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THE CLASS OF BI-ANISOTROPIC IB-MEDIA

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Abstract—Representation of electromagnetic expressions in terms of the four-dimensional differential-form formalism has been recently shown to allow simple analysis to problems involving general classes of linear electromagnetic media. In the present paper, another class of media is defined by expressing the medium dyadic representing the mapping between the electromagnetic two-forms in terms of one dyadic representing mapping between two four-vectors. Thus, the class, labeled as that of IB-media, is represented by 16 parameters instead of the 36 of the most general bi-anisotropic medium. Condition for the medium dyadic is derived and and properties of fields in the IB-medium are discussed.

1. INTRODUCTION

It has been demonstrated that differential-form representation is of advantage not only for the basic Maxwell equations[†] [1–5],

$$\mathbf{d} \wedge \boldsymbol{\Phi} = 0, \qquad \mathbf{d} \wedge \boldsymbol{\Psi} = \boldsymbol{\gamma}, \tag{1}$$

relating the source and field multiforms

$$\boldsymbol{\Phi} = \mathbf{B} + \mathbf{E} \wedge \mathbf{d}\tau, \qquad \boldsymbol{\Psi} = \mathbf{D} - \mathbf{H} \wedge \mathbf{d}\tau, \qquad \boldsymbol{\gamma} = \boldsymbol{\varrho} - \mathbf{J} \wedge \mathbf{d}\tau, \tag{2}$$

but also when analyzing various classes of linear electromagnetic media. In fact, expressing the linear relation between the electromagnetic two-forms in terms of the medium dyadic $\overline{\overline{\mathsf{M}}}$ as

$$\boldsymbol{\Psi} = \overline{\overline{\mathsf{M}}}|\boldsymbol{\Phi}, \tag{3}$$

[†] Notation adopted here follows that of [5]. A brief introduction can be found in the Appendix of [6].

it has been shown that many important and interesting classes of media represented in a complicated form in terms of the classical Gibbsian three-dimensional dyadics take a simple form in the four-dimensional differential-form representation. As an example we may first mention the PEMC medium, a generalization of PEC and PMC media, which appears quite pathological in the light of Gibbsian medium parameters ϵ, μ, ξ, ζ but is represented by a simple scalar factor in place of $\overline{\overline{\mathsf{M}}}$ in (3) [7–9]. As another example, the class of decomposable media, defined through a complicated set of Gibbsian dyadic expressions in [10–12] can be expressed in quite simple terms [5, 13].

In the present study, a class of electromagnetic media, simply defined in differential-form formalism, is analyzed for the existence of electromagnetic fields. The analysis is based on the extension of Gibbsian three-dimensional vectors and dyadics to four-dimensional multivectors and dyadics. Because of space limitation, the reader should consult [5] for details of in the methods used in this analysis. However, an interpretation of the basic results will be also given in terms of Gibbsian vectors and dyadic medium parameters.

2. IB MEDIUM

In continuation of the research of simple classes of electromagnetic media using four-dimensional differential-form formalism we consider a class which can be defined in terms of a single dyadic $\overline{\overline{\mathsf{B}}} \in \mathbb{E}_1\mathbb{F}_1$ expressed in the form

$$\overline{\overline{\mathsf{M}}} = \left(\overline{\overline{\mathsf{I}}} {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{B}}}\right)^T = \left(\overline{\overline{\mathsf{B}}} {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{I}}}\right)^T = \overline{\overline{\mathsf{I}}}^T {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{B}}}^T. \tag{4}$$

Such a class of media will be labeled here as that of IB-media. Assuming that the eigenvectors $\mathbf{e}_i$, $i = 1 \cdots 4$ of the dyadic $\overline{\overline{\mathsf{B}}}$ form a basis and their dual basis is denoted by $\boldsymbol{\varepsilon}_i$, we can write

$$\overline{\overline{\mathsf{B}}}|\mathbf{e}_i = B_i \mathbf{e}_i, \qquad \overline{\overline{\mathsf{B}}} = \sum_i B_i \mathbf{e}_i \boldsymbol{\varepsilon}_i, \tag{5}$$

with B_i denoting the eigenvalues. Applying the identity

$$\begin{aligned} \left(\overline{\overline{\mathsf{I}}} {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{B}}}\right) |\left(\mathbf{e}_i \wedge \mathbf{e}_j\right) &= \left(\overline{\overline{\mathsf{I}}}|\mathbf{e}_i\right) \wedge \left(\overline{\overline{\mathsf{B}}}|\mathbf{e}_j\right) + \left(\overline{\overline{\mathsf{B}}}|\mathbf{e}_i\right) \wedge \left(\overline{\overline{\mathsf{I}}}|\mathbf{e}_j\right) \\ &= \left(B_i + B_j\right) \mathbf{e}_i \wedge \mathbf{e}_j, \end{aligned} \tag{6}$$

the left eigenbivectors of $\overline{\overline{\mathsf{M}}}$ can be identified as $\mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_{ij}$, corresponding to the eigenvalues $B_i + B_j$ and the expansion

$$\overline{\overline{\mathsf{M}}} = \sum_{i<j} \left(B_i + B_j\right) \boldsymbol{\varepsilon}_{ij} \mathbf{e}_{ij}. \tag{7}$$

From this the following relation between the scalar invariants is found:

$$\mathrm{tr}\overline{\overline{\mathsf{M}}} = \sum_{i<j} (B_i + B_j) = 3\sum B_i = 3\mathrm{tr}\overline{\overline{\mathsf{B}}}. \tag{8}$$

Because the dyadic $\overline{\overline{\mathsf{B}}}$ has $4 \times 4 = 16$ free parameters, the dyadic $\overline{\overline{\mathsf{M}}}$ is defined by the same number of parameters, instead of $6 \times 6 = 36$ which corresponds to the most general medium. Thus, $\overline{\overline{\mathsf{M}}}$ must satisfy some restricting condition. From the result (A7) with $\overline{\overline{\mathsf{Y}}}$ replaced by $\overline{\overline{\mathsf{M}}}^T$ in Appendix A we see that the condition for a medium dyadic to be of the form (4) is

$$6\overline{\overline{\mathsf{M}}} + 2\overline{\overline{\mathsf{I}}}^{(2)T}\left(\mathrm{tr}\overline{\overline{\mathsf{M}}}\right) - 3\left(\overline{\overline{\mathsf{M}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}\right) {}^\wedge_\wedge \overline{\overline{\mathsf{I}}}^T = 0, \tag{9}$$

or

$$\overline{\overline{\mathsf{M}}} = -\frac{1}{6}\overline{\overline{\mathsf{I}}}^T {}^\wedge_\wedge \overline{\overline{\mathsf{I}}}^T \left(\mathrm{tr}\overline{\overline{\mathsf{M}}}\right) + \frac{1}{2}\overline{\overline{\mathsf{I}}}^T {}^\wedge_\wedge \left(\overline{\overline{\mathsf{M}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}\right). \tag{10}$$

If this is satisfied by $\overline{\overline{\mathsf{M}}}$, the dyadic $\overline{\overline{\mathsf{B}}}$ can be identified as

$$\overline{\overline{\mathsf{B}}} = \frac{1}{2}\overline{\overline{\mathsf{M}}}^T\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T - \frac{1}{6}\left(\mathrm{tr}\overline{\overline{\mathsf{M}}}\right)\overline{\overline{\mathsf{I}}}. \tag{11}$$

As a special case, the perfect electromagnetic conductor (PEMC) [7] or axion medium [4] is included in the class of IB media. In fact, assuming

$$\overline{\overline{\mathsf{B}}} = \frac{M}{2}\overline{\overline{\mathsf{I}}}, \qquad M = \frac{1}{2}\mathrm{tr}\overline{\overline{\mathsf{B}}}, \tag{12}$$

we have

$$\overline{\overline{\mathsf{M}}} = M\overline{\overline{\mathsf{I}}}^{(2)T}, \tag{13}$$

which is the definition of the PEMC medium. Obviously, we can extract the axion part from the general IB medium dyadic and write

$$\overline{\overline{\mathsf{M}}} = M\overline{\overline{\mathsf{I}}}^{(2)T} + \overline{\overline{\mathsf{M}}}_o, \qquad M = \frac{1}{6}\mathrm{tr}\overline{\overline{\mathsf{M}}}, \tag{14}$$

where $\overline{\overline{\mathsf{M}}}_o$ is trace-free.

Nature of the dyadic $\overline{\overline{\mathsf{M}}}_o$ can be conveniently studied by applying the identity (B9) from Appendix B, which for $\overline{\overline{\mathsf{C}}} = \overline{\overline{\mathsf{M}}}^T$ can be written as

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{M}}} = \mathrm{tr}\left(\overline{\overline{\mathsf{M}}}\right)\overline{\overline{\mathsf{I}}}^{(2)} - \left(\overline{\overline{\mathsf{M}}}^T\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right) {}^\wedge_\wedge \overline{\overline{\mathsf{I}}} + \overline{\overline{\mathsf{M}}}^T. \tag{15}$$

This is valid for any medium dyadic $\overline{\overline{\mathsf{M}}}$. For the present IB-medium with $\overline{\overline{\mathsf{M}}} = (\overline{\overline{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}})^T$ and $\mathrm{tr}\overline{\overline{\mathsf{B}}} = \mathrm{tr}\overline{\overline{\mathsf{M}}}/3$ (15) takes on the simpler special form (B10) as

$$\overline{\overline{\mathsf{I}}}{}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{M}}} = \frac{1}{3}\mathrm{tr}\left(\overline{\overline{\mathsf{M}}}\right)\overline{\overline{\mathsf{I}}}{}^{(2)} - \overline{\overline{\mathsf{M}}}{}^T. \tag{16}$$

Inserting $\overline{\overline{\mathsf{I}}}{}^{(4)} = \mathbf{e}_N\boldsymbol{\varepsilon}_N$ with $\mathbf{e}_N = \mathbf{e}_{1234}$ a quadrivector and $\boldsymbol{\varepsilon}_N = \boldsymbol{\varepsilon}_{1234}$ its reciprocal dual quadrivector, (16) can be transformed to

$$\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}} = \frac{\mathrm{tr}\overline{\overline{\mathsf{M}}}}{3}\overline{\overline{\mathsf{I}}}{}^{(2)}\rfloor\mathbf{e}_N - \left(\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}\right)^T. \tag{17}$$

Defining the modified (metric) medium dyadic as $\overline{\overline{\mathsf{M}}}_g = \mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}$ [5], (17) now becomes

$$\overline{\overline{\mathsf{M}}}_g = \frac{\mathrm{tr}\overline{\overline{\mathsf{M}}}}{3}\overline{\overline{\mathsf{I}}}{}^{(2)}\rfloor\mathbf{e}_N - \overline{\overline{\mathsf{M}}}{}_g^T, \tag{18}$$

which is valid for the modified medium dyadic of any IB medium. Noting that

$$\begin{aligned}\overline{\overline{\mathsf{I}}}{}^{(2)}\rfloor\mathbf{e}_N &= \sum_{i<j}\mathbf{e}_{ij}\boldsymbol{\varepsilon}_{ij}\rfloor\mathbf{e}_{1234} \\ &= \mathbf{e}_{12}\mathbf{e}_{34} - \mathbf{e}_{13}\mathbf{e}_{24} + \mathbf{e}_{14}\mathbf{e}_{23} + \mathbf{e}_{23}\mathbf{e}_{14} - \mathbf{e}_{24}\mathbf{e}_{13} + \mathbf{e}_{34}\mathbf{e}_{12}\end{aligned} \tag{19}$$

is a symmetric dyadic, from (18) we find that the modified medium dyadic corresponding to a trace-free IB medium dyadic is antisymmetric.

In summary, the medium dyadic $\overline{\overline{\mathsf{M}}}$ of an IB-medium was seen to consist of two components: a multiple of the unit dyadic $\overline{\overline{\mathsf{I}}}{}^{(2)T}$ and a trace-free part $\overline{\overline{\mathsf{M}}}_o$, which modified by $\mathbf{e}_N\lfloor$ gives an antisymmetric dyadic. In a decomposition of the medium dyadic in three irreducible pieces, Hehl and Obukhov have called the former component by the name axion, and the latter component by the name skewon [4, 14–16]. The third piece, called principal by the same authors and defined by trace-free $\overline{\overline{\mathsf{M}}}$ and symmetric $\overline{\overline{\mathsf{M}}}_g$, is thus missing from the IB-medium. Conversely, any medium with zero principal part can be represented as an IB medium in the form $\overline{\overline{\mathsf{M}}} = (\overline{\overline{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}})^T$.

3. FIELD AND POTENTIAL EQUATIONS

Let us consider the field two-forms $\mathbf{\Phi}$, $\mathbf{\Psi}$ generated by the electric source $\boldsymbol{\gamma}$ in a homogeneous IB medium. Since there is no magnetic source, the field $\mathbf{\Phi}$ satisfying the first of the Maxwell equations (1) can be (locally) represented in terms of a potential one-form $\boldsymbol{\alpha}$ as

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{d} \wedge \boldsymbol{\alpha}(\mathbf{x}). \tag{20}$$

Invoking the rule [5]

$$\left(\overline{\overline{\mathsf{A}}}^T {}^\wedge_\wedge \overline{\overline{\mathsf{B}}}^T\right) \lfloor \left(\boldsymbol{\beta}_1 \wedge \boldsymbol{\beta}_2\right) = \left(\overline{\overline{\mathsf{A}}}^T|\boldsymbol{\beta}_1\right) \wedge \left(\overline{\overline{\mathsf{B}}}^T|\boldsymbol{\beta}_2\right) + \left(\overline{\overline{\mathsf{B}}}^T|\boldsymbol{\beta}_1\right) \wedge \left(\overline{\overline{\mathsf{A}}}^T|\boldsymbol{\beta}_2\right), \tag{21}$$

valid for any dyadics $\overline{\overline{\mathsf{A}}}$, $\overline{\overline{\mathsf{B}}} \in \mathbb{E}_1\mathbb{F}_1$ and one-forms $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2 \in \mathbb{F}_1$, we can expand

$$\begin{aligned}\mathbf{\Psi}(\mathbf{x}) &= \overline{\overline{\mathsf{M}}}|\mathbf{\Phi}(\mathbf{x}) = \left(\overline{\overline{\mathsf{I}}}^T {}^\wedge_\wedge \overline{\overline{\mathsf{B}}}^T\right) \lfloor \left(\mathbf{d} \wedge \boldsymbol{\alpha}(\mathbf{x})\right) \\ &= \mathbf{d} \wedge \left(\overline{\overline{\mathsf{B}}}^T|\boldsymbol{\alpha}(\mathbf{x})\right) + \left(\overline{\overline{\mathsf{B}}}^T|\mathbf{d}\right) \wedge \boldsymbol{\alpha}(\mathbf{x}),\end{aligned} \tag{22}$$

whence the second of the Maxwell equations (1) becomes

$$\mathbf{d} \wedge \overline{\overline{\mathsf{M}}}|\mathbf{\Phi}(\mathbf{x}) = \mathbf{d} \wedge \left(\mathbf{d}|\overline{\overline{\mathsf{B}}}\right) \wedge \boldsymbol{\alpha}(\mathbf{x}) = -\left(\mathbf{d}|\overline{\overline{\mathsf{B}}}\right) \wedge \mathbf{d} \wedge \boldsymbol{\alpha}(\mathbf{x}) = \boldsymbol{\gamma}(\mathbf{x}), \tag{23}$$

which for the field two-form reads

$$\left(\mathbf{d}|\overline{\overline{\mathsf{B}}}\right) \wedge \mathbf{\Phi}(\mathbf{x}) = -\boldsymbol{\gamma}(\mathbf{x}). \tag{24}$$

Because of charge conservation, the source three-form $\boldsymbol{\gamma}(\mathbf{x})$ must satisfy $\mathbf{d} \wedge \boldsymbol{\gamma}(\mathbf{x}) = 0$. (24) requires that, in an IB-medium, it must also satisfy a second condition,

$$\left(\mathbf{d}|\overline{\overline{\mathsf{B}}}\right) \wedge \boldsymbol{\gamma}(\mathbf{x}) = 0, \tag{25}$$

because otherwise there would be no solution $\mathbf{\Phi}$ for (24). In other words, a source not satisfying (25) cannot exist in an IB medium. This is not too extraordinary as it was previously seen that certain sources could not exist within the PEMC medium.

Considering solutions of the potential equation (23), one can note that if $\boldsymbol{\alpha}_1(\mathbf{x})$ is one solution, so is the one-form

$$\boldsymbol{\alpha}(\mathbf{x}) = \boldsymbol{\alpha}_1(\mathbf{x}) + c_1\mathbf{d}\phi_1(\mathbf{x}) + c_2\mathbf{d}|\overline{\overline{\mathsf{B}}}\phi_2(\mathbf{x}) \tag{26}$$

for any two scalar functions $\phi_1(\mathbf{x})$, $\phi_2(\mathbf{x})$ and scalar coefficients c_1, c_2. To find these coefficients it appears necessary to have two extra conditions corresponding to the single Lorenz condition defined for the Q-media, for example [5].

4. PLANE WAVES

To convince oneself that there may exist nonzero fields in a homogeneous IB medium, let us consider plane-wave type of fields

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{\Phi}_o \exp(\boldsymbol{\nu}|\mathbf{x}), \qquad \mathbf{\Psi}(\mathbf{x}) = \mathbf{\Psi}_o \exp(\boldsymbol{\nu}|\mathbf{x}), \tag{27}$$

satisfying the Maxwell equations (1) with $\gamma(\mathbf{x}) = 0$ in the region of interest. Here, $\boldsymbol{\nu} \in \mathbb{F}_1$ is the propagation dual vector and $\mathbf{\Phi}_o$, $\mathbf{\Psi}_o \in \mathbb{F}_2$ are the field-amplitude dual bivectors. Equations (1) and (24) now require that $\boldsymbol{\nu}$ and $\mathbf{\Phi}_o$ satisfy

$$\boldsymbol{\nu} \wedge \mathbf{\Phi}_o = 0, \qquad \left(\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}}\right) \wedge \mathbf{\Phi}_o = 0. \tag{28}$$

Substituting the field amplitude in terms of a potential-amplitude dual vector $\boldsymbol{\alpha}_o$ as

$$\mathbf{\Phi}_o = \boldsymbol{\nu} \wedge \boldsymbol{\alpha}_o, \tag{29}$$

leads to the plane-wave equation

$$\left(\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}}\right) \wedge \boldsymbol{\nu} \wedge \boldsymbol{\alpha}_o = 0, \tag{30}$$

which means that the three dual vectors must be linearly dependent. Beyond (30), there are no restrictions concerning the choice of the two dual vectors $\boldsymbol{\nu}$ and $\boldsymbol{\alpha}_o$. In fact, we could start by choosing the dual vector $\boldsymbol{\nu}$. Assuming that $\boldsymbol{\nu}$ and $\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}}$ are linearly independent: $\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}} \wedge \boldsymbol{\nu} \neq 0$ i.e., $\boldsymbol{\nu}$ is not a left dual eigenvector of the dyadic $\overline{\overline{\mathsf{B}}}$, there must exist two scalars c_1, c_2 such that we can write

$$\boldsymbol{\alpha}_o = c_1\boldsymbol{\nu} + c_2\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}}. \tag{31}$$

In this case the field amplitude of the plane wave has the form

$$\mathbf{\Phi}_o = \boldsymbol{\nu} \wedge \boldsymbol{\alpha}_o = c_2\boldsymbol{\nu} \wedge \left(\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}}\right) \tag{32}$$

and from (22) the other field amplitude is obtained as

$$\mathbf{\Psi}_o = \boldsymbol{\nu} \wedge \left(\overline{\overline{\mathsf{B}}}^T|\boldsymbol{\alpha}_o\right) + \left(\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}}\right) \wedge \boldsymbol{\alpha}_o = c_2\boldsymbol{\nu} \wedge \left(\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}}^2\right). \tag{33}$$

However, if the propagation dual vector is chosen to satisfy

$$\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}} = \lambda\boldsymbol{\nu}, \qquad \lambda \neq 0, \tag{34}$$

(30) is valid for any $\boldsymbol{\alpha}_o$ and the two equations (28) become the same. In this case there is a lot of freedom for $\mathbf{\Phi}_o$ to choose from, because it is only required to satisfy $\boldsymbol{\nu} \wedge \mathbf{\Phi}_o = 0$.

To conclude, an IB medium defined by the dyadic $\overline{\overline{\mathsf{B}}}$ can support the following electromagnetic field:

$$\mathbf{\Phi}(\mathbf{x}) = c\boldsymbol{\nu} \wedge \left(\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}}\right) \exp\left(\boldsymbol{\nu}|\mathbf{x}\right), \qquad \mathbf{\Psi}(\mathbf{x}) = c\boldsymbol{\nu} \wedge \left(\boldsymbol{\nu}|\overline{\overline{\mathsf{B}}}^2\right) \exp\left(\boldsymbol{\nu}|\mathbf{x}\right), \tag{35}$$

satisfying the Maxwell equations

$$\mathbf{d} \wedge \mathbf{\Phi}(\mathbf{x}) = \boldsymbol{\nu} \wedge \mathbf{\Phi}(\mathbf{x}) = 0, \qquad \mathbf{d} \wedge \mathbf{\Psi}(\mathbf{x}) = \boldsymbol{\nu} \wedge \mathbf{\Psi}(\mathbf{x}) = 0, \tag{36}$$

as well as the medium condition

$$\mathbf{\Psi}(\mathbf{x}) = \overline{\overline{\mathsf{M}}}|\mathbf{\Phi}(\mathbf{x}) = \left(\overline{\mathsf{I}} {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{B}}}\right)^T |\mathbf{\Phi}(\mathbf{x}), \tag{37}$$

for any $\boldsymbol{\nu}$ except when it is chosen as a dual eigenvector of $\overline{\overline{\mathsf{B}}}$.

5. 3D REPRESENTATION OF THE IB MEDIUM

To explore the properties of an IB-medium, we extract the temporal vector $\mathbf{e}_4$ and dual vector $\boldsymbol{\varepsilon}_4$ in the expansion

$$\overline{\overline{\mathsf{B}}} = \overline{\overline{\mathsf{A}}} + \mathbf{a}\boldsymbol{\varepsilon}_4 + \mathbf{e}_4\boldsymbol{\alpha} + a\mathbf{e}_4\boldsymbol{\varepsilon}_4, \tag{38}$$

where $\overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{F}_1$ is a three-dimensional (spatial) dyadic, $\mathbf{a}$ is a three-dimensional vector, $\boldsymbol{\alpha}$ is a three-dimensional dual vector and a is a scalar. Also, denoting

$$\mathbf{d} = \mathbf{d}_s + \boldsymbol{\varepsilon}_4\partial_\tau, \tag{39}$$

where $\mathbf{d}_s$ differentiates along the three-dimensional spatial coordinates, the condition (25) for the source three-form $\boldsymbol{\gamma}$ can be expanded as

$$\begin{aligned}\left(\mathbf{d}|\overline{\overline{\mathsf{B}}}\right) \wedge \boldsymbol{\gamma}(\mathbf{x}) &= \left(\mathbf{d}_s|\overline{\overline{\mathsf{A}}} + \mathbf{d}_s|\mathbf{a}\boldsymbol{\varepsilon}_4 + \partial_\tau\boldsymbol{\alpha} + a\partial_\tau\boldsymbol{\varepsilon}_4\right) \wedge \left(\boldsymbol{\varrho} - \mathbf{J} \wedge \boldsymbol{\varepsilon}_4\right) \\ &= -\mathbf{d}_s|\overline{\overline{\mathsf{A}}} \wedge \mathbf{J} \wedge \boldsymbol{\varepsilon}_4 + (\mathbf{d}_s|\mathbf{a})\, \boldsymbol{\varepsilon}_4 \wedge \boldsymbol{\varrho} - \partial_\tau\boldsymbol{\alpha} \wedge \mathbf{J} \wedge \boldsymbol{\varepsilon}_4 \\ &\quad + a\partial_\tau\boldsymbol{\varepsilon}_4 \wedge \boldsymbol{\varrho} \\ &= 0, \end{aligned} \tag{40}$$

or

$$\mathbf{d}_s|\overline{\overline{\mathsf{A}}} \wedge \mathbf{J} + (\mathbf{d}_s|\mathbf{a})\boldsymbol{\varrho} + \boldsymbol{\alpha} \wedge \partial_\tau\mathbf{J} + a\partial_\tau\boldsymbol{\varrho} = 0. \tag{41}$$

Combining with

$$\mathbf{d} \wedge \boldsymbol{\gamma} = 0 \qquad \Rightarrow \qquad \partial_\tau\boldsymbol{\varrho} + \mathbf{d}_s \wedge \mathbf{J} = 0, \tag{42}$$

we obtain a condition for the current two-from $\mathbf{J}$:

$$\left(\mathbf{d}_s|\overline{\overline{\mathsf{A}}}\partial_\tau + \boldsymbol{\alpha}\partial_\tau^2 - \mathbf{a}|\mathbf{d}_s\mathbf{d}_s - a\mathbf{d}_s\partial_\tau\right) \wedge \mathbf{J} = 0. \tag{43}$$

For example, $\mathbf{J}$ constant in time and space satisfies the condition.

Expanding the unit dyadic as

$$\overline{\overline{\mathsf{I}}} = \overline{\overline{\mathsf{I}}}_s + \mathbf{e}_4\boldsymbol{\varepsilon}_4, \tag{44}$$

the medium dyadic expressions (4) can be written as

$$\overline{\overline{\mathsf{M}}} = \overline{\overline{\mathsf{I}}}_s^T {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{A}}}^T + \overline{\overline{\mathsf{I}}}_s^T {}^{\wedge}_{\wedge} \left(\boldsymbol{\varepsilon}_4\mathbf{a}\right) + \overline{\overline{\mathsf{I}}}_s^T {}^{\wedge}_{\wedge} \left(\boldsymbol{\alpha}\mathbf{e}_4\right) + \left(\overline{\overline{\mathsf{A}}}^T + a\overline{\overline{\mathsf{I}}}_s^T\right) {}^{\wedge}_{\wedge} \boldsymbol{\varepsilon}_4\mathbf{e}_4. \tag{45}$$

The constitutive equations in terms of three-dimensional fields are

$$\begin{aligned} \mathbf{D} - \mathbf{H}\wedge\boldsymbol{\varepsilon}_4 &= \overline{\overline{\mathsf{M}}}|(\mathbf{B} + \mathbf{E}\wedge\boldsymbol{\varepsilon}_4) \\ &= \left(\overline{\overline{\mathsf{I}}}_s^T {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{A}}}^T - \boldsymbol{\varepsilon}_4\wedge\overline{\overline{\mathsf{I}}}_s^T\wedge\mathbf{a}\right)|\mathbf{B} \\ &\quad -\left(\boldsymbol{\alpha}\wedge\overline{\overline{\mathsf{I}}}_s^T + \boldsymbol{\varepsilon}_4\wedge\left(\overline{\overline{\mathsf{A}}}^T + a\overline{\overline{\mathsf{I}}}_s^T\right)\right)|\mathbf{E}. \end{aligned} \tag{46}$$

They can be split in two equations as

$$\mathbf{D} = \left(\overline{\overline{\mathsf{I}}}_s^T {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{A}}}^T\right)|\mathbf{B} - \boldsymbol{\alpha}\wedge\mathbf{E}, \tag{47}$$

$$\mathbf{H} = -\mathbf{a}\rfloor\mathbf{B} - \overline{\overline{\mathsf{A}}}^T|\mathbf{E} - a\mathbf{E}. \tag{48}$$

Comparing with the representation [5]

$$\mathbf{D} = \overline{\overline{\alpha}}|\mathbf{B} + \overline{\overline{\epsilon}}'|\mathbf{E}, \tag{49}$$

$$\mathbf{H} = \overline{\overline{\mu}}^{-1}|\mathbf{B} + \overline{\overline{\beta}}'|\mathbf{E}, \tag{50}$$

we can identify the three-dimensional medium dyadics as

$$\overline{\overline{\alpha}} = \overline{\overline{\mathsf{I}}}_s^T {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{A}}}^T = \mathrm{tr}\overline{\overline{\mathsf{A}}}\,\overline{\overline{\mathsf{I}}}_s^{(2)T} - \boldsymbol{\varepsilon}_{123}\mathbf{e}_{123}\lfloor\lfloor\overline{\overline{\mathsf{A}}}, \tag{51}$$

$$\overline{\overline{\epsilon}}' = -\boldsymbol{\alpha}\wedge\overline{\overline{\mathsf{I}}}_s^T = -\overline{\overline{\mathsf{I}}}^{(2)}\lfloor\boldsymbol{\alpha}, \tag{52}$$

$$\overline{\overline{\mu}}^{-1} = -\mathbf{a}\rfloor\overline{\overline{\mathsf{I}}}_s^{(2)T} = -\overline{\overline{\mathsf{I}}}_s^T\wedge\mathbf{a}, \tag{53}$$

$$\overline{\overline{\beta}} = -\left(\overline{\overline{\mathsf{A}}}^T + a\overline{\overline{\mathsf{I}}}_s^T\right). \tag{54}$$

Because of $\overline{\overline{\epsilon}}'|\boldsymbol{\alpha} = 0$ and $\mathbf{a}|\overline{\overline{\mu}}^{-1} = 0$, the dyadics $\overline{\overline{\epsilon}}'^{-1}$ and $\overline{\overline{\mu}}$ do not exist. Thus, even if we can invert the equations and express ($\mathbf{B}$, $\mathbf{E}$) in terms of ($\mathbf{D}$, $\mathbf{H}$), we cannot write the equations in the engineering form as ($\mathbf{D}$, $\mathbf{B}$) in terms of ($\mathbf{E}$, $\mathbf{H}$), at least not in a straightforward manner.

The number of free parameters in an IB-medium can be checked from $\overline{\overline{\mathsf{A}}} \rightarrow 3\times 3 = 9$, $\boldsymbol{\alpha} \rightarrow 3$, $\mathbf{a} \rightarrow 3$ and $a \rightarrow 1$, which makes 16 in total. This coincides with that of the four-dimensional dyadic $\overline{\overline{\mathsf{B}}} \rightarrow 4\times 4 = 16$.

6. GIBBSIAN REPRESENTATION

To see the connection to a classical representation in terms of three-dimensional (spatial) Gibbsian vectors and dyadics, let us define the vector counterparts of the field two-forms as

$$\mathbf{D}_g = \mathbf{e}_{123}\lfloor\mathbf{D}, \qquad \mathbf{B}_g = \mathbf{e}_{123}\lfloor\mathbf{B}, \tag{55}$$

and of the field one-forms as

$$\mathbf{E}_g = \overline{\overline{\mathsf{G}}}|\mathbf{E}, \qquad \mathbf{H}_g = \overline{\overline{\mathsf{G}}}|\mathbf{H}, \tag{56}$$

where $\overline{\overline{\mathsf{G}}} = \sum_1^3 \mathbf{e}_i\mathbf{e}_i$ is a three-dimensional metric dyadic. Moreover, we can define

$$\overline{\overline{\mathsf{A}}}_g = \overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{G}}} \in \mathbb{E}_1\mathbb{E}_1, \qquad \boldsymbol{\alpha}_g = \overline{\overline{\mathsf{G}}}|\boldsymbol{\alpha} \in \mathbb{E}_1 \tag{57}$$

corresponding to the dyadic $\overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{F}_1$ and dual vector $\boldsymbol{\alpha} \in \mathbb{F}_1$. Let us further define the dot product as

$$\boldsymbol{\beta}|\mathbf{b} = \boldsymbol{\beta}|\overline{\overline{\mathsf{G}}}\cdot\mathbf{b} = \boldsymbol{\beta}_g\cdot\mathbf{b}, \tag{58}$$

and the cross product as

$$\mathbf{e}_N\lfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) = \boldsymbol{\beta}_g\times\boldsymbol{\gamma}_g. \tag{59}$$

for any dual vecors $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and vector $\mathbf{b}$.

Expanding now

$$\begin{aligned}\mathbf{e}_{123}\lfloor\left(\overline{\mathsf{I}}_s{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}}\right)^T|B &= \left(\overline{\mathsf{I}}_s^{(3)}\lfloor\lfloor\left(\overline{\mathsf{I}}_s{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}}\right)^T\right)|\mathbf{B}_g = \left(\left(\overline{\mathsf{I}}_s^{(3)}\lfloor\lfloor\overline{\mathsf{I}}_s^T\right)\lfloor\lfloor\overline{\overline{\mathsf{A}}}^T\right)|\mathbf{B}_g \\ &= \left(\overline{\mathsf{I}}_s^{(2)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^T\right)|\mathbf{B}_g = \left(\mathrm{tr}\overline{\overline{\mathsf{A}}}\,\overline{\mathsf{I}} - \overline{\overline{\mathsf{A}}}\right)|\mathbf{B}_g = \left(\mathrm{tr}\overline{\overline{\mathsf{A}}}\right)\mathbf{B}_g - \overline{\overline{\mathsf{A}}}_g\cdot\mathbf{B}_g,\end{aligned} \tag{60}$$

(47) and (48) are transformed to the Gibbsian vector equations

$$\mathbf{D}_g = \left(\mathrm{tr}\overline{\overline{\mathsf{A}}}\right)\mathbf{B}_g - \overline{\overline{\mathsf{A}}}_g\cdot\mathbf{B}_g - \boldsymbol{\alpha}_g\times\mathbf{E}_g, \tag{61}$$

$$\mathbf{H}_g = -\mathbf{a}\times\mathbf{B}_g - \overline{\overline{\mathsf{A}}}_g^T\cdot\mathbf{E}_g - a\mathbf{E}_g. \tag{62}$$

Let us check the previously derived plane-wave condition from the Gibbsian form. Assuming a time-harmonic plane wave with $\exp(-j\mathbf{k}\cdot\mathbf{r})$ dependence and denoting $\mathbf{p} = \mathbf{k}/\omega$, the Maxwell equations become

$$\mathbf{p}\times\mathbf{E}_g = \mathbf{B}_g, \tag{63}$$

$$\mathbf{p}\times\mathbf{H}_g = -\mathbf{D}_g. \tag{64}$$

Substituting (63) in (61) and (62), (64) then takes the form

$$\left(\mathrm{tr}\overline{\overline{\mathsf{A}}}+\mathbf{a}\cdot\mathbf{p}-a\right)\mathbf{p}\times\mathbf{E}-\boldsymbol{\alpha}_g\times\mathbf{E}-\left(\mathbf{p}\times\overline{\overline{\mathsf{A}}}_g^T+\overline{\overline{\mathsf{A}}}_g\times\mathbf{p}\right)\cdot\mathbf{E}=0. \quad (65)$$

Now the dyadic in brackets is actually antisymmetric and can be expanded as

$$\mathbf{p}\times\overline{\overline{\mathsf{A}}}_g^T+\overline{\overline{\mathsf{A}}}_g\times\mathbf{p}=\left(\mathrm{tr}\overline{\overline{\mathsf{A}}}\mathbf{p}-\mathbf{p}\cdot\overline{\overline{\mathsf{A}}}_g\right)\times\overline{\overline{\mathsf{I}}}, \quad (66)$$

which is seen by writing $\overline{\overline{\mathsf{A}}}_g=\sum\mathbf{r}_i\mathbf{s}_i$, expanding

$$\mathbf{p}\times\mathbf{s}_i\mathbf{r}_i+\mathbf{r}_i\mathbf{s}_i\times\mathbf{p}=(\mathbf{r}_i\times(\mathbf{p}\times\mathbf{s}_i))\times\overline{\overline{\mathsf{I}}}=(\mathbf{p}(\mathbf{r}_i\cdot\mathbf{s}_i)-\mathbf{p}\cdot\mathbf{r}_i\mathbf{s}_i)\times\overline{\overline{\mathsf{I}}}, \quad (67)$$

and noting that $\mathrm{tr}\overline{\overline{\mathsf{A}}}=\sum\mathbf{r}_i\cdot\mathbf{s}_i$. Inserting (66) in (65) we obtain the final equation as

$$\overline{\overline{\mathsf{D}}}(\mathbf{p})\cdot\mathbf{E}_g=\left[\left(\mathbf{a}\cdot\mathbf{p}-a\right)\mathbf{p}-\boldsymbol{\alpha}_g+\mathbf{p}\cdot\overline{\overline{\mathsf{A}}}_g\right]\times\mathbf{E}_g=0. \quad (68)$$

In more conventional media we proceed by requiring $\det\overline{\overline{\mathsf{D}}}(\mathbf{p})=0$ which is called the dispersion equation, from which the possible values for the propagation vector $\mathbf{p}$ are obtained. In the present case $\overline{\overline{\mathsf{D}}}(\mathbf{p})=\mathbf{q}(\mathbf{p})\times\overline{\overline{\mathsf{I}}}$ is antisymmetric with

$$\mathbf{q}(\mathbf{p})=(\mathbf{a}\cdot\mathbf{p}-a)\mathbf{p}-\boldsymbol{\alpha}_g+\mathbf{p}\cdot\overline{\overline{\mathsf{A}}}_g, \quad (69)$$

and $\det\overline{\overline{\mathsf{D}}}(\mathbf{p})=0$ is satisfied for any $\mathbf{p}$. This means that the plane wave is possible for any chosen $\mathbf{p}$. From (68) we see that the corresponding electric field of a plane wave must be parallel to the vector $\mathbf{q}(\mathbf{p})$ and, the $\mathbf{B}$ field, parallel to $\mathbf{p}\times\mathbf{q}(\mathbf{p})=\boldsymbol{\alpha}_g\times\mathbf{p}-\mathbf{p}\cdot\overline{\overline{\mathsf{A}}}_g\times\mathbf{p}$. However, if $\mathbf{p}$ is chosen so that $\mathbf{q}(\mathbf{p})=0$, there is no restriction to the field vector $\mathbf{E}_g$ due to (68).

7. EXAMPLE

As a concrete example of an IB-medium let us consider the simplest generalization of the PEMC (axion) medium (13) defined by two scalar parameters A and B through the $\overline{\overline{\mathsf{B}}}$ dyadic

$$\overline{\overline{\mathsf{B}}}=A\overline{\overline{\mathsf{I}}}_s+B\mathbf{e}_4\boldsymbol{\varepsilon}_4. \quad (70)$$

Expressing it as

$$\overline{\overline{\mathsf{B}}}=\frac{M}{2}\overline{\overline{\mathsf{I}}}+\frac{N}{2}\left(\overline{\overline{\mathsf{I}}}_s-3\mathbf{e}_4\boldsymbol{\varepsilon}_4\right), \quad (71)$$

with

$$M = \frac{3A+B}{2}, \qquad N = \frac{A-B}{2}, \tag{72}$$

the medium dyadic becomes

$$\overline{\overline{\mathsf{M}}} = \left(\bar{\mathsf{I}} \,{}^{\wedge}_{\wedge}\, \overline{\overline{\mathsf{B}}}\right)^T = M\bar{\mathsf{I}}^{(2)T} + N\left(\bar{\mathsf{I}}_s^{(2)T} - \bar{\mathsf{I}}_s^T \,{}^{\wedge}_{\wedge}\, \boldsymbol{\varepsilon}_4\mathbf{e}_4\right). \tag{73}$$

Obviously, for $N = 0$ we have the axion medium while for $M = 0$ the dyadics $\overline{\overline{\mathsf{B}}}$ and $\overline{\overline{\mathsf{M}}}$ are trace-free and, thus, the medium falls to the class of skewons. A medium defined by (70) was called spatially isotropic in [5], pp.128–129, while the skewon medium with $M = 0$ was considered in [4], pp.261–262.

The modified medium dyadic corresponding to (73) can be expanded as

$$\begin{aligned} \overline{\overline{\mathsf{M}}}_g \;\; = \;\; & (M+N)(\mathbf{e}_{34}\mathbf{e}_{12} + \mathbf{e}_{14}\mathbf{e}_{23} + \mathbf{e}_{24}\mathbf{e}_{31}) \\ & + (M-N)(\mathbf{e}_{23}\mathbf{e}_{14} + \mathbf{e}_{31}\mathbf{e}_{24} + \mathbf{e}_{12}\mathbf{e}_{34}). \end{aligned} \tag{74}$$

Obviously, this dyadic is symmetric for $N = 0$ and antisymmetric for $M = 0$. The medium conditions can be represented by

$$\mathbf{D} = (M+N)\mathbf{B}, \qquad \mathbf{H} = (N-M)\mathbf{E}, \tag{75}$$

which appear generalizations of those of the PEMC medium [7]. In fact, the Poynting two-form $\mathbf{E} \wedge \mathbf{H}$ vanishes just like in the PEMC medium, which means that no power can propagate in such a medium. However, this does not mean that the fields must vanish within the medium. Also, it must be emphasized that the Poynting two-form does not vanish in general IB-media.

7.1. Time-harmonic Fields

Let us now consider time-harmonic fields in such a medium in terms of Gibbsian vectors in a region outside the sources. The Maxwell equations then become

$$\nabla \times \mathbf{E}_g = -j\omega\mathbf{B}_g, \qquad \nabla \times \mathbf{H}_g = j\omega\mathbf{D}_g. \tag{76}$$

Combining these as

$$\nabla \times (\mathbf{H}_g - (N-M)\mathbf{E}_g) = j\omega(\mathbf{D}_g + (N-M)\mathbf{B}_g), \tag{77}$$

we see that the left-hand side vanishes for the present medium. Thus, outside sources we must also have $\mathbf{D}_g + (N-M)\mathbf{B}_g = 0$. But taking into account the first condition (75), we have

$$\mathbf{D}_g = M\mathbf{B}_g, \qquad N\mathbf{B}_g = 0. \tag{78}$$

Assuming finite M and nonzero N we end up in the quite restricting conditions

$$\mathbf{D}_g = 0, \qquad \mathbf{B}_g = 0. \tag{79}$$

Equations (76) now require that both $\mathbf{H}_g$ and $\mathbf{E}_g$ are irrotational fields outside the sources and they are related by (75).

Considering a plane-wave solution in this medium, the vector $\mathbf{q}(\mathbf{p})$ of (69) can be expanded after defining the Gibbsian medium parameters as

$$\overline{\overline{\mathsf{A}}}_g = \frac{M+N}{2}\sum_1^3 \mathbf{e}_i\mathbf{e}_i, \quad \mathbf{a} = 0, \quad \boldsymbol{\alpha}_g = 0, \quad a = \frac{M-3N}{2}, \tag{80}$$

whence (69) gives us $\mathbf{q}(\mathbf{p}) = 2N\mathbf{p} = 2N\mathbf{k}/\omega$. Thus, in a medium with $N \neq 0$ the Gibbsian plane-wave fields are of the form

$$\mathbf{E}_g(\mathbf{r}) = E_o\mathbf{k}\exp(-j\mathbf{k}\cdot\mathbf{r}), \tag{81}$$

$$\mathbf{H}_g(\mathbf{r}) = (N-M)E_o\mathbf{k}\exp(-j\mathbf{k}\cdot\mathbf{r}), \tag{82}$$

$$\mathbf{D}_g(\mathbf{r}) = \mathbf{B}_g(\mathbf{r}) = 0. \tag{83}$$

It is easy to verify that the fields satisfy the Maxwell equations for any $\mathbf{k}$. This kind of pathological longitudinal electromagnetic waves are not encountered in 'ordinary' media. They do not carry any energy when $\mathbf{k}$ is real because then $\mathbf{E}_g \times \mathbf{H}_g^* = 0$.

7.2. Bi-isotropic Representation

Following the pattern of PEMC medium representation given in [7], we can show that the medium of our example can also be defined as a bi-isotropic medium in the Gibbsian vector representation [17]

$$\begin{pmatrix} \mathbf{D}_g \\ \mathbf{B}_g \end{pmatrix} = \begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix}\begin{pmatrix} \mathbf{E}_g \\ \mathbf{H}_g \end{pmatrix} \tag{84}$$

in terms of the relative Tellegen and chirality parameters χ_r, κ_r as [18]

$$\begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix} = \sqrt{\epsilon\mu}\begin{pmatrix} \sqrt{\epsilon/\mu} & \chi_r - j\kappa_r \\ \chi_r + j\kappa_r & \sqrt{\mu/\epsilon} \end{pmatrix}. \tag{85}$$

In fact, assuming

$$\chi_r^2 + \kappa_r^2 = 1, \tag{86}$$

from (84) and (85) we have

$$\mathbf{D}_g = \sqrt{\epsilon/\mu}(\chi_r - j\kappa_r)\mathbf{B}_g, \tag{87}$$

$$\mathbf{B}_g = \sqrt{\mu\epsilon}((\chi_r + j\kappa_r)\mathbf{E}_g + \sqrt{\mu/\epsilon}\mathbf{H}_g). \tag{88}$$

When we assume $\sqrt{\mu\epsilon} \rightarrow \infty$, and finite $\mathbf{B}_g$, the latter condition requires

$$\mathbf{H}_g = -\sqrt{\epsilon/\mu}(\chi_r + j\kappa_r)\mathbf{E}_g. \tag{89}$$

Comparing with the conditions (75) we can identify the relations between the two sets of parameters as

$$M = \sqrt{\epsilon/\mu}\,\chi_r, \qquad N = -j\sqrt{\epsilon/\mu}\,\kappa_r. \tag{90}$$

The conditions (86) corresponds to

$$M^2 - N^2 = \epsilon/\mu \tag{91}$$

between the parameters M and N. As a summary, the simple axion-skewon medium studied in this example can be interpreted as a bi-isotropic medium whose relative Tellegen and chiral parameters satisfy the condition (86). The parameters ϵ and μ have infinite magnitudes but finite ratio ϵ/μ.

8. CONCLUSION

A class of linear media whose medium dyadic $\overline{\overline{\mathsf{M}}} \in \mathbb{F}_2\mathbb{E}_2$ can be defined through a dyadic $\overline{\overline{\mathsf{B}}} \in \mathbb{E}_1\mathbb{F}_1$ was studied in four and three-dimensional representations. It turns out that the medium coincides with the one called axion-skewon medium in [4]. Plane-wave solution in such a medium was found to exist for given wave-vectors. As an example, a simple axion-skewon medium was shown to be equivalent to a bi-isotropic medium with infinite medium parameters.

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APPENDIX A.

Let us find the solution $\overline{\overline{\mathsf{X}}} \in \mathbb{E}_1\mathbb{F}_1$ of the dyadic equation

$$\overline{\mathsf{I}} {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{X}}} = \overline{\overline{\mathsf{Y}}} \tag{A1}$$

for a given dyadic $\overline{\overline{\mathsf{Y}}} \in \mathbb{E}_2\mathbb{F}_2$. Obviously, since $\overline{\overline{\mathsf{X}}}$ involves 16 scalars and the most general $\overline{\overline{\mathsf{Y}}}$ 36 scalars, such a solution does not exist unless $\overline{\overline{\mathsf{Y}}}$

satisfies some condition. Assuming that a solution exists, we can apply a rule from [5] with $\mathrm{tr}\overline{\overline{\mathsf{I}}} = 4$:

$$\left(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{X}}}\right)\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T = \overline{\overline{\mathsf{X}}}\left(\mathrm{tr}\overline{\overline{\mathsf{I}}}\right)+\overline{\overline{\mathsf{I}}}\left(\mathrm{tr}\overline{\overline{\mathsf{X}}}\right)-\overline{\overline{\mathsf{I}}}|\overline{\overline{\mathsf{I}}}|\overline{\overline{\mathsf{X}}}-\overline{\overline{\mathsf{X}}}|\overline{\overline{\mathsf{I}}}|\overline{\overline{\mathsf{I}}} = 2\overline{\overline{\mathsf{X}}}+\overline{\overline{\mathsf{I}}}\left(\mathrm{tr}\overline{\overline{\mathsf{X}}}\right), \quad \text{(A2)}$$

and another rule,

$$\mathrm{tr}\left(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{X}}}\right) = 3\mathrm{tr}\overline{\overline{\mathsf{X}}}, \quad \text{(A3)}$$

which is obtained by equating the trace of left and right sides of (A2):

$$\begin{aligned}\mathrm{tr}\left(\left(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{X}}}\right)\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right) &= \left(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{X}}}\right)||\left(\overline{\overline{\mathsf{I}}}^T{}^\wedge_\wedge\overline{\overline{\mathsf{I}}}^T\right) = 2\mathrm{tr}\left(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{X}}}\right), && \text{(A4)}\\ \mathrm{tr}\left(2\overline{\overline{\mathsf{X}}}+\overline{\overline{\mathsf{I}}}\left(\mathrm{tr}\overline{\overline{\mathsf{X}}}\right)\right) &= 2\mathrm{tr}\overline{\overline{\mathsf{X}}}+4\mathrm{tr}\overline{\overline{\mathsf{X}}} = 6\mathrm{tr}\overline{\overline{\mathsf{X}}}. && \text{(A5)}\end{aligned}$$

(A3) implies $\mathrm{tr}\overline{\overline{\mathsf{X}}} = \mathrm{tr}\overline{\overline{\mathsf{Y}}}/3$ which inserted in (A2) yields the solution to (A1):

$$\overline{\overline{\mathsf{X}}} = \frac{1}{2}\left(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{X}}}\right)\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T - \frac{1}{2}\overline{\overline{\mathsf{I}}}\left(\mathrm{tr}\overline{\overline{\mathsf{X}}}\right) = \frac{1}{2}\overline{\overline{\mathsf{Y}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T - \frac{1}{6}\overline{\overline{\mathsf{I}}}\left(\mathrm{tr}\overline{\overline{\mathsf{Y}}}\right). \quad \text{(A6)}$$

Inserting this in the original equation (A1), the dyadic condition

$$6\overline{\overline{\mathsf{Y}}}+2\overline{\overline{\mathsf{I}}}^{(2)}\mathrm{tr}\overline{\overline{\mathsf{Y}}}-3\left(\overline{\overline{\mathsf{Y}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right){}^\wedge_\wedge\overline{\overline{\mathsf{I}}} = 0 \quad \text{(A7)}$$

for $\overline{\overline{\mathsf{Y}}}$ is obtained. As a check of this condition we can expand the trace of the left-hand side and notice that it vanishes for any dyadic $\overline{\overline{\mathsf{Y}}} \in \mathbb{E}_2\mathbb{F}_2$:

$$6\mathrm{tr}\overline{\overline{\mathsf{Y}}}+2\mathrm{tr}\overline{\overline{\mathsf{I}}}^{(2)}\mathrm{tr}\overline{\overline{\mathsf{Y}}}-3\mathrm{tr}\left(\left(\overline{\overline{\mathsf{Y}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right){}^\wedge_\wedge\overline{\overline{\mathsf{I}}}\right) = 6\mathrm{tr}\overline{\overline{\mathsf{Y}}}+2\times 6\mathrm{tr}\overline{\overline{\mathsf{Y}}}-3\times 6\mathrm{tr}\overline{\overline{\mathsf{Y}}} = 0. \quad \text{(A8)}$$

For the last term we have applied the relation

$$\begin{aligned}\mathrm{tr}\left(\left(\overline{\overline{\mathsf{Y}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right){}^\wedge_\wedge\overline{\overline{\mathsf{I}}}\right) &= \left(\left(\overline{\overline{\mathsf{Y}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right){}^\wedge_\wedge\overline{\overline{\mathsf{I}}}\right)||\overline{\overline{\mathsf{I}}}^{(2)T}\\ &= \mathrm{tr}\left(\overline{\overline{\mathsf{Y}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right)\mathrm{tr}\overline{\overline{\mathsf{I}}}-\left(\overline{\overline{\mathsf{Y}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right)||\overline{\overline{\mathsf{I}}}^T\\ &= 6\mathrm{tr}\overline{\overline{\mathsf{Y}}} && \text{(A9)}\end{aligned}$$

based on

$$\mathrm{tr}\left(\overline{\overline{\mathsf{Y}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right) = \left(\overline{\overline{\mathsf{Y}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right)||\overline{\overline{\mathsf{I}}}^T = \overline{\overline{\mathsf{Y}}}||\left(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{I}}}\right)^T = 2\mathrm{tr}\overline{\overline{\mathsf{Y}}}. \quad \text{(A10)}$$

Now, if $\overline{\overline{\mathsf{Y}}}$ does not satisfy (A7), a contradiction arises and $\overline{\overline{\mathsf{Y}}}$ cannot be expressed in the form (A1). Thus, (A7) is a necessary condition for the solution $\overline{\overline{\mathsf{X}}}$ to exist. (A7) is also a sufficient condition because its form shows that $\overline{\overline{\mathsf{Y}}}$ can be expressed as $\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{Z}}}$ for some dyadic $\overline{\overline{\mathsf{Z}}} \in \mathbb{E}_1\mathbb{F}_1$.

APPENDIX B.

Let us derive a useful identity for a dyadic $\overline{\overline{\mathsf{C}}} \in \mathbb{E}_2\mathbb{F}_2$ by starting from the following contraction identities for two dyadics $\overline{\overline{\mathsf{A}}}$, $\overline{\overline{\mathsf{B}}} \in \mathbb{E}_1\mathbb{F}_1$ which can be derived through basis expansions [19]

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^T = \left(\mathrm{tr}\overline{\overline{\mathsf{A}}}\right)\overline{\overline{\mathsf{I}}}^{(3)} - \overline{\overline{\mathsf{I}}}^{(2)}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}, \quad \text{(B1)}$$

$$\overline{\overline{\mathsf{I}}}^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^T = \left(\mathrm{tr}\overline{\overline{\mathsf{A}}}\right)\overline{\overline{\mathsf{I}}}^{(2)} - \overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}, \quad \text{(B2)}$$

$$\left(\overline{\overline{\mathsf{I}}}^{(2)}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}\right)\lfloor\lfloor\overline{\overline{\mathsf{B}}}^T = \left(\mathrm{tr}\left(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}}\right)\right)\overline{\overline{\mathsf{I}}}^{(2)} + \overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}\mathrm{tr}\overline{\overline{\mathsf{B}}} - \overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\left(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}} + \overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{A}}}\right) - \overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}. \quad \text{(B3)}$$

From (B1) we have

$$\begin{aligned}\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\left(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}\right)^T &= \left(\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^T\right)\lfloor\lfloor\overline{\overline{\mathsf{B}}}^T \\ &= \left(\mathrm{tr}\overline{\overline{\mathsf{A}}}\right)\overline{\overline{\mathsf{I}}}^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{B}}}^T - \left(\overline{\overline{\mathsf{I}}}^{(2)}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}\right)\lfloor\lfloor\overline{\overline{\mathsf{B}}}^T. \end{aligned} \quad \text{(B4)}$$

Applying (B2) and (B3) we have

$$\begin{aligned}\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\left(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}\right)^T &= \left(\mathrm{tr}\overline{\overline{\mathsf{A}}}\mathrm{tr}\overline{\overline{\mathsf{B}}} - \mathrm{tr}\left(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}}\right)\right)\overline{\overline{\mathsf{I}}}^{(2)} \\ &\quad + \overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\left(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}} + \overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{A}}} - \left(\mathrm{tr}\overline{\overline{\mathsf{A}}}\right)\overline{\overline{\mathsf{B}}} - \left(\mathrm{tr}\overline{\overline{\mathsf{B}}}\right)\overline{\overline{\mathsf{A}}}\right) \\ &\quad + \overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}. \end{aligned} \quad \text{(B5)}$$

This identity can be made more compact through the rules [5]

$$\left(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}\right)\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T = \left(\mathrm{tr}\overline{\overline{\mathsf{A}}}\right)\overline{\overline{\mathsf{B}}} + \left(\mathrm{tr}\overline{\overline{\mathsf{B}}}\right)\overline{\overline{\mathsf{A}}} - \overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}} - \overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{A}}}, \quad \text{(B6)}$$

$$\mathrm{tr}\left(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}\right) = \frac{1}{2}\left(\left(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}\right)\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right)||\overline{\overline{\mathsf{I}}}^T = \mathrm{tr}\overline{\overline{\mathsf{A}}}\mathrm{tr}\overline{\overline{\mathsf{B}}} - \mathrm{tr}\left(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}}\right). \quad \text{(B7)}$$

Inserted in (B5) we have

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\left(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}\right)^T = \mathrm{tr}\left(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}\right)\overline{\overline{\mathsf{I}}}^{(2)} - \left(\left(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}\right)\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right){}^\wedge_\wedge\overline{\overline{\mathsf{I}}} + \overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}. \quad \text{(B8)}$$

Because (B8) is linear in $\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}$, this dyadic can be replaced by an arbitrary dyadic $\overline{\overline{\mathsf{C}}} \in \mathbb{E}_2\mathbb{F}_2$, whence (B8) finally takes the form of the identity

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{C}}}^T = \left(\mathrm{tr}\overline{\overline{\mathsf{C}}}\right)\overline{\overline{\mathsf{I}}}^{(2)} - \left(\overline{\overline{\mathsf{C}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right){}^\wedge_\wedge\overline{\overline{\mathsf{I}}} + \overline{\overline{\mathsf{C}}}. \quad \text{(B9)}$$

As a check, taking the trace of this identity gives $\mathrm{tr}\overline{\overline{\mathsf{C}}} = 6\mathrm{tr}\overline{\overline{\mathsf{C}}} - 6\mathrm{tr}\overline{\overline{\mathsf{C}}} + \mathrm{tr}\overline{\overline{\mathsf{C}}}$. Also, operating both sides of (B9) by $\overline{\mathsf{I}}^{(4)T}\lfloor\lfloor$ and applying (B1), (B2) can be shown to give the same identity (B9) transposed.

For the special case $\overline{\overline{\mathsf{C}}} = \overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}$ (B9) becomes

$$\begin{aligned} \overline{\mathsf{I}}^{(4)}\lfloor\lfloor\left(\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}\right)^T &= \mathrm{tr}\left(\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}\right)\overline{\mathsf{I}}^{(2)} - \left(\mathrm{tr}\overline{\overline{\mathsf{B}}}\,\overline{\mathsf{I}} + 2\overline{\overline{\mathsf{B}}}\right){}^\wedge_\wedge\overline{\mathsf{I}} + \overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}} \\ &= \mathrm{tr}\overline{\overline{\mathsf{B}}}\,\overline{\mathsf{I}}^{(2)} - \overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}, \end{aligned} \tag{B10}$$

when applying results from Appendix A. For the trace-free special case, $\overline{\overline{\mathsf{C}}} = \overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o$ with $\mathrm{tr}\overline{\overline{\mathsf{B}}}_o = 0$, (111) reduces to

$$\overline{\mathsf{I}}^{(4)}\lfloor\lfloor\left(\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o\right)^T = -\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o. \tag{B11}$$

APPENDIX C.

We can prove that a dyadic of the form $\left(\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o\right)\rfloor\mathbf{e}_N \in \mathbb{E}_2\mathbb{E}_2$ is antisymmetric if the dyadic $\overline{\overline{\mathsf{B}}}_o \in \mathbb{E}_1\mathbb{F}_1$ is trace free. For this purpose we apply (B11) by inserting $\overline{\mathsf{I}}^{(4)} = \mathbf{e}_N\boldsymbol{\varepsilon}_N$

$$\mathbf{e}_N\lfloor\left(\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o\right)^T\rfloor\boldsymbol{\varepsilon}_N = -\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o. \tag{C1}$$

Multiplying by $\rfloor\mathbf{e}_N$ from the right we have

$$\left(\left(\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o\right)\rfloor\mathbf{e}_N\right)^T = -\left(\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o\right)\rfloor\mathbf{e}_N, \tag{C2}$$

which shows us that $\left(\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o\right)\rfloor\mathbf{e}_N$ must be antisymmetric.

We can also prove the converse: any antisymmetric dyadic $\overline{\overline{\mathsf{D}}} \in \mathbb{E}_2\mathbb{E}_2$ can be represented in the form

$$\overline{\overline{\mathsf{D}}} = \left(\overline{\mathsf{I}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}_o\right)\rfloor\mathbf{e}_N, \tag{C3}$$

with trace-free $\overline{\overline{\mathsf{B}}}_o$. To show this, we start from (B9), which multiplied by $\rfloor\mathbf{e}_N$ gives

$$\mathbf{e}_N\lfloor\overline{\overline{\mathsf{C}}}^T = \left(\mathrm{tr}\overline{\overline{\mathsf{C}}}\right)\left(\overline{\mathsf{I}}^{(2)}\rfloor\mathbf{e}_N\right) - \left(\left(\overline{\overline{\mathsf{C}}}\lfloor\lfloor\overline{\mathsf{I}}^T\right){}^\wedge_\wedge\overline{\mathsf{I}}\right)\rfloor\mathbf{e}_N + \overline{\overline{\mathsf{C}}}\rfloor\mathbf{e}_N. \tag{C4}$$

Writing $\overline{\overline{\mathsf{D}}} = \overline{\overline{\mathsf{C}}}\rfloor\mathbf{e}_N$ or $\overline{\overline{\mathsf{C}}} = \overline{\overline{\mathsf{D}}}\rfloor\boldsymbol{\varepsilon}_N$, (C4) takes the form

$$\overline{\overline{\mathsf{D}}} - \overline{\overline{\mathsf{D}}}^T = -\left(\mathrm{tr}\overline{\overline{\mathsf{C}}}\right)\left(\overline{\mathsf{I}}^{(2)}\rfloor\mathbf{e}_N\right) + \left(\left(\overline{\overline{\mathsf{C}}}\lfloor\lfloor\overline{\mathsf{I}}^T\right){}^\wedge_\wedge\overline{\mathsf{I}}\right)\rfloor\mathbf{e}_N. \tag{C5}$$

Now if $\overline{\overline{\mathsf{D}}}$ is antisymmetric, one can show that $\mathrm{tr}\left(\overline{\overline{\mathsf{D}}}\rfloor\boldsymbol{\varepsilon}_N\right) = \mathrm{tr}\overline{\overline{\mathsf{C}}} = 0$. In fact,

$$\mathrm{tr}\overline{\overline{\mathsf{C}}} = \left(\overline{\overline{\mathsf{D}}}\rfloor\boldsymbol{\varepsilon}_N\right)||\bar{\bar{\mathsf{I}}}^{(2)T} = \overline{\overline{\mathsf{D}}}||\left(\bar{\bar{\mathsf{I}}}^{(2)T}\rfloor\boldsymbol{\varepsilon}_N\right), \tag{C6}$$

which vanishes because $\bar{\bar{\mathsf{I}}}^{(2)T}\rfloor\boldsymbol{\varepsilon}_N \in \mathbb{F}_2\mathbb{F}_2$ is symmetric, being the dual of (19).

$$\begin{aligned} \bar{\bar{\mathsf{I}}}^{(2)T}\rfloor\boldsymbol{\varepsilon}_N &= \sum_{i<j}\boldsymbol{\varepsilon}_{ij}\mathbf{e}_{ij}\rfloor\boldsymbol{\varepsilon}_{1234} \\ &= \boldsymbol{\varepsilon}_{12}\boldsymbol{\varepsilon}_{34} - \boldsymbol{\varepsilon}_{13}\boldsymbol{\varepsilon}_{24} + \boldsymbol{\varepsilon}_{14}\boldsymbol{\varepsilon}_{23} + \boldsymbol{\varepsilon}_{23}\boldsymbol{\varepsilon}_{14} - \boldsymbol{\varepsilon}_{24}\boldsymbol{\varepsilon}_{13} + \boldsymbol{\varepsilon}_{34}\boldsymbol{\varepsilon}_{12} \end{aligned} \tag{C7}$$

Thus, an antisymmetric $\overline{\overline{\mathsf{D}}}$ satisfies from (C5), (A10)

$$\overline{\overline{\mathsf{D}}} = \left(\left(\frac{1}{2}\overline{\overline{\mathsf{C}}}\lfloor\lfloor\bar{\bar{\mathsf{I}}}^T\right) {}^{\wedge}_{\wedge}\bar{\bar{\mathsf{I}}}\right)\rfloor\mathbf{e}_N, \tag{C8}$$

which is of the form $\left(\bar{\bar{\mathsf{I}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}_o\right)\rfloor\mathbf{e}_N$ with

$$\overline{\overline{\mathsf{B}}}_o = \frac{1}{2}\left(\overline{\overline{\mathsf{C}}}\lfloor\lfloor\bar{\bar{\mathsf{I}}}^T\right), \qquad \mathrm{tr}\overline{\overline{\mathsf{B}}}_o = \mathrm{tr}\overline{\overline{\mathsf{C}}} = 0. \tag{C9}$$

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