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BROADBAND SPATIOTEMPORAL DIFFERENTIAL-OPERATOR REPRESENTATIONS FOR VELOCITY-DEPENDENT SCATTERING

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Abstract—A novel approach based on spatiotemporal differential-operators is developed here for broadband, velocity-dependent scattering. Unlike the spectral-domain representations, the new method facilitates a compact formulation for scattering by arbitrary excitation signals, in the presence of moving objects. In free space (vacuum), relativistically exact formulas are developed.

After developing the general theory, analysis of relativistically exact free-space scattering by cylinders, and a half-plane, are examined. For cylinders the analysis shows that in the far field pulses are located on circles in the co-moving reference-frame where the object is at-rest. In other reference frames this feature is valid only as an approximation. These results apply also to the diffractive part of the half-plane scattered field. The geometrical-optics contribution is associated with plane-waves and obeys the appropriate transformations. The various zones for these fields in an arbitrary reference-frame are analyzed.

1. INTRODUCTION

An arbitrary space-time dependent function $f(\boldsymbol{R})$ can be represented in terms of a four-fold Fourier integral

$$f(\boldsymbol{R}) = q\int(d^4\boldsymbol{K})f(\boldsymbol{K})e^{i\boldsymbol{K}\cdot\boldsymbol{R}},\ q=(2\pi)^{-4},\ d^4\boldsymbol{K}=dk_xdk_ydk_zdi\omega/c$$
$$\boldsymbol{R}=(\boldsymbol{r},ict)=(x,y,z,ict),\ \boldsymbol{K}=(\boldsymbol{k},i\omega/c)=(k_x,k_y,k_z,i\omega/c) \quad (1)$$

compactly symbolized by using in (1) Minkowski four-vectors $\boldsymbol{K}$, $\boldsymbol{R}$, and usually the four integrations extend over the range $-\infty$ to ∞, and all the provisos for the existence of the integrals are met.

Functions as in (1) are involved in Maxwell's equations in sourceless domains

$$\begin{aligned} \partial_{\boldsymbol{r}} \times \boldsymbol{E} &= -\partial_t \boldsymbol{B},\ \partial_{\boldsymbol{r}} \times \boldsymbol{H} = \partial_t \boldsymbol{D} \\ \partial_{\boldsymbol{r}} \cdot \boldsymbol{D} &= 0, \quad \partial_{\boldsymbol{r}} \cdot \boldsymbol{B} = 0 \end{aligned} \tag{2}$$

where in the spatiotemporal domain $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{R})$, etc. See for example [1] for notation. In order to solve (2) one needs constitutive relations, e.g.,

$$\boldsymbol{D} = \tilde{\boldsymbol{\varepsilon}} \cdot \boldsymbol{E}, \quad \boldsymbol{B} = \tilde{\boldsymbol{\mu}} \cdot \boldsymbol{H} \tag{3}$$

where in simple nondispersive media the dyadics (matrices) $\tilde{\boldsymbol{\varepsilon}}$, $\tilde{\boldsymbol{\mu}}$, reduce to constant scalars ε, μ, respectively.

Substituting the constitutive relations, e.g., (3), in (2) yields the associated wave-equations. In sourceless domains we have homogeneous wave-equations, which can be symbolized by an operator $F(\partial_{\boldsymbol{R}})$

$$F(\partial_{\boldsymbol{R}}) f(\boldsymbol{R}) = 0, \quad \partial_{\boldsymbol{R}} = (\partial_{\boldsymbol{r}}, -i\partial_t/c) \tag{4}$$

where in (4) $f(\boldsymbol{R})$ can stand for any Cartesian component of the fields $\boldsymbol{E}$, $\boldsymbol{D}, \boldsymbol{B}$, $\boldsymbol{H}$, in (2), and $\partial_{\boldsymbol{R}}$ is the Minkowski four-gradient vector. The wave-equation (4) is obtained from Maxwell's equations in sourceless domains (2), including the constitutive equations, e.g., (3), by repeated substitution reduction, or equivalently, by equating to zero the symbolic determinant of the system.

Applying the wave operator $F(\partial_{\boldsymbol{R}})$ to the integral (1) and equating to zero defines $f(\boldsymbol{R})$ as a solution of (4). Now, interchanging order of integration and differentiation, and applying the derivatives to the exponential, yields

$$F(\partial_{\boldsymbol{R}}) f(\boldsymbol{R}) = q \int (d^4\boldsymbol{K}) f(\boldsymbol{K}) F(i\boldsymbol{K}) e^{i\boldsymbol{K}\cdot\boldsymbol{R}} = 0 \tag{5}$$

where in (5) the component-derivatives of $\partial_{\boldsymbol{R}}$ are simply replaces by the corresponding components of the Minkowski vector $i\boldsymbol{K}$.

It follows that the algebraic expression, the so-called dispersion-relation

$$F(i\boldsymbol{K}) = 0 \tag{6}$$

must be satisfied. The constraint (6) can be incorporated into the integral (1) in the form

$$f(\boldsymbol{R}) = q \int (d^4\boldsymbol{K}) g(\boldsymbol{K}) e^{i\boldsymbol{K}\cdot\boldsymbol{R}} = 0,\ g(\boldsymbol{K}) = f(\boldsymbol{K}) \delta(F(i\boldsymbol{K})) \tag{7}$$

where the Dirac delta function δ indicates that the integrals vanish unless (6) is satisfied. Alternatively, the four-fold integral can be

collapsed to a three-fold integral, e.g.,

$$\begin{aligned} f(\boldsymbol{R}) &= q\int (d^3\boldsymbol{k})\overline{f}(\boldsymbol{k})e^{i\boldsymbol{K}\cdot\boldsymbol{R}} = 0 \\ \overline{f}(\boldsymbol{k}) &= f(\boldsymbol{k}, i\omega(\boldsymbol{k})/c),\ \boldsymbol{K}\cdot\boldsymbol{R} = \boldsymbol{k}\cdot\boldsymbol{r} - \omega(\boldsymbol{k})t \end{aligned} \tag{8}$$

with $\boldsymbol{K}$ expressed in terms of $\boldsymbol{k}$, ω and ω eliminated by substituting from (6) in the form

$$F(i\boldsymbol{K}) = F(i\boldsymbol{k}, -\omega(\boldsymbol{k})/c) = 0 \tag{9}$$

Note that (8) is no more a Fourier-integral proper, and therefore no inverse can be given.

