\right]\right\}, \; z < z' \end{aligned} \tag{31b}$$

In Eq. (31a)–(31b), α_I and α_{II} are the eigenvalues, $\hat{e}_{nI}(\pm k_{zI})$ and $\hat{e}_{nII}(\pm k_{zI})$ are the orthonormal eigenvectors, which physically represent two characteristic electric fields for the type I and type II waves that exist in a gyroelectric medium and they are defined as

$$\hat{e}_{nI}(\pm k_{zI}) = \frac{\overline{e}_I(\pm k_{zI})}{norm\left(\overline{e}_I(\pm k_{zI})\right)} \tag{32a}$$

$$\hat{e}_{nII}(\pm k_{zII}) = \frac{\overline{e}_{II}(\pm k_{zII})}{norm\left(\overline{e}_{II}(\pm k_{zII})\right)} \tag{32b}$$

where

$$\overline{e}_I(\pm k_{zI}) = \begin{bmatrix} 1 \\ \dfrac{A_{13}A_{21} + A_{23}\alpha_I - A_{23}A_{11}}{\alpha_I A_{13} - A_{22}A_{13} + A_{23}A_{12}} \\ \dfrac{-A_{12}}{A_{13}}\left[\dfrac{A_{13}A_{21} + A_{23}\alpha_I - A_{23}A_{11}}{\alpha_I A_{13} - A_{22}A_{13} + A_{13}A_{12}}\right] + \dfrac{\alpha_I - A_{11}}{A_{13}} \end{bmatrix} \tag{33a}$$

$$\overline{e}_{II}(\pm k_{zII}) = \begin{bmatrix} 1 \\ \dfrac{A_{13}A_{21} + A_{23}\alpha_{II} - A_{23}A_{11}}{\alpha_{II}A_{13} - A_{22}A_{13} + A_{23}A_{12}} \\ \dfrac{-A_{12}}{A_{13}}\left[\dfrac{A_{13}A_{21} + A_{23}\alpha_{II} - A_{23}A_{11}}{\alpha_{II}A_{13} - A_{22}A_{13} + A_{13}A_{12}}\right] + \dfrac{\alpha_{II} - A_{11}}{A_{13}} \end{bmatrix} \tag{33b}$$

where the elements of A_{ij}, $(i,j) = 1,2,3$ are defined in Appendix A by Eqs. (A1)–(A9). The forms of $\overline{e}_I$, $\overline{e}_{II}$ given by Eq. (33a)–(33b) are valid when the x component of the electric field is not zero. The forms should be adjusted when the x component of the electric field is zero. For each eigenvector, the corresponding eigenvalues are given by

$$\begin{aligned} \alpha_I &= k_I^4 - k_I^2 k_0^2 \left[\varepsilon_1 \left(3 - \cos^2\theta\right) + \varepsilon_3 \left(1 + \cos^2\theta\right)\right] \\ &\quad + k_0^4 \left(\varepsilon_1^2 - \varepsilon_2^2 + 2\varepsilon_1\varepsilon_3\right) \end{aligned} \tag{34a}$$

$$\begin{aligned} \alpha_{II} &= k_{II}^4 - k_{II}^2 k_0^2 \left[\varepsilon_1 \left(3 - \cos^2\theta\right) + \varepsilon_3 \left(1 + \cos^2\theta\right)\right] \\ &\quad + k_0^4 \left(\varepsilon_1^2 - \varepsilon_2^2 + 2\varepsilon_1\varepsilon_3\right) \end{aligned} \tag{34b}$$

where θ is an angle between the wave vector $\left(\overline{k}_I, \overline{k}_{II}\right)$ and the direction of applied dc magnetic field $\left(\hat{b}_0\right)$. k_I, k_{II} are the wavenumbers for the type I and type II waves, and they are given by

$$\begin{aligned} k_I^2 &= k_\rho^2 + k_{zI}^2 \\ k_{II}^2 &= k_\rho^2 + k_{zII}^2, \end{aligned}$$

2.2. Magnetic Type DGF $\overline{\overline{G}}{}_{mm}^{e}(\overline{r}, \overline{r}')$ for an Electrically Gyrotropic Medium

The second order dyadic differential equation for the magnetic type DGF is given by Eq. (10b) as

$$\left[\nabla \times \overline{\overline{\varepsilon}}^{-1} \cdot \nabla \times \overline{\overline{I}} - k_0^2 \overline{\overline{I}}\right] \cdot \overline{\overline{G}}{}_{mm}^{e}(\overline{r}, \overline{r}') = i\omega\varepsilon_0 \overline{\overline{I}} \delta(\overline{r} - \overline{r}') \tag{35}$$

To find the DGF $\overline{\overline{G}}{}_{mm}^{e}(\overline{r}, \overline{r}')$, which is a solution of the dyadic differential equation given by equation (35), we transform the problem into *k-domain*. For this purpose, we introduce the Fourier transform pair of the DGF $\overline{\overline{G}}{}_{mm}^{e}(\overline{r}, \overline{r}')$ as follows.

$$\overline{\overline{G}}{}_{mm}^{e}(\overline{r}, \overline{r}') = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \overline{\overline{G}}{}_{mm}^{e}(\overline{k}, \overline{r}') e^{i\overline{k}\cdot\overline{r}} d^3\overline{k} \tag{36a}$$

and

$$\overline{\overline{G}}_{mm}^{e}(\overline{k},\,\overline{r}') = \int_{-\infty}^{\infty} \overline{\overline{G}}_{mm}^{e}(\overline{r},\,\overline{r}')e^{-i\overline{k}\cdot\overline{r}}d^3\overline{r} \tag{36b}$$

Substituting Eq. (36a) into (35) and using the identity given by Eq. (13), we obtain the Fourier-transformed DGF as

$$\overline{\overline{G}}_{mm}^{e}(\overline{k},\,\overline{r}') = -i\omega\varepsilon_0 \left[\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{I}}\right]^{-1} e^{-i\overline{k}\cdot\overline{r}'} \tag{37}$$

The DGF $\overline{\overline{G}}_{mm}^{e}(\overline{k},\,\overline{r}')$ given in Eq. (37) can be expressed in terms of the magnetic wave matrix $\overline{\overline{W}}_H$ as

$$\overline{\overline{G}}_{mm}^{e}(\overline{k},\,\overline{r}') = -i\omega\varepsilon_0\overline{\overline{W}}_H^{-1} e^{-i\overline{k}\cdot\overline{r}'} \tag{38}$$

where we introduce

$$\overline{\overline{W}}_H = \left[\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{I}}\right] \tag{39}$$

as a *magnetic wave matrix* for an electrically gyrotropic medium. When we substitute Eq. (38) into Eq. (36a), we obtain

$$\overline{\overline{G}}_{mm}^{e}(\overline{r},\,\overline{r}') = \frac{-i\omega\varepsilon_0}{(2\pi)^3}\int_{-\infty}^{\infty} \overline{\overline{W}}_H^{-1} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')}d^3\overline{k}$$

or

$$\overline{\overline{G}}_{mm}^{e}(\overline{r},\,\overline{r}') = \frac{-i\omega\varepsilon_0}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{adj\left(\overline{\overline{W}}_H\right)}{\left|\overline{\overline{W}}_H\right|} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')}d^3\overline{k} \tag{40}$$

So, the problem of finding the DGF $\overline{\overline{G}}_{mm}^{e}(\overline{r},\,\overline{r}')$ again reduces to finding the inverse of the magnetic wave matrix $\overline{\overline{W}}_H$ which is equal to

$$\overline{\overline{W}}_H^{-1} = \frac{adj\left(\overline{\overline{W}}_H\right)}{\left|\overline{\overline{W}}_H\right|} \tag{41}$$

Instead of taking the inverse of the magnetic wave matrix with the method described in Section 2.1, we can utilize the following relation between $\overline{\overline{W}}_E$ and $\overline{\overline{W}}_H$ to find the inverse of $\overline{\overline{W}}_H$.

$$k_0^2\overline{\overline{W}}_H^{-1} = \overline{\overline{I}} - \overline{\overline{k}}\,\overline{\overline{W}}_E^{-1}\overline{\overline{k}} \tag{42}$$

This relationship is derived in Appendix C. From Eq. (42),

$$\overline{\overline{W}}_H^{-1} = \frac{1}{k_0^2}\frac{\left[\left|\overline{\overline{W}}_E\right|\overline{\overline{I}} - \overline{\overline{k}}adj\overline{\overline{W}}_E\overline{\overline{k}}\right]}{\left|\overline{\overline{W}}_E\right|} \tag{43}$$

We can now rewrite $\overline{\overline{W}}_H^{-1}$ in matrix form as

$$\overline{\overline{W}}_H^{-1} = \frac{\overline{\overline{B}}}{\left|\overline{\overline{W}}_E\right|} \tag{44}$$

where

$$\overline{\overline{B}} = \frac{\left[\left|\overline{\overline{W}}_E\right|\overline{\overline{I}} - \overline{\overline{k}} adj \overline{\overline{W}}_E \overline{\overline{k}}\right]}{k_0^2}$$

or

$$\overline{\overline{B}} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \tag{45}$$

The elements of B_{ij}, $(i, j) = 1, 2, 3$ in Eq. (45) are given in Appendix A by Eqs. (A10)–(A18).

We perform the integration over k_z after substituting Eqs. (20), (45) into Eq. (40). The poles of the integrand occur at the zeros of $\left|\overline{\overline{W}}_E\right|$ denoted by $k_z = \pm k_{zI}$ and $k_z = \pm k_{zII}$ where k_{zI} and k_{zII} are defined by Eqs. (21a)–(21b). Assuming the medium to be slightly lossy, i.e., $\mathrm{Im}k_z \ll \mathrm{Re}k_z$, $\mathrm{Im}k_z > 0$ and performing the contour integration over k_z, we obtain the following result for $z > z'$:

$$\begin{aligned} \overline{\overline{G}}{}^e_{mm}(\overline{r}, \overline{r}') &= \frac{-i\omega\varepsilon_0}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \\ &\left\{ \frac{1}{k_0^2 \varepsilon_3 \left(k_{zI}^2 - k_{zII}^2\right)} \left[\frac{\overline{\overline{B}}(k_{zI})}{k_{zI}} e^{i\overline{k}_I \cdot (\overline{r} - \overline{r}')} - \frac{\overline{\overline{B}}(k_{zII})}{k_{zII}} e^{i\overline{k}_{II} \cdot (\overline{r} - \overline{r}')} \right] \right\}, \\ & z > z' \end{aligned} \tag{46}$$

Similarly, when $z < z'$, $\overline{\overline{G}}{}^e_{mm}(\overline{r}, \overline{r}')$ can be obtained by assuming $\mathrm{Im}(-k_{zI}) < 0$ and $\mathrm{Im}(-k_{zII}) < 0$ as

$$\begin{aligned} \overline{\overline{G}}{}^e_{mm}(\overline{r}, \overline{r}') &= \frac{-i\omega\varepsilon_0}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \\ &\left\{ \frac{1}{k_0^2 \varepsilon_3 \left(k_{zI}^2 - k_{zII}^2\right)} \left[\frac{\overline{\overline{B}}(-k_{zI})}{k_{zI}} e^{i\overline{\kappa}_I \cdot (\overline{r} - \overline{r}')} - \frac{\overline{\overline{B}}(-k_{zII})}{k_{zII}} e^{i\overline{\kappa}_{II} \cdot (\overline{r} - \overline{r}')} \right] \right\}, \\ & z < z' \end{aligned} \tag{47}$$

where $\overline{k}_I$, $\overline{k}_{II}$, $\overline{\kappa}_I$, and $\overline{\kappa}_{II}$ are defined by Eqs. (29a)–(29d).

Because the matrix $\overline{\overline{B}}$ has the same matrix properties as $adj\overline{\overline{W}}_E$, it is *Hermitian*. Hence $\overline{\overline{B}}$ satisfies the following relation

$$\overline{\overline{B}} = \overline{\overline{B}}^{\dagger} \tag{48}$$

We can then represent the DGF $\overline{\overline{G}}^{e}_{mm}(\overline{r}, \overline{r}')$ given by Eq. (46) in dyadic form when $z > z'$ as

$$\begin{aligned}\overline{\overline{G}}^{e}_{mm}(\overline{r}, \overline{r}') &= \frac{-\omega\varepsilon_0}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \\ &\left\{\frac{1}{k_0^2\varepsilon_3(k_{zI}^2 - k_{zII}^2)}\left[\frac{\beta_I}{k_{zI}}\hat{h}_{nI}(k_{zI})\hat{h}^*_{nI}(k_{zI})e^{i\overline{k}_I\cdot(\overline{r}-\overline{r}')}\right.\right. \\ &\left.\left. - \frac{\beta_{II}}{k_{zII}}\hat{h}_{nII}(k_{zII})\hat{h}^*_{nII}(k_{zII})e^{i\overline{k}_{II}\cdot(\overline{r}-\overline{r}')}\right]\right\}\end{aligned} \tag{49a}$$

Similarly, the DGF $\overline{\overline{G}}^{e}_{mm}(\overline{r}, \overline{r}')$ given by Eq. (47) in dyadic form when $z < z'$ as

$$\begin{aligned}\overline{\overline{G}}^{e}_{mm}(\overline{r}, \overline{r}') &= \frac{-\omega\varepsilon_0}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \\ &\left\{\frac{1}{k_0^2\varepsilon_3(k_{zI}^2 - k_{zII}^2)}\left[\frac{\beta_I}{k_{zI}}\hat{h}_{nI}(-k_{zI})\hat{h}^*_{nI}(-k_{zI})e^{i\overline{\kappa}_I\cdot(\overline{r}-\overline{r}')}\right.\right. \\ &\left.\left. - \frac{\beta_{II}}{k_{zII}}\hat{h}_{nII}(-k_{zII})\hat{h}^*_{nII}(-k_{zII})e^{i\overline{\kappa}_{II}\cdot(\overline{r}-\overline{r}')}\right]\right\}\end{aligned} \tag{49b}$$

where β_I and β_{II} are the eigenvalues and $\hat{h}_{nI}(\pm k_{zI})$ and $\hat{h}_{nII}(\pm k_{zI})$ are the orthonormal eigenvectors, which physically represent two characteristic *magnetic* fields for the type I and type II waves that exist in a gyroelectric medium and defined as

$$\hat{h}_{nI}(\pm k_{zI}) = \frac{\overline{h}_I(\pm k_{zI})}{norm\left(\overline{h}_I\left(\pm k_{zI}\right)\right)} \tag{50a}$$

$$\hat{h}_{nII}(\pm k_{zII}) = \frac{\overline{h}_{II}(\pm k_{zII})}{norm\left(\overline{h}_{II}\left(\pm k_{zII}\right)\right)} \tag{50b}$$

where

$$\overline{h}_I(\pm k_{zI}) = \begin{bmatrix} 1 \\ \dfrac{B_{13}B_{21} + B_{23}\beta_I - B_{23}B_{11}}{\beta_I B_{13} - B_{22}B_{13} + B_{23}B_{12}} \\ \dfrac{-B_{12}}{B_{13}}\left[\dfrac{B_{13}B_{21} + B_{23}\beta_I - B_{23}B_{11}}{\beta_I B_{13} - B_{22}B_{13} + B_{13}B_{12}}\right] + \dfrac{\beta_I - B_{11}}{B_{13}} \end{bmatrix} \tag{51a}$$

$$\overline{h}_{II}(\pm k_{zII}) = \begin{bmatrix} 1 \\ \dfrac{B_{13}B_{21} + B_{23}\beta_{II} - B_{23}B_{11}}{\beta_{II}B_{13} - B_{22}B_{13} + B_{23}B_{12}} \\ \dfrac{-B_{12}}{B_{13}}\left[\dfrac{B_{13}B_{21} + B_{23}\beta_{II} - B_{23}B_{11}}{\beta_{II}B_{13} - B_{22}B_{13} + B_{13}B_{12}}\right] + \dfrac{\beta_{II} - B_{11}}{B_{13}} \end{bmatrix} \tag{51b}$$

The elements of B_{ij}, $(i, j) = 1, 2, 3$ are defined in Appendix A by Eqs. (A10)–(A18). The forms of $\overline{h}_I$, $\overline{h}_{II}$ given by Eq. (51a)–(51b) are valid when the x component of the magnetic field is not zero. For each eigenvector, the corresponding eigenvalues are given by

$$\begin{aligned} \beta_I &= k_I^4\left[\varepsilon_3\cos^2\theta + \varepsilon_1\sin^2\theta\right] - 2k_I^2k_0^2\left[\varepsilon_1\varepsilon_3\left(1 + \cos^2\theta\right)\right. \\ &+ \left.\left(\varepsilon_1^2 - \varepsilon_2^2\right)\sin^2\theta\right] + 3k_0^4\varepsilon_3\left(\varepsilon_1^2 - \varepsilon_2^2\right) \end{aligned} \tag{52a}$$

$$\begin{aligned} \beta_{II} &= k_{II}^4\left[\varepsilon_3\cos^2\theta + \varepsilon_1\sin^2\theta\right] - 2k_{II}^2k_0^2\left[\varepsilon_1\varepsilon_3\left(1 + \cos^2\theta\right)\right. \\ &+ \left.\left(\varepsilon_1^2 - \varepsilon_2^2\right)\sin^2\theta\right] + 3k_0^4\varepsilon_3\left(\varepsilon_1^2 - \varepsilon_2^2\right) \end{aligned} \tag{52b}$$

2.3. Electric-Magnetic Type DGF $\overline{\overline{G}}_{em}^{e}(\overline{r}, \overline{r}')$ and Magnetic-Electric Type DGF $\overline{\overline{G}}_{me}^{e}(\overline{r}, \overline{r}')$ for an Electrically Gyrotropic Medium

Since we derived the explicit expressions for the magnetic type DGF $\overline{\overline{G}}_{mm}^{e}(\overline{r}, \overline{r}')$ and the electric type DGF $\overline{\overline{G}}_{ee}^{e}(\overline{r}, \overline{r}')$, we can use them to find the electric-magnetic type DGF $\overline{\overline{G}}_{em}^{e}(\overline{r}, \overline{r}')$ and the magnetic-electric type DGF $\overline{\overline{G}}_{me}^{e}(\overline{r}, \overline{r}')$, respectively.

To relate the electric-magnetic type DGF $\overline{\overline{G}}_{em}^{e}(\overline{r}, \overline{r}')$ to the magnetic type DGF $\overline{\overline{G}}_{mm}^{e}(\overline{r}, \overline{r}')$, substitute Eqs. (6)–(7) with $\overline{J} = 0$ into Eq. (5). We obtain,

$$\nabla \times \left[\int_{V'} \overline{\overline{G}}_{mm}^{e}(\overline{r}, \overline{r}') \cdot \overline{M}(\overline{r}')d^3\overline{r}'\right] = -i\omega\varepsilon_0\overline{\overline{\varepsilon}} \cdot \int_{V'} \overline{\overline{G}}_{em}^{e}(\overline{r}, \overline{r}') \cdot \overline{M}(\overline{r}')d^3\overline{r}'$$

or

$$\nabla \times \overline{\overline{G}}_{mm}^{e}(\overline{r}, \overline{r}') = -i\omega\varepsilon_0\overline{\overline{\varepsilon}} \cdot \overline{\overline{G}}_{em}^{e}(\overline{r}, \overline{r}') \tag{53}$$

Using Eq. (53), the electric-magnetic type DGF $\overline{\overline{G}}_{em}^{e}(\overline{r}, \overline{r}')$ can be expressed as

$$\overline{\overline{G}}_{em}^{e}(\overline{r}, \overline{r}') = \frac{i}{\omega\varepsilon_0}\overline{\overline{\varepsilon}}^{-1} \cdot \nabla \times \overline{\overline{G}}_{mm}^{e}(\overline{r}, \overline{r}') \tag{54}$$

We relate the magnetic-electric type DGF $\overline{\overline{G}}^e_{me}(\overline{r}, \overline{r}')$ to the electric type DGF $\overline{\overline{G}}^e_{ee}(\overline{r}, \overline{r}')$ by substituting Eqs. (6)–(7) with $\overline{M} = 0$ into Eq. (4). We obtain,

$$\nabla \times \left[\int_{V'} \overline{\overline{G}}^e_{ee}(\overline{r}, \overline{r}') \cdot \overline{J}(\overline{r}')d^3\overline{r}'\right] = i\omega\mu_0\mu \cdot \int_{V'} \overline{\overline{G}}^e_{me}(\overline{r}, \overline{r}') \cdot \overline{J}(\overline{r}')d^3\overline{r}'$$

or

$$\nabla \times \overline{\overline{G}}^e_{ee}(\overline{r}, \overline{r}') = i\omega\mu_0\mu \cdot \overline{\overline{G}}^e_{me}(\overline{r}, \overline{r}') \tag{55}$$

Similarly, using Eq. (55) the magnetic-electric type DGF $\overline{\overline{G}}^e_{me}(\overline{r}, \overline{r}')$ can be expressed as

$$\overline{\overline{G}}^e_{me}(\overline{r}, \overline{r}') = \frac{-i}{\omega\mu_0\mu}\nabla \times \overline{\overline{G}}^e_{ee}(\overline{r}, \overline{r}') \tag{56}$$

3. CONCLUSION

In this paper, the complete set of dyadic Green's functions for an unbounded electrically gyrotropic or a gyroelectric medium is derived using a new formulation technique. The analytic expressions for the electric-type and the magnetic-type DGFs are presented in a unique form in terms of characteristic field vectors for the type I and type II waves that exist in a gyroelectric medium. In the forms that are used to express $\overline{\overline{G}}^e_{ee}(\overline{r}, \overline{r}')$ and $\overline{\overline{G}}^e_{mm}(\overline{r}, \overline{r}')$, $\hat{e}_n(\pm k_z)$ and $\hat{h}_n(\pm k_z)$ are the eigenvectors representing the characteristic electric and magnetic field vectors, respectively. The electric-magnetic type and magnetic-electric type DGFs, $\overline{\overline{G}}^e_{em}(\overline{r}, \overline{r}')$ and $\overline{\overline{G}}^e_{me}(\overline{r}, \overline{r}')$, are expressed in terms of $\overline{\overline{G}}^e_{mm}(\overline{r}, \overline{r}')$ and $\overline{\overline{G}}^e_{ee}(\overline{r}, \overline{r}')$, respectively. It is shown that the dyadic decomposition greatly facilitates the calculation of an inverse operation, which is crucial in derivation of Green's function. The method introduced here can be used in solving the problems of ionospheric propagation, radiation and scattering involving new types of anisotropic materials such as ceramics and advanced composites, which are widely used in high frequency electromagnetic and optical applications.

APPENDIX A. MATRIX ELEMENTS OF $adj\left(\overline{\overline{W}}_E\right)$ AND $\overline{\overline{B}}$

The elements of the matrices $adj\left(\overline{\overline{W}}_E\right)$ and $\overline{\overline{B}}$ are given by Eqs. (A1)–(A9) and Eqs. (A10)–(A18), respectively.

$$A_{11} = \left(k_\rho^2 + k_z^2\right) k_x^2 - k_0^2 \left[\varepsilon_1 k_\rho^2 + \varepsilon_3 \left(k_x^2 + k_z^2\right)\right] + k_0^4 \varepsilon_1 \varepsilon_3 \tag{A1}$$

$$A_{12} = \left(k_\rho^2 + k_z^2\right) k_x k_y - k_0^2 \left[i\varepsilon_2 k_\rho^2 + \varepsilon_3 k_x k_y\right] + ik_0^4 \varepsilon_2 \varepsilon_3 \quad \text{(A2)}$$

$$A_{13} = \left(k_\rho^2 + k_z^2\right) k_x k_z - k_0^2 \left[\varepsilon_1 k_x k_z + i\varepsilon_2 k_y k_z\right] \quad \text{(A3)}$$

$$A_{21} = \left(k_\rho^2 + k_z^2\right) k_x k_y - k_0^2 \left[-i\varepsilon_2 k_\rho^2 + \varepsilon_3 k_x k_y\right] - ik_0^4 \varepsilon_2 \varepsilon_3 \quad \text{(A4)}$$

$$A_{22} = \left(k_\rho^2 + k_z^2\right) k_y^2 - k_0^2 \left[\varepsilon_1 k_\rho^2 + \varepsilon_3 \left(k_y^2 + k_z^2\right)\right] + k_0^4 \varepsilon_1 \varepsilon_3 \quad \text{(A5)}$$

$$A_{23} = \left(k_\rho^2 + k_z^2\right) k_y k_z - k_0^2 \left[\varepsilon_1 k_y k_z - i\varepsilon_2 k_x k_z\right] \quad \text{(A6)}$$

$$A_{31} = \left(k_\rho^2 + k_z^2\right) k_x k_z - k_0^2 \left[\varepsilon_1 k_x k_z + i\varepsilon_2 k_y k_z\right] \quad \text{(A7)}$$

$$A_{32} = \left(k_\rho^2 + k_z^2\right) k_y k_z - k_0^2 \left[\varepsilon_1 k_y k_z - i\varepsilon_2 k_x k_z\right] \quad \text{(A8)}$$

$$A_{33} = \left(k_\rho^2 + k_z^2\right) k_z^2 - k_0^2 \left[\varepsilon_1 \left(k_\rho^2 + 2k_z^2\right)\right] + k_0^4 \left[\varepsilon_1^2 - \varepsilon_2^2\right] \quad \text{(A9)}$$

$$\begin{aligned} B_{11} = &\ k_z^2 k_x^2 \varepsilon_3 + k_0^4 \varepsilon_3 \left(\varepsilon_1^2 - \varepsilon_3^2\right) \\ &- k_0^2 \left[k_x^2 \left(\varepsilon_1^2 - \varepsilon_2^2\right) + \left(k_\rho^2 + k_z^2\right) \varepsilon_1 \varepsilon_3\right] + k_\rho^2 k_x^2 \varepsilon_1 \end{aligned} \quad \text{(A10)}$$

$$B_{12} = k_x k_y \left[\varepsilon_1 k_\rho^2 + \varepsilon_3 k_z^2\right] - k_0^2 \left[k_x k_y \left(\varepsilon_1^2 - \varepsilon_2^2\right) - i\varepsilon_2 \varepsilon_3 k_z^2\right] \quad \text{(A11)}$$

$$B_{13} = k_x k_z \left[\varepsilon_1 k_\rho^2 + \varepsilon_3 k_z^2\right] - k_0^2 \left[k_x k_z \varepsilon_1 \varepsilon_3 + ik_z k_y \varepsilon_2 \varepsilon_3\right] \quad \text{(A12)}$$

$$B_{21} = k_x k_y \left[\varepsilon_1 k_\rho^2 + \varepsilon_3 k_z^2\right] - k_0^2 \left[k_x k_y \left(\varepsilon_1^2 - \varepsilon_3^2\right) + i\varepsilon_2 \varepsilon_3 k_z^2\right] \quad \text{(A13)}$$

$$\begin{aligned} B_{22} = &\ k_z^2 k_y^2 \varepsilon_3 + k_0^4 \varepsilon_3 \left(\varepsilon_1^2 - \varepsilon_3^2\right) \\ &- k_0^2 \left[k_y^2 \left(\varepsilon_1^2 - \varepsilon_2^2\right) + \left(k_\rho^2 + k_z^2\right) \varepsilon_1 \varepsilon_3\right] + k_\rho^2 k_y^2 \varepsilon_1 \end{aligned} \quad \text{(A14)}$$

$$B_{23} = k_z k_y \left[\varepsilon_1 k_\rho^2 + \varepsilon_3 k_z^2\right] - k_0^2 \left[k_z k_y \varepsilon_1 \varepsilon_3 + ik_x k_z \varepsilon_2 \varepsilon_3\right] \quad \text{(A15)}$$

$$B_{31} = k_x k_z \left[\varepsilon_1 k_\rho^2 + \varepsilon_3 k_z^2\right] - k_0^2 \left[k_x k_z \varepsilon_1 \varepsilon_3 + ik_z k_y \varepsilon_2 \varepsilon_3\right] \quad \text{(A16)}$$

$$B_{32} = k_z k_y \left[\varepsilon_1 k_\rho^2 + \varepsilon_3 k_z^2\right] - k_0^2 \left[k_z k_y \varepsilon_1 \varepsilon_3 + ik_x k_z \varepsilon_2 \varepsilon_3\right] \quad \text{(A17)}$$

$$\begin{aligned} B_{33} = &\ k_z^4 \varepsilon_3 + k_0^4 \varepsilon_3 \left(\varepsilon_1^2 - \varepsilon_3^2\right) - k_0^2 \left[k_\rho^2 \left(\varepsilon_1^2 - \varepsilon_2^2\right) + 2k_z^2 \varepsilon_1 \varepsilon_3\right] \\ &+ k_\rho^2 k_z^2 \varepsilon_1 \end{aligned} \quad \text{(A18)}$$

APPENDIX B. DYADIC REPRESENTATION OF $adj\left(\overline{\overline{W}}_E\right)$

When the elements of the matrix $adj\left(\overline{\overline{W}}_E\right)$ is reviewed, it is seen that it satisfies the following relation

$$adj\left(\overline{\overline{W}}_E\right) = \left(adj\left(\overline{\overline{W}}_E\right)\right)^{\dagger} \tag{B1}$$

where † denotes the conjugate transpose of the matrix.

This requires $adj\left(\overline{\overline{W}}_E\right)$ to be a *Hermitian* matrix. The eigenvalues of a Hermitian matrix are real and the eigenvectors corresponding to distinct eigenvalues are orthogonal in the sense that the Hermitian dot product vanishes. In other words every Hermitian matrix possesses a complete set of orthonormal eigenvectors. In this case, the completeness relation [13] becomes

$$\overline{\overline{I}} = \hat{u}_1\hat{u}_1^* + \hat{u}_2\hat{u}_2^* + \hat{u}_3\hat{u}_3^* \tag{B2}$$

where

$$\hat{u}_i^* \cdot \hat{u}_j = \delta_{ij} \tag{B3}$$

and $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$ are the orthonormal eigenvectors of the Hermitian matrix. Then the dyadic decomposition of the matrix $adj\left(\overline{\overline{W}}_E\right)$ takes the form as

$$adj\left(\overline{\overline{W}}_E\right) = \lambda_1\hat{u}_1\hat{u}_1^* + \lambda_2\hat{u}_2\hat{u}_2^* + \lambda_3\hat{u}_3\hat{u}_3^* \tag{B4}$$

The characteristic equation of $adj\left(\overline{\overline{W}}_E\right)$, i.e., $f(\lambda)$ can be expressed as

$$\begin{aligned} f(\lambda) &= \left|\lambda I - adj\overline{\overline{W}}_E\right| \\ &= \lambda^3 - tr\left(adj\overline{\overline{W}}_E\right)\lambda^2 + tr\left(adj\left(adj\overline{\overline{W}}_E\right)\lambda - \left|adj\overline{\overline{W}}_E\right|\right. \\ &= 0 \end{aligned} \tag{B5}$$

where tr stands for the trace of the matrix. Using the following identities [14],

$$\left|adj\overline{\overline{W}}_E\right| = \left|\overline{\overline{W}}_E\right|^2 \tag{B6}$$

$$adj\left(adj\overline{\overline{W}}_E\right) = \left|\overline{\overline{W}}_E\right|\overline{\overline{W}}_E \tag{B7}$$

$f(\lambda)$ can be written as

$$f(\lambda) = \lambda^3 - tr\left(adj\overline{\overline{W}}_E\right)\lambda^2 + tr\left(\left|\overline{\overline{W}}_E\right|\overline{\overline{W}}_E\right)\lambda - \left|\overline{\overline{W}}_E\right|^2 = 0 \quad \text{(B8)}$$

Since $\left|\overline{\overline{W}}_E\left(k_z\right)\right|$ is zero when $k_z = \pm k_{zI}$ or $k_z = \pm k_{zII}$, then characteristic equation for $adj\left(\overline{\overline{W}}_E\right)$ reduces to

$$f(\lambda) = \lambda^3 - tr\left(adj\overline{\overline{W}}_E\right)\lambda^2 \quad \text{(B9)}$$

Hence, the eigenvalues for $adj\left(\overline{\overline{W}}_E\right)$ are

$$\lambda_1 = tr\left(adj\overline{\overline{W}}_E\right), \ \lambda_2 = \lambda_3 = 0 \quad \text{(B10)}$$

As a result, using Eq. (B4) we can express $adj\overline{\overline{W}}_E$ as a single dyad in the following form for the adjoint matrices of the type I and the type II waves as follows.

$$adj\overline{\overline{W}}_E(\pm k_{zI}) = \alpha_I\left[\hat{e}_{nI}\left(\pm k_{zI}\right)\hat{e}^*_{nI}\left(\pm k_{zI}\right)\right] \quad \text{(B11a)}$$

$$adj\overline{\overline{W}}_E(\pm k_{zII}) = \alpha_{II}\left[\hat{e}_{nII}\left(\pm k_{zII}\right)\hat{e}^*_{nII}\left(\pm k_{zII}\right)\right] \quad \text{(B11b)}$$

$\lambda_I = \alpha_I$, $\lambda_{II} = \alpha_{II}$ are the eigenvalues and are defined in Eqs. (34a)–(34b). $\hat{e}_{nI}\left(\pm k_{zI}\right)$ and $\hat{e}_{nII}\left(\pm k_{zI}\right)$ are the orthonormal eigenvectors representing two characteristic electric fields for the type I and type II waves that exist in a gyroelectric medium and are defined in Eqs. (32a)–(32b).

APPENDIX C. RELATIONS BETWEEN $\overline{\overline{W}}_E^{-1}$ AND $\overline{\overline{W}}_H^{-1}$

The electric wave matrix $\overline{\overline{W}}_E$ and magnetic wave matrix $\overline{\overline{W}}_H$ for a general anisotropic medium can be expressed as

$$\overline{\overline{W}}_E = \left[\overline{\overline{k}}\overline{\overline{\mu}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\varepsilon}}\right] \quad \text{(C1)}$$

$$\overline{\overline{W}}_H = \left[\overline{\overline{k}}\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\mu}}\right] \quad \text{(C2)}$$

We can relate the magnetic wave matrix and the electric wave matrix $\overline{\overline{W}}_H$ as follows. When we perform the matrix multiplication of Eq. (C2) from the left hand side (LHS) with $\overline{\overline{k}}\overline{\overline{\mu}}^{-1}$, we obtain

$$\overline{\overline{k}}\overline{\overline{\mu}}^{-1}\overline{\overline{W}}_H = \overline{\overline{k}}\overline{\overline{\mu}}^{-1}\left[\overline{\overline{k}}\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\mu}}\right] = \overline{\overline{k}}\overline{\overline{\mu}}^{-1}\overline{\overline{k}}\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{k}} \quad \text{(C3)}$$

Now we perform the matrix multiplication of Eq. (C1) from right hand side (RHS) with $\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}}$ as

$$\overline{\overline{W}}_E\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} = \left[\overline{\overline{k}}\overline{\overline{\mu}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\varepsilon}}\right]\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}}$$

or

$$\overline{\overline{W}}_E\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} = \overline{\overline{k}}\overline{\overline{\mu}}^{-1}\overline{\overline{k}}\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{k}} \tag{C4}$$

When Eq. (C3–C4) are compared, it is seen that

$$\overline{\overline{k}}\overline{\overline{\mu}}^{-1}\overline{\overline{W}}_H = \overline{\overline{W}}_E\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}}$$

or

$$\overline{\overline{W}}_E^{-1}\overline{\overline{k}}\overline{\overline{\mu}}^{-1} = \overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}}\,\overline{\overline{W}}_H^{-1} \tag{C5}$$

Another useful relation can be obtained as follows. When we perform the matrix multiplication of Eq. (C2) from the LHS with $\overline{\overline{\mu}}^{-1}$, we obtain

$$\overline{\overline{\mu}}^{-1}\overline{\overline{W}}_H = \overline{\overline{\mu}}^{-1}\overline{\overline{k}}\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{I}} \tag{C6}$$

Now we perform the matrix multiplication of Eq. (C6) from the RHS by $\overline{\overline{W}}_H^{-1}$ as

$$\overline{\overline{\mu}}^{-1} = \overline{\overline{\mu}}^{-1}\overline{\overline{k}}\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}}\,\overline{\overline{W}}_H^{-1} + k_0^2\overline{\overline{W}}_H^{-1}$$

or

$$k_0^2\overline{\overline{W}}_H^{-1} = \overline{\overline{\mu}}^{-1} - \overline{\overline{\mu}}^{-1}\overline{\overline{k}}\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}}\,\overline{\overline{W}}_H^{-1} \tag{C7}$$

Now we substitute Eq. (C5) into Eq. (C7) and we obtain

$$k_0^2\overline{\overline{W}}_H^{-1} = \overline{\overline{\mu}}^{-1} - \overline{\overline{\mu}}^{-1}\overline{\overline{k}}\,\overline{\overline{W}}_E^{-1}\overline{\overline{k}}\overline{\overline{\mu}}^{-1} \tag{C8}$$

Since for a gyroelectric case, $\overline{\overline{\mu}} = \overline{\overline{I}}$, then Eq. (C8) reduces to

$$k_0^2\overline{\overline{W}}_H^{-1} = \overline{\overline{I}} - \overline{\overline{k}}\,\overline{\overline{W}}_E^{-1}\overline{\overline{k}} \qquad \text{for } \textit{gyroelectric} \text{ medium} \tag{C9}$$

Eq. (C9) relates the magnetic wave matrix $\overline{\overline{W}}_H$ and the electric wave matrix $\overline{\overline{W}}_E$ through their inverses in the most practical manner such that we can utilize the existing results that we already have derived.

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