We need a few elements from Einstein's Special-Relativity theory [2]. For more detail see also [3, 4]. Accordingly if (2) is valid in an inertial reference-frame Γ, it is also valid in another arbitrary inertial reference-frame Γ' in the form

$$\begin{aligned} \partial_{\boldsymbol{r}'}\times\boldsymbol{E}' &= -\partial_{t'}\boldsymbol{B}',\ \partial_{\boldsymbol{r}'}\times\boldsymbol{H}' = \partial_{t'}\boldsymbol{D}' \\ \partial_{\boldsymbol{r}'}\cdot\boldsymbol{D}' &= 0, \quad \partial_{\boldsymbol{r}'}\cdot\boldsymbol{B}' = 0 \end{aligned} \tag{10}$$

where in the present spatiotemporal domain $\boldsymbol{E}' = \boldsymbol{E}'(\boldsymbol{R}')$, etc. The coordinates are related by the Lorentz transformation $\boldsymbol{R}' = \boldsymbol{R}'[\boldsymbol{R}]$ in the form

$$\begin{aligned} \boldsymbol{r}' &= \tilde{\boldsymbol{U}}\cdot(\boldsymbol{r} - \boldsymbol{v}t),\ t' = \gamma(t - \boldsymbol{v}\cdot\boldsymbol{r}/c^2) \\ \gamma &= (1-\beta^2)^{-1/2},\ \beta = v/c,\ v = |\boldsymbol{v}|, \\ \tilde{\boldsymbol{U}} &= \tilde{\boldsymbol{I}} + (\gamma - 1)\hat{\boldsymbol{v}}\hat{\boldsymbol{v}}, \quad \hat{\boldsymbol{v}} = \boldsymbol{v}/v \end{aligned} \tag{11}$$

and the associated transformation for the derivatives $\partial_{\boldsymbol{R}'} = \partial_{\boldsymbol{R}'}[\partial_{\boldsymbol{R}}]$ is given by

$$\partial_{\boldsymbol{r}'} = \tilde{\boldsymbol{U}}\cdot(\partial_{\boldsymbol{r}} + \boldsymbol{v}\partial_t/c^2), \quad \partial_{t'} = \gamma(\partial_t + \boldsymbol{v}\cdot\partial_{\boldsymbol{r}}) \tag{12}$$

Consequently the fields transformations are prescribed by $\boldsymbol{F}' = \boldsymbol{F}'[\boldsymbol{F}]$ in the form

$$\begin{aligned} \boldsymbol{E}' &= \tilde{\boldsymbol{V}}\cdot(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}),\ \boldsymbol{B}' = \tilde{\boldsymbol{V}}\cdot(\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{E}/c^2) \\ \boldsymbol{D}' &= \tilde{\boldsymbol{V}}\cdot(\boldsymbol{D} + \boldsymbol{v}\times\boldsymbol{H}/c^2),\ \boldsymbol{H}' = \tilde{\boldsymbol{V}}\cdot(\boldsymbol{H} - \boldsymbol{v}\times\boldsymbol{D}) \\ \tilde{\boldsymbol{V}} &= \gamma\tilde{\boldsymbol{I}} + (1-\gamma)\hat{\boldsymbol{v}}\hat{\boldsymbol{v}} \end{aligned} \tag{13}$$

Postulating the phase-invariance "principle" (which is actually superfluous if we already have declared $\boldsymbol{K}$, $\boldsymbol{R}$, to be Minkowski four-vectors [3, 4]), we have

$$\boldsymbol{K}\cdot\boldsymbol{R} = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t = \boldsymbol{K}'\cdot\boldsymbol{R}' = \boldsymbol{k}'\cdot\boldsymbol{r}' - \omega' t' \tag{14}$$

Consequently (14) yields the relativistic Doppler-effect $\boldsymbol{K}' = \boldsymbol{K}'[\boldsymbol{K}]$ in the form

$$\boldsymbol{k}' = \tilde{\boldsymbol{U}} \cdot (\boldsymbol{k} - \boldsymbol{v}\omega/c^2), \quad \omega' = \gamma(\omega - \boldsymbol{v} \cdot \boldsymbol{k}) \tag{15}$$

The formulas $\boldsymbol{R}' = \boldsymbol{R}'[\boldsymbol{R}]$, $\partial_{\boldsymbol{R}'} = \partial_{\boldsymbol{R}'}[\partial_{\boldsymbol{R}}]$, $\boldsymbol{F}' = \boldsymbol{F}'[\boldsymbol{F}]$, $\boldsymbol{K}' = \boldsymbol{K}'[\boldsymbol{K}]$ (11), (12), (13), (15), have inverses $\boldsymbol{R} = \boldsymbol{R}[\boldsymbol{R}']$, $\partial_{\boldsymbol{R}} = \partial_{\boldsymbol{R}}[\partial_{\boldsymbol{R}'}]$, $\boldsymbol{F} = \boldsymbol{F}[\boldsymbol{F}']$, $\boldsymbol{K} = \boldsymbol{K}[\boldsymbol{K}']$, respectively, obtained by exchanging primed and unprimed symbols and noting that $\boldsymbol{v}' = -\boldsymbol{v}$.

2. SPATIOTEMPORAL DIFFERENTIAL-OPERATORS

Recently differential-operators representations for velocity-dependent scattering have been developed [3, 5, 6], with emphasis on harmonic excitation. Presently we aim at developing corresponding spatiotemporal domain operators. To that end we substitute the Maxwell equations (2) in (13). For example, the first formula (2) is recast in the form $\boldsymbol{B} = -\partial_t^{-1}\partial_{\boldsymbol{r}} \times \boldsymbol{E}$, consequently the first formula (13) becomes

$$\begin{aligned} \boldsymbol{E}' &= \tilde{\boldsymbol{V}} \cdot (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = \tilde{\boldsymbol{V}} \cdot (\boldsymbol{E} - \boldsymbol{v} \times \partial_{\boldsymbol{r}} \times \partial_t^{-1}\boldsymbol{E}) = \tilde{\boldsymbol{W}} \cdot \boldsymbol{E} \\ \tilde{\boldsymbol{W}} &= \tilde{\boldsymbol{V}} \cdot (\tilde{\boldsymbol{I}} - \boldsymbol{v} \times \partial_t^{-1}\partial_{\boldsymbol{r}} \times \tilde{\boldsymbol{I}}) \end{aligned} \tag{16}$$

defining the new dyadic differential-operator $\tilde{\boldsymbol{W}}$. In (16) ∂_t^{-1} denotes the primitive time-integration (indefinite integral). Similarly we find

$$\boldsymbol{H}' = \tilde{\boldsymbol{V}} \cdot (\boldsymbol{H} - \boldsymbol{v} \times \boldsymbol{D}) = \tilde{\boldsymbol{W}} \cdot \boldsymbol{H} \tag{17}$$

with the same dyadic differential-operator as given in (16). We can also introduce

$$\begin{aligned} \boldsymbol{B}' &= \tilde{\boldsymbol{V}} \cdot (\boldsymbol{B} - \boldsymbol{v} \times \boldsymbol{E}/c^2) = -\tilde{\boldsymbol{Q}} \cdot \boldsymbol{E} \\ \boldsymbol{D}' &= \tilde{\boldsymbol{V}} \cdot (\boldsymbol{D} + \boldsymbol{v} \times \boldsymbol{H}/c^2) = \tilde{\boldsymbol{Q}} \cdot \boldsymbol{H} \\ \tilde{\boldsymbol{Q}} &= \tilde{\boldsymbol{V}} \cdot \left(\partial_t^{-1}\partial_{\boldsymbol{r}} \times \tilde{\boldsymbol{I}} + \boldsymbol{v} \times \tilde{\boldsymbol{I}}/c^2\right) \end{aligned} \tag{18}$$

completing the general formulas.

Inasmuch as in (16)–(18) no constitutive relations are incorporated, inverse formulas are obtained by interchanging primed and unprimed fields and coordinates, and replacing $\boldsymbol{v}$ by $-\boldsymbol{v}$, yielding

$$\begin{aligned} &\boldsymbol{E} = \tilde{\boldsymbol{W}}' \cdot \boldsymbol{E},\ \boldsymbol{H} = \tilde{\boldsymbol{W}}' \cdot \boldsymbol{H}',\ \tilde{\boldsymbol{W}}' = \tilde{\boldsymbol{V}} \cdot \left(\tilde{\boldsymbol{I}} + \boldsymbol{v} \times \partial_{t'}^{-1}\partial_{\boldsymbol{r}'} \times \tilde{\boldsymbol{I}}\right) \\ &\boldsymbol{B} = -\tilde{\boldsymbol{Q}}' \cdot \boldsymbol{E}',\ \boldsymbol{D} = \tilde{\boldsymbol{Q}}' \cdot \boldsymbol{H}',\ \tilde{\boldsymbol{Q}}' = \tilde{\boldsymbol{V}} \cdot \left(\partial_{t'}^{-1}\partial_{\boldsymbol{r}'} \times \tilde{\boldsymbol{I}} - \boldsymbol{v} \times \tilde{\boldsymbol{I}}/c^2\right) \end{aligned} \tag{19}$$

It seems interesting that in (18) the electric field is derived from the magnetic one, and *vice-versa*. Of course one cannot expect that in (18), say in the first formula, $\boldsymbol{E}$ be eliminated by substituting $\boldsymbol{B} = -\partial_{t'}^{-1}\partial_{\boldsymbol{r}} \times \boldsymbol{E}$, because an inverse operation to $\partial_{\boldsymbol{r}}\times$ does not exist. By manipulation of (16)–(19) we obtain additional relations

$$\begin{aligned} &\boldsymbol{D} = \tilde{\boldsymbol{S}} \cdot \boldsymbol{H},\ \boldsymbol{D}' = \tilde{\boldsymbol{S}}' \cdot \boldsymbol{H}',\ \boldsymbol{B} = -\tilde{\boldsymbol{S}} \cdot \boldsymbol{E},\ \boldsymbol{B}' = -\tilde{\boldsymbol{S}}' \cdot \boldsymbol{E}' \\ &\tilde{\boldsymbol{S}} = \tilde{\boldsymbol{Q}}' \cdot \tilde{\boldsymbol{W}},\ \tilde{\boldsymbol{S}}' = \tilde{\boldsymbol{Q}} \cdot \tilde{\boldsymbol{W}}',\ \tilde{\boldsymbol{W}} \cdot \tilde{\boldsymbol{W}}' = \tilde{\boldsymbol{W}}' \cdot \tilde{\boldsymbol{W}} = \tilde{\boldsymbol{I}} \end{aligned} \tag{20}$$

In (20) the first four relations look very much like some new kind of constitutive relations, but of course they are not, because no material considerations have been included so far. If we do admit constitutive relations, e.g., (3), then by substitution we find

$$\tilde{\boldsymbol{Q}}' \cdot \boldsymbol{H}' = \tilde{\varepsilon} \cdot \tilde{\boldsymbol{W}}' \cdot \boldsymbol{E}', \quad \tilde{\boldsymbol{Q}}' \cdot \boldsymbol{E}' = -\tilde{\boldsymbol{\mu}} \cdot \tilde{\boldsymbol{W}}' \cdot \boldsymbol{H}' \tag{21}$$

The equations (21) are indeed constitutive-relations which should reduce to the well-known Minkowski constitutive-relations. For more detail see e.g., [1, 7].

Below, the new tools (16), (17) and their inverses will serve us in discussing spatiotemporal velocity-dependent scattering.

3. THE SCATTERING ALGORITHM

Corresponding to (7), consider a solution of the Maxwell equations (2), in the form

$$\left\{ \begin{array}{c} \boldsymbol{E}(\boldsymbol{R}) \\ \boldsymbol{H}(\boldsymbol{R}) \end{array} \right\} = q \int (d^4\boldsymbol{K}) \left\{ \begin{array}{c} \boldsymbol{g}_{\boldsymbol{E}}(\boldsymbol{K}) \\ \boldsymbol{g}_{\boldsymbol{H}}(\boldsymbol{K}) \end{array} \right\} e^{i\boldsymbol{K}\cdot\boldsymbol{R}} \tag{22}$$

where in (22) $\boldsymbol{E}, \boldsymbol{H}$, hence respectively $\boldsymbol{g}_{\boldsymbol{E}}, \boldsymbol{g}_{\boldsymbol{H}}$, are related by constitutive equations, e.g., (3).

Specific waves of the form (22) will serve us as the initial excitation-wave specified in the "laboratory" reference-frame Γ. In order to be able to discuss scattering problems in the reference-frame Γ' in which the scatterer is at rest, we need to apply the operators (16), (17) to (22), yielding

$$\begin{aligned} \left\{ \begin{array}{c} \boldsymbol{E}'(\boldsymbol{R}) \\ \boldsymbol{H}'(\boldsymbol{R}) \end{array} \right\} &= \tilde{\boldsymbol{W}}_{\boldsymbol{R}} \cdot \left\{ \begin{array}{c} \boldsymbol{E}(\boldsymbol{R}) \\ \boldsymbol{H}(\boldsymbol{R}) \end{array} \right\} = q \int (d^4\boldsymbol{K}) e^{i\boldsymbol{K}\cdot\boldsymbol{R}} \tilde{\boldsymbol{W}}_{\boldsymbol{K}} \cdot \left\{ \begin{array}{c} \boldsymbol{g}_{\boldsymbol{E}}(\boldsymbol{K}) \\ \boldsymbol{g}_{\boldsymbol{H}}(\boldsymbol{K}) \end{array} \right\} \\ \beta &= v/v_{ph}, \quad v_{ph} = \omega/k \\ \tilde{\boldsymbol{W}}_{\boldsymbol{R}} &= \tilde{\boldsymbol{V}} \cdot (\tilde{\boldsymbol{I}} - \boldsymbol{v} \times \partial_t^{-1} \partial_{\boldsymbol{r}} \times \tilde{\boldsymbol{I}}),\ \tilde{\boldsymbol{W}}_{\boldsymbol{K}} = \tilde{\boldsymbol{V}} \cdot (\tilde{\boldsymbol{I}} + \beta \hat{\boldsymbol{v}} \times \hat{\boldsymbol{k}} \times \tilde{\boldsymbol{I}}) \end{aligned} \tag{23}$$

Incorporating (14) in the exponent in (23), it is possible to express the fields in terms of the native coordinates of Γ', denoted for brevity by $\boldsymbol{E}'(\boldsymbol{R}')$, $\boldsymbol{H}'(\boldsymbol{R}')$

$$\left\{ \begin{array}{c} \boldsymbol{E}'(\boldsymbol{R}') \\ \boldsymbol{H}'(\boldsymbol{R}') \end{array} \right\} = q \int (d^4\boldsymbol{K}) e^{i\boldsymbol{K}'\cdot\boldsymbol{R}'} \tilde{\boldsymbol{W}}_{\boldsymbol{K}} \cdot \left\{ \begin{array}{c} \boldsymbol{g}_{\boldsymbol{E}}(\boldsymbol{K}) \\ \boldsymbol{g}_{\boldsymbol{H}}(\boldsymbol{K}) \end{array} \right\} \tag{24}$$

where in (24) $\boldsymbol{K}'$ is related to $\boldsymbol{K}$ through the relativistic Doppler-effect $\boldsymbol{K}' = \boldsymbol{K}'[\boldsymbol{K}]$, (15).

We can do even better than that: the integral (24) can be transformed into $\boldsymbol{K}'$ coordinates by using the Jacobian determinant $\det[\partial_{\boldsymbol{K}}\boldsymbol{K}']$, which for the present case equals unity, e.g., see [1, 8]

$$\begin{aligned} \left\{ \begin{array}{c} \boldsymbol{E}'(\boldsymbol{R}') \\ \boldsymbol{H}'(\boldsymbol{R}') \end{array} \right\} &= q \int (d^4\boldsymbol{K}) e^{i\boldsymbol{K}'\cdot\boldsymbol{R}'} \left\{ \begin{array}{c} \boldsymbol{g}'_{\boldsymbol{E}} \\ \boldsymbol{g}'_{\boldsymbol{H}} \end{array} \right\} \\ \left\{ \begin{array}{c} \boldsymbol{g}'_{\boldsymbol{E}} \\ \boldsymbol{g}'_{\boldsymbol{H}} \end{array} \right\} &= \left\{ \begin{array}{c} \boldsymbol{g}'_{\boldsymbol{E}}(\boldsymbol{K}') \\ \boldsymbol{g}'_{\boldsymbol{H}}(\boldsymbol{K}') \end{array} \right\} = \left[\tilde{\boldsymbol{W}}_{\boldsymbol{K}} \cdot \left\{ \begin{array}{c} \boldsymbol{g}_{\boldsymbol{E}}(\boldsymbol{K}) \\ \boldsymbol{g}_{\boldsymbol{H}}(\boldsymbol{K}) \end{array} \right\} \right]_{\boldsymbol{K}=\boldsymbol{K}[\boldsymbol{K}']} \\ d^4\boldsymbol{K}' &= \det[\partial_{\boldsymbol{K}}\boldsymbol{K}'] d^4\boldsymbol{K} = d^4\boldsymbol{K} \end{aligned} \tag{25}$$

where in (25) we used the fact that both $\boldsymbol{K}$ and $\boldsymbol{K}'$ are integrated over the entire four-dimensional spatiotemporal spaces, and $\boldsymbol{g}'_{\boldsymbol{E}}(\boldsymbol{K}')$, $\boldsymbol{g}'_{\boldsymbol{H}}(\boldsymbol{K}')$ are obtained as a function of $\boldsymbol{K}'$ by substituting the Doppler-effect formulas $\boldsymbol{K} = \boldsymbol{K}[\boldsymbol{K}']$, i.e., the inverse of (15).

Consider (22) to be the excitation wave specified in reference-frame Γ. Then (25) provides the fields measured in the reference-frame Γ' of the object at-rest. In many cases scattering by harmonic plane-wave excitation is analyzed. This makes (25) a convenient starting point. In response to the plane-wave excitation under the integral sign, the scattering wave is created, chosen in a manner prescribed by the boundary-conditions and behaving as an outgoing wave at large distances.

Without going into the details of the solution of the boundary-value problem, the scattered wave created by a spectrum of plane waves (25) can be presented in the form

$$\left\{ \begin{array}{c} \boldsymbol{E}'_{sc}(\boldsymbol{R}') \\ \boldsymbol{H}'_{sc}(\boldsymbol{R}') \end{array} \right\} = q \int (d^4\boldsymbol{K}') \left\{ \begin{array}{c} \boldsymbol{E}'_{sc}(\boldsymbol{R}', \boldsymbol{g}'_{\boldsymbol{E}}(\boldsymbol{K}'), \boldsymbol{K}') \\ \boldsymbol{H}'_{sc}(\boldsymbol{R}', \boldsymbol{g}'_{\boldsymbol{H}}(\boldsymbol{K}'), \boldsymbol{K}') \end{array} \right\} \tag{26}$$

where in the integral (26) in $\boldsymbol{E}'_{sc}(\boldsymbol{R}', \boldsymbol{g}'_{\boldsymbol{E}}(\boldsymbol{K}'), \boldsymbol{K}')$, $\boldsymbol{H}'_{sc}(\boldsymbol{R}', \boldsymbol{g}'_{\boldsymbol{H}}(\boldsymbol{K}'), \boldsymbol{K}')$ the plane-wave exponential $e^{i\boldsymbol{K}'\cdot\boldsymbol{R}'}$ is already included. The dependence on $\boldsymbol{K}'$ is due both to this exponential, and via the complex

vector weight-functions $\boldsymbol{g}'_{\boldsymbol{E}}(\boldsymbol{K}')$, $\boldsymbol{g}'_{\boldsymbol{H}}(\boldsymbol{K}')$, with their absolute value and direction of polarization included.

In order to derive the scattered wave in the initial reference-frame Γ, the inverse spatiotemporal operator (cf. (23)) is applied to (26)

$$\begin{aligned}
&\left\{ \begin{array}{c} \boldsymbol{E}_{sc}(\boldsymbol{R}') \\ \boldsymbol{H}_{sc}(\boldsymbol{R}') \end{array} \right\} = \tilde{\boldsymbol{W}}'_{\boldsymbol{R}'} \cdot \left\{ \begin{array}{c} \boldsymbol{E}'_{sc}(\boldsymbol{R}') \\ \boldsymbol{H}'_{sc}(\boldsymbol{R}') \end{array} \right\} \\
&= q \int (d^4\boldsymbol{K}') \tilde{\boldsymbol{W}}'_{\boldsymbol{R}'} \cdot \left\{ \begin{array}{c} \boldsymbol{E}_{sc}(\boldsymbol{R}', \boldsymbol{g}'_{\boldsymbol{E}}(\boldsymbol{K}'), \boldsymbol{K}') \\ \boldsymbol{H}_{sc}(\boldsymbol{R}', \boldsymbol{g}'_{\boldsymbol{H}}(\boldsymbol{K}'), \boldsymbol{K}') \end{array} \right\} \\
&\tilde{\boldsymbol{W}}'_{\boldsymbol{R}'} = \tilde{\boldsymbol{V}} \cdot (\tilde{\boldsymbol{I}} + \boldsymbol{v} \times \partial_{t'}^{-1} \partial_{\boldsymbol{r}'} \times \tilde{\boldsymbol{I}})
\end{aligned} \tag{27}$$

Obviously in (27) the Γ fields are still expressed in terms of reference-frame Γ' native coordinates. If so wished, the Lorentz transformation (11) can be exploited to derive the fields in terms of Γ native coordinates.

4. FREE-SPACE IMPULSIVE PLANE-WAVE EXCITATION

As a concrete example for broadband, velocity-dependent scattering, consider propagation in free-space, i.e., in (3)

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \ \boldsymbol{B} = \mu \boldsymbol{H}, \ \varepsilon = \varepsilon_0, \ \mu = \mu_0, \ \varepsilon_0 \mu_0 = 1/c \tag{28}$$

where in (28) ε_0, μ_0 are constants, valid for all inertial systems, and c is the speed of light in free-space.

For broadband excitation-waves we consider the extreme case of an impulsive, transversal plane-wave, whose time-dependence is stated by a Dirac delta-function

$$\begin{aligned}
&\left\{ \begin{array}{c} \boldsymbol{E} \\ \boldsymbol{H} \end{array} \right\} = \left\{ \begin{array}{c} \hat{\boldsymbol{E}} e \\ \hat{\boldsymbol{H}} h \end{array} \right\} \delta(\tau), \ \delta(\tau) = \int_{-\infty}^{\infty} e^{-i\omega\tau} d\omega / 2\pi \\
&\tau = t - \hat{\boldsymbol{k}} \cdot \boldsymbol{r}/c, \quad \omega\tau = \omega t - \boldsymbol{k} \cdot \boldsymbol{r} \\
&\omega/k = v_{ph} = (\mu\varepsilon)^{-1/2}, \quad e/h = Z = (\mu/\varepsilon)^{1/2} \\
&\hat{\boldsymbol{E}} \cdot \hat{\boldsymbol{H}} = \hat{\boldsymbol{E}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{H}} \cdot \hat{\boldsymbol{k}} = 0, \quad \hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}} = \hat{\boldsymbol{k}}
\end{aligned} \tag{29}$$

We could also derive (29) from the general form (22). Assume a Cartesian coordinate system with the wave vector $\boldsymbol{k} = \hat{\xi} k_\xi$ pointing in the ξ-direction, perpendicular to coordinates η, ζ. We thus have to include in the integral constraints $\delta(k_\eta)$, $\delta(k_\zeta)$. Furthermore, we

include a constraint $\delta(k_\xi - \omega/c)$ which is the dispersion-equation (6) appearing in the integral (5). We now have

$$\left\{ \begin{array}{c} \boldsymbol{E} \\ \boldsymbol{H} \end{array} \right\} = \left\{ \begin{array}{c} \hat{\boldsymbol{E}}e \\ \hat{\boldsymbol{H}}h \end{array} \right\} q \int (d^4\boldsymbol{K}) g e^{i\boldsymbol{K}\cdot\boldsymbol{R}} = \left\{ \begin{array}{c} \hat{\boldsymbol{E}}e \\ \hat{\boldsymbol{H}}h \end{array} \right\} \delta(\tau) \tag{30}$$
$$g = -ic\delta(k_\xi - \omega/c)\delta(k_\eta)\delta(k_\zeta)(2\pi)^3$$

where in (30) the arbitrary weight function contains the factors $(2\pi)^3$ and $-ic$ compensating for the factor i/c due to the definition of the normalized frequency in (1), and adjusting (30) to (29).

In accordance with (25), we have the wave (30) transformed into reference-frame Γ'. Inasmuch as we have only a single direction of propagation we can use the operator $\tilde{\boldsymbol{W}}_{\boldsymbol{K}}$ defined in (23). Therefore in Γ' we now have

$$\left\{ \begin{array}{c} \boldsymbol{E}' \\ \boldsymbol{H}' \end{array} \right\} = \tilde{\boldsymbol{W}}_{\boldsymbol{K}} \cdot \left\{ \begin{array}{c} \hat{\boldsymbol{E}}e \\ \hat{\boldsymbol{H}}h \end{array} \right\} \delta(\tau) \tag{31}$$

Exploiting the phase-invariance (14) which prescribes $\omega'\tau' = \omega\tau$, and changing the integration variable, (31) is rewritten as

$$\left\{ \begin{array}{c} \boldsymbol{E}' \\ \boldsymbol{H}' \end{array} \right\} = \left\{ \begin{array}{c} \hat{\boldsymbol{E}}'e' \\ \hat{\boldsymbol{H}}'h' \end{array} \right\} \delta(\tau'), \quad \left\{ \begin{array}{c} \hat{\boldsymbol{E}}'e' \\ \hat{\boldsymbol{H}}'h' \end{array} \right\} = \tilde{\boldsymbol{W}}_{\boldsymbol{K}} \cdot \left\{ \begin{array}{c} \hat{\boldsymbol{E}}e \\ \hat{\boldsymbol{H}}h \end{array} \right\}$$
$$\tau' = t' - \hat{\boldsymbol{k}}' \cdot \boldsymbol{r}'/c, \quad \omega'\tau' = \omega t \tag{32}$$
$$\delta(\tau) = \int_{-\infty}^{\infty} e^{-i\omega\tau} d\omega/2\pi = \int_{-\infty}^{\infty} e^{-i\omega'\tau'} p d\omega'/2\pi = p\delta(\tau')$$
$$p = d\omega/d\omega' = 1/(\gamma(1 - \beta\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}}))$$

It follows from (32) that in Γ' the transformation leads once again to an impulse plane-wave, whose direction and amplitude are determined by the factors shown in (31), (32). Furthermore, since the spectrum is flat and infinite, there is no way of detecting frequency shifts according to the Doppler-effect formulas (15). Of course, there will be a change in the amplitude and the direction, but compared to frequency shifts, these parameters are usually much more complicated to measure.

It also follows that reflections from moving plane interfaces, as long as the material is nondispersive, will lead in the initial reference-frame Γ to scattered impulsive plane-waves having the same spectral structure, hence once again for delta-function impulses no Doppler frequency shifts are detectable.

5. BROADBAND SCATTERING BY CYLINDERS AT-REST

Intuition suggests that an impulsive plane-wave excitation as in (29), when hitting a small object at-rest, will be scattered as circular-cylindrical, or spherical-waves, in the two- or three-dimensional cases, respectively. Careful scrutiny shows that this problem is much more complicated, compared to reflection from plane interfaces, for objects at rest and more so for objects in motion.

We consider the two-dimensional case of scattering by cylinders at rest in Γ'. The excitation wave in Γ is given by (29), (30), polarized along the cylindrical z-axis, i.e., for say the $\boldsymbol{E}$-field, we have $\hat{\boldsymbol{E}} = \hat{\boldsymbol{z}}$. The motion is considered in the plane perpendicular to $\hat{\boldsymbol{z}}$, i.e., $\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{z}} = 0$. Using the vector identities for $\partial_{\boldsymbol{r}} \times (\boldsymbol{v} \times \tilde{\boldsymbol{I}})$ and $\hat{\boldsymbol{v}} \times (\hat{\boldsymbol{k}} \times \tilde{\boldsymbol{I}})$, in (23), (27), the operators reduce to

$$\begin{aligned} &\tilde{\boldsymbol{W}}_{\boldsymbol{R}} = \gamma \left(1 + (\boldsymbol{v} \cdot \partial_{\boldsymbol{r}})\partial_t^{-1}\right) \tilde{\boldsymbol{Z}},\ \tilde{\boldsymbol{W}}_{\boldsymbol{K}} = \gamma(1 - \beta \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}})\tilde{\boldsymbol{Z}} \\ &\tilde{\boldsymbol{W}}'_{\boldsymbol{R}'} = \gamma \left(1 - (\boldsymbol{v} \cdot \partial_{\boldsymbol{r}'})\partial_{t'}^{-1}\right) \tilde{\boldsymbol{Z}},\ \tilde{\boldsymbol{W}}'_{\boldsymbol{K}'} = \gamma(1 + \beta \hat{\boldsymbol{v}} \cdot k')\tilde{\boldsymbol{Z}},\ \tilde{\boldsymbol{Z}} = \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \end{aligned} \tag{33}$$

In the initial reference-frame Γ the excitation wave is provided by (30) with $\hat{\boldsymbol{E}} = \hat{\boldsymbol{z}}$. The scatterer is at rest in Γ', hence the transformed excitation wave is given by (31), (32) with $\hat{\boldsymbol{E}}' = \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}'$. It is noted that in the present case the factor $\gamma(1 - \beta \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}})$ appearing in (32) in the denominator cancels with the same factor, appearing according to (33) in the numerator. However, it must be remembered that this is a special case, for the field of a plane wave polarized perpendicularly with respect to $\boldsymbol{v}$. It will not work for the $\boldsymbol{H}$-field of the same wave, unless it too satisfies $\boldsymbol{v} \cdot \boldsymbol{H} = 0$, i.e., $\hat{\boldsymbol{v}} = \hat{\boldsymbol{k}}$.

Various representations for the scattered wave are summarized in [3], in particular see Twersky (ref. [9] there), who discusses the special-function representation of the scattered wave and shows that it exists at least (there exists also a weaker condition) outside the circle circumscribing the scatterer's cross-section.

Accordingly, for $\hat{\boldsymbol{E}}' = \hat{\boldsymbol{z}}$, and a scatterer whose geometry is defined relative to the spatial origin of Γ', (26) can be written as

$$\begin{aligned} &\boldsymbol{E}'_{sc}(\boldsymbol{R}') = \hat{\boldsymbol{z}} e' \int_{-\infty}^{\infty} d\omega' E'_{sc}(\boldsymbol{R}', \omega')/2\pi \\ &E'_{sc}(\boldsymbol{R}', \omega') = e^{-i\omega' t'} \Sigma_m i^m a_m(\omega', \alpha') H_m(\rho') e^{im\varphi'} \\ &= \int_{\psi'} e^{i\rho' C_{\varphi' - \psi'} - i\omega' t'} g(\psi') d\psi'/\pi,\ C_{\varphi' - \psi'} = \cos(\varphi' - \psi') \end{aligned} \tag{34}$$

$$\int_{\psi'} = \int_{\psi'=\varphi'-(\pi/2)+i\infty}^{\psi'=\varphi'+(\pi/2)-i\infty}, \ \rho' = \omega' r'/c,$$
$$\Sigma_m = \Sigma_{m=-\infty}^{m=\infty}, \quad g(\varphi') = \Sigma_m a_m e^{im\varphi'}$$

In (34) φ', α' are the azimuthal and the incidence angles, respectively. The Hankel functions of the first kind $H_m = H_m^{(1)}$, together with the time-factor $e^{-i\omega' t'}$ provide for outgoing waves. The coefficients $a_m(\omega', \alpha')$ are given, e.g., by solving the pertinent boundary-value problem. The complex integral representation is recognized as a plane-wave integral-representation based on the Sommerfeld integral-representation for the Hankel functions (e.g., see [9]). Note also the compact notation for the trigonometric functions. The function $\boldsymbol{E}'_{sc}(\boldsymbol{R}', \omega')$ denotes the integrand in (34), and its dependence on the frequency ω'.

In order to find the spatiotemporal function $\boldsymbol{E}'_{sc}(\boldsymbol{R}')$ the integral must be evaluated, which seems analytically infeasible. Even far-field expressions in terms of the asymptotic approximation of the Hankel function are inapplicable, because the infinite range of ω' prescribing an unbounded argument for $H_m = H_m^{(1)}$.

Furthermore, consider the example of scattering by a perfectly conducting circular cylinder of radius a and $\hat{\boldsymbol{E}} = \hat{\boldsymbol{z}}$ polarization. For this case (e.g., see for example Kong [10]), we have

$$\begin{aligned} a_m(\omega', \alpha') &= -e^{im\alpha'} J_m(\rho'_a)/H_m(\rho'_a) \\ \rho'_a &= \omega' a/c \end{aligned} \tag{35}$$

For thin cylinders with infinitesimally small a we find

$$\begin{aligned} a_0 &\approx -\frac{i\pi}{2\ln(2/(\delta\rho'_a))}, \quad \delta = 1.78107 \\ a_m &\approx -e^{-im\alpha'} i\pi m(\rho'_a/2)2^{2m}/(m!)^2, \ m = 1, 2, \ldots \end{aligned} \tag{36}$$

where in (36) δ is Euler's constant (e.g., see [9], p. 358). For fixed ω' and very small a the dominant term in (36) is a_0, hence the scattered wave will display monopole behavior, i.e., the scattered field will be independent of directions.

For $\hat{\boldsymbol{H}} = \hat{\boldsymbol{z}}$ polarization, instead of (35) we find

$$a_m(\omega', \alpha') = -e^{-im\alpha'} \partial_{\rho'_a} J_m(\rho'_a)/\partial_{\rho'_a} H_m(\rho'_a) \tag{37}$$

For this case, instead of (36) we find

$$\begin{aligned} a_0 &\approx -i\pi(\rho'_a/2)^2 \\ a_m &\approx -e^{-im\alpha'} i\pi m(\rho'_a/2)2^{2m}/(m!)^2, \ m = 1, 2, \ldots \end{aligned} \tag{38}$$

hence according to (38) the dominant multipoles are both the monopole a_0 and the dipole a_1 which is already dependent on directions. In general the scattering-amplitude terms a_m govern the azimuthal dependence of the field.

Obviously in view of the infinite range of ω' the term "thin cylinder" is inapplicable in (34)–(38). Moreover, upon trying to implement the far-field asymptotic approximation (e.g., see [9], p. 359) in (34) in the form

$$i^m H_m(\rho') \sim (2/i\pi\rho')^{1/2} e^{i\rho'} \tag{39}$$

we have to conclude that it is inapplicable because of the infinite range of ω'.

To be able to implement the above approximations, we need to assume a limited but sufficiently broadband spectrum centered about some arbitrary ω'_0, so that the impulsive nature of the peak amplitude is significant, but the band is limited in order to allow the approximations. This is implemented by assuming a spectral window $W(\omega')$ instead of the delta-function (29)–(32).

We assume that in (34) and the far-field approximation (39) only the exponentials vary rapidly, hence we conclude that the slowly-varying factors are well-approximated as constants at ω'_0 and we finally approximate (34) in the form

$$\begin{aligned}
&\boldsymbol{E}'_{sc}(\boldsymbol{R}', \omega'_0) \sim \hat{\boldsymbol{z}} e' L g S(\overline{\tau}'),\\
&\overline{\tau}' = t' - r'/c,\ L = (2/i\pi\rho'_0)^{1/2},\ \rho'_0 = \omega'_0 r'/c\\
&S(\overline{\tau}') = \int_{-\infty}^{\infty} W(\omega') e^{-i\omega'\overline{\tau}'} d\omega'/2\pi\\
&g = g(\varphi', \omega'_0) = \Sigma_m a_m(\omega'_0, \alpha') e^{im\varphi'}
\end{aligned} \tag{40}$$

Subject to the approximations, (40) displays in the time-domain a spiked impulsive scattered wave, spatially modulated according to $g = g(\varphi', \omega'_0)$. If $W(\omega')$ is sufficiently broad, the pulse is increasingly spiked in the vicinity of $\overline{\tau}' = 0$. The equation of motion

$$\overline{\tau}' = t' - r'/c = 0 \tag{41}$$

displays circular outgoing waves, propagating with a radial group-velocity $dr'/dt' = c$.

At a first glance (41) looks trivial, but actually it is not self-evident. The dispersion-relation (6) is derived for plane waves assumed in (5). Accordingly, in free space (29) prescribes $\omega/k = v_{ph} = c$. The reason for the simplicity of (40) stems from the fact that we are dealing

with a far-field form where the outgoing wave with a slow-varying amplitude locally resembles a plane wave. Furthermore, in free-space the phase- and group-velocities are identical.

Special cases are provided by (36), (38). Without delving into the complicated mathematics of electromagnetic scattering by three-dimensional objects (e.g., see [9] for a discussion of the Mie problem), it is expected that the results be similar, i.e., for a band-limited spectrum the impulsive scattered-wave will display the spatial behavior prescribed by the scattering-amplitude, and outgoing spherical waves governed by (40) with r' now denoting the spherical radius.

6. BROADBAND SCATTERING BY MOVING CYLINDERS

The transition from $\boldsymbol{E}'_{sc}(\boldsymbol{R}')$, the scattered field measured in Γ' in terms of the coordinates $\boldsymbol{R}'$ native to this reference-system, to $\boldsymbol{E}_{sc}(\boldsymbol{R}')$, the field measured in Γ, but expressed in terms of the non-native coordinates $\boldsymbol{R}'$, is effected by applying the differential operator indicated in (27), (33) to $\boldsymbol{E}'_{sc}(\boldsymbol{R})$ in (34). The harmonic case has been discussed recently [5] in relation to elliptical coordinates. We obtain

$$\begin{aligned}
&\boldsymbol{E}_{sc}(\boldsymbol{R}') = \hat{\boldsymbol{z}} e' \int_{-\infty}^{\infty} d\omega' E_{sc}(\boldsymbol{R}', \omega')/2\pi \\
&E_{sc}(\boldsymbol{R}', \omega') = e^{-i\omega' t'} B_{\xi'} \Sigma_m i^m a_m(\omega', \alpha') H_m(\rho') e^{im\varphi'} \\
&B_{\xi'} = \gamma(1 - i\beta(C_{\varphi'-\xi'} \partial_{\rho'} (S_{\varphi'-\xi'}/\rho')\partial_{\varphi'})),\ C_{\xi'} = \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{x}}'
\end{aligned} \tag{42}$$

where in (42) ξ' is the angle subtended by the two unit vectors $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{x}}'$ in Γ'.

For $\xi' = 0$, we have a simpler expression, already obtained before (without employing the differential-operator forms) [1, 11]

$$\begin{aligned}
E_{sc}(\boldsymbol{R}', \omega') &= e^{-i\omega' t'} B \Sigma_m i^m a_m(\omega', \alpha') H_m(\rho') e^{im\varphi'} \\
&= e^{-i\omega' t'} \gamma \Sigma_m i^m b_m(\omega', \alpha') H_m(\rho') e^{im\varphi'} \\
B &= \gamma(1 - i\beta(C_{\varphi'} \partial_{\rho'} - (S_{\varphi'}/\rho')\partial_{\varphi'})) \\
b_m &= a_m + (a_{m-1} + a_{m+1})\beta/2
\end{aligned} \tag{43}$$

In (42), (43) it is demonstrated that the application of the differential operator modifies the scattering coefficients. It has been noted before [1, 11] that this mode-coupling displays new, velocity-dependent, multipole terms.

Subject to (43), the analog of (40) is now obtained simply by replacing a_m (40) by b_m, (43). This result is remarkable in that

the pulse spectrum of the scattered wave $\boldsymbol{E}_{sc}(\boldsymbol{R}')$, measured in reference-frame Γ, but expressed in terms of Γ' native coordinates, is unchanged, except for its space-dependent amplitude as determined by the scattering amplitude g.

What do we intuitively expect for the equation of motion of the pulse in Γ, in terms of the Γ native coordinates? This point is discussed in many textbooks, e.g., see [12]. According to a somewhat oversimplified argument, once the wave is emitted by the thin cylinder, it propagates in free space, and therefore pulse speed should be c. The simple argument then continues, claiming that for moving sources we should get non-concentric circles (or non-concentric spherical surfaces in the corresponding three-dimensional case), with centers indicating the position from which the scattered wave initially emanated.

The present discussion is more detailed. It reveals that, expressed in terms of Γ' native coordinates, the far-field forms of (42), (43) leave the equation of motion (41) unchanged. Upon squaring terms in (41) and substituting (11) we now find

$$\begin{aligned} t'^2 &= \gamma^2\left(t - \boldsymbol{v}\cdot\boldsymbol{r}/c^2\right)^2 = \gamma^2\left(t^2 - 2vr_{\parallel}/c^2 + (vr_{\parallel}/c^2)^2\right) \\ &= (r'/c)^2 = (\gamma(\boldsymbol{r}_{\parallel} - \boldsymbol{v}t) + \boldsymbol{r}_{\perp})\cdot(\gamma(\boldsymbol{r}_{\parallel} - \boldsymbol{v}t) + \boldsymbol{r}_{\perp})/c^2 \\ &= \left(\gamma^2\left(r_{\parallel}^2 - 2vr_{\parallel}t + v^2t^2\right) + r_{\perp}^2\right)/c^2 \end{aligned} \tag{44}$$

where in (44) $\perp$, $\parallel$, denote components perpendicular, parallel, to the velocity, correspondingly, and it is noted that the term $-2vr_{\parallel}t$ cancels on both sides of the equation. Obviously this does not conform with the notion that in Γ we should find once again circular, spherical, wavefronts, for the two, three dimensional cases, respectively.

To gain more insight, let us retain in (44) only first order effects in v/c. This yields

$$\begin{aligned} t'^2 &= \left(t - \boldsymbol{v}\cdot\boldsymbol{r}/c^2\right)^2 = \left(t^2 - 2vr_{\parallel}/c^2\right) \\ &= \left(r_{\parallel}^2 - 2vr_{\parallel}t + v^2t^2 + r_{\perp}^2\right)/c^2 = \left(r^2 - 2vr_{\parallel}t + v^2t^2\right)/c^2 \end{aligned} \tag{45}$$

and upon simplifying (45) we obtain (cf. (41))

$$t - r/c = 0 \tag{46}$$

This means that only in the far field, and to the first order in v/c, the intuitively expected results of circular, spherical, wave-surfaces, are indeed valid.

7. BROADBAND SCATTERING BY A HALF-PLANE AT-REST

The solution for the problem of scattering of a monochromatic plane wave by a moving half-plane is historically attributed to Sommerfeld [13], see also [14, 15] for a discussion of the problem and the related Wiener-Hopf method, as well as early references. Presently we consider the simple problem of scattering by a perfectly-conducting half-plane, with the excitation-wave given by (29), (30) in reference-frame Γ, hence in Γ' we have the corresponding transformed (31), (32).

We rely mainly on formulas given by [14], therefore some remarks on notation are needed. Throughout the present study, positive angles are measured in the x', y' plane off the x'-axis towards the y'-axis. This convention is also used in [14], except for the incident wave, see Fig. 11.6, p. 566 there, which also indicates the scatterer half-plane, taken along the positive x-axis. The direction of the incident wave is indicated in [14] by the angle α_0, which for positive values is measured off the $-x$ direction towards the $-y$ direction. To conform with (29), (30) we have to assume in the Γ reference-frame

$$\alpha - \alpha_0 = \pi, \quad C_\alpha = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{x}}, \quad C_{\alpha_0} = -\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{x}} \tag{47}$$

This would carry over to all subsequent formulas given in [14], but in order to discuss the problem for an object at-rest in Γ', apostrophes have to be judicially added. Hence referring to (31), (32) we are dealing with

$$\alpha' - \alpha_0' = \pi,\ C_{\alpha'} = \hat{\boldsymbol{k}}' \cdot \hat{\boldsymbol{x}}',\ C_{\alpha_0'} = -\hat{\boldsymbol{k}}' \cdot \hat{\boldsymbol{x}}',\ \hat{\boldsymbol{x}}' = \hat{\boldsymbol{x}} \tag{48}$$

where in (48) we made a choice $\hat{\boldsymbol{x}}' = \hat{\boldsymbol{x}}$, and the angles in (47), (48), are related by the Doppler-effect formulas (15) which can be displayed as the aberration formula for the angles of a plane wave in two inertial reference-frames [2].

With all this in mind, for a monochromatic incident wave, the wave scattered by a half-plane at-rest in Γ' is given by (cf. [14], Ch. 11.5, Eqn. (8))

$$\begin{aligned} \boldsymbol{E}'_{sc}(\boldsymbol{R}', \omega') &= \hat{\boldsymbol{z}} e' e^{-i\omega' t'} \int_{\psi'} e^{i\rho' C_{\varphi' \mp \psi'}} g(\psi') d\psi'/\pi \\ g(\psi') &= i S_{\alpha_0'/2} S_{\psi'/2} / (C_{\psi'} + C_{\alpha_0'}) \Sigma_m a_m(\alpha_0') e^{im\psi'} \end{aligned} \tag{49}$$

where in (49) the sign $\mp$ applies to $y' \gtrless 0$ half-planes, respectively. The full solution (49) must include the contribution of the simple pole at $\psi' = \pi - \alpha_0'$.

It is noted that unlike the cylinder case, (34), here $g(\psi')$ is frequency-independent. Also note the symmetry properties

$$g(\psi') = -g(-\psi') = -g(\sigma'),\ \sigma' = -\psi' \tag{50}$$

enabling us to rewrite the part of (49) for $y' < 0$, with the + in the exponent, in the form

$$\boldsymbol{E}'_{sc}(\boldsymbol{R}', \omega') = -\hat{\boldsymbol{z}} e' e^{-i\omega' t'} \int_{\sigma'} e^{i\rho' C_{\varphi'-\sigma'}} g(\sigma') d\sigma'/\pi \tag{51}$$

Exploiting the similarity between (34), and (49), (51), we can rewrite (49) using the Hankel-Fourier series representation in the form

$$\begin{aligned} &\boldsymbol{E}'_{sc}(\boldsymbol{R}', \omega') = \pm\hat{\boldsymbol{z}} e' e^{-i\omega' t'} \int_{\psi'} e^{i\rho' C_{\varphi'-\psi'}} g(\psi') d\psi'/\pi \\ &= \pm\hat{\boldsymbol{z}} e' e^{-i\omega' t'} \Sigma_m i^m a_m(\alpha'_0) H_m(\rho') e^{im\varphi'} \\ &\sim \pm\hat{\boldsymbol{z}} e' (2/i\pi\rho')^{1/2} e^{i\rho' - i\omega' t'} g(\varphi') \\ &g(\varphi') = iS_{\alpha'_0/2} S_{\varphi'/2}/(C_{\varphi'} + C_{\alpha'_0}) = \Sigma_m a_m(\alpha'_0) e^{im\varphi'} \end{aligned} \tag{52}$$

where in (52) the sign $\pm$ applies to $y' \gtrless 0$, respectively. We must hasten and say that although $H_m(\rho')$ are singular at $\rho' = 0$, in (52) the scattered field at the origin is finite. In (52) the series must be considered as a whole and not term by term. This "illusory" effect of the "luminous edge", as referred to by Sommerfeld [13] are also discussed experimentally and from the point of view of the eye's physiology. Indeed, it is shown (e.g., see [13, 14]), that the total field $\boldsymbol{E}'_{sc} + \boldsymbol{E}'$ can be represented in terms of the Fresnel-integrals functions F in the form

$$\begin{aligned} \boldsymbol{E}'_{sc} + \boldsymbol{E}' \ = \ &\hat{\boldsymbol{z}} e' e^{-i\omega' t'} (i\pi)^{-1/2} \Big[e^{-i\rho' C_{\varphi'-\alpha'_0}} F(\rho'' C_{(\varphi'-\alpha'_0)/2}) \\ &- e^{-i\rho' C_{\varphi'+\alpha'_0}} F(\rho'' C_{(\varphi'+\alpha'_0)/2}) \Big],\ \rho'' = -(2\rho')^{1/2} \end{aligned} \tag{53}$$

and since F in (53) are finite at the origin, including both diffractive and geometrical-optics contributions, so also is $\boldsymbol{E}'_{sc}$ (53).

We are now ready to discuss the broadband behavior of the scattered field. Again the incident wave is an impulse as given by (31), (32). The diffractive scattered waves corresponding to (49)–(52) are now considered in the far field, in the same way (40) was fashioned, expressed in terms of the impulsive signal $S(\overline{\tau}')$, (40). It will be advantageous to express the asymptotic behavior of the field

in terms of the appropriate formulas for the Fresnel-integral functions. Accordingly, the pulse peaks at circles as given in (41).

For the various domains, we now need to consider the contribution prescribed by the simple pole at $\psi' = \pi - \alpha'_0$, causing $g(\psi')$ in (49) to become infinite. Inasmuch as $g(\psi')$ is now frequency-independent, the additional impulsive waves will replicate the excitation wave, i.e., if excitation-wave is taken as a delta-function impulse as in (31), (32), this structure will be preserved. Within the "reflection region" this prescribes [14] (cf. (31), (32))

$$\begin{aligned} \boldsymbol{E}'_{ref} &= -\hat{\boldsymbol{E}}' e' \delta(\tau'_{ref}),\ \tau'_{ref} = t' - \hat{\boldsymbol{k}}'_{ref} \cdot \boldsymbol{r}'/c \\ \hat{\boldsymbol{k}}_{\|ref} &= \hat{\boldsymbol{k}}'_{\|},\ \hat{\boldsymbol{k}}'_{\perp ref} = -\hat{\boldsymbol{k}}'_{\perp},\ 0 < \varphi' < \pi - \alpha'_0 \end{aligned} \tag{54}$$

where in (54) the definitions (48) are employed and $\|$, $\perp$ refer to components parallel, perpendicular, with respect to the half-plane scatterer. Thus (54) describes the geometrical-optics reflected wave. This wave exists only in the sector indicated in (54), subtended by the scatterer half-plane $\varphi' = 0$, and the angle $\varphi' = \pi - \alpha'_0 = 2\pi - \alpha'$.

In the region below the scatterer, i.e., for $y' < 0,\ \pi < \varphi' < 2\pi$, there exists a sector

$$\pi < \varphi' < \alpha', \quad \alpha' = \pi + \alpha'_0 \tag{55}$$

illuminated by the excitation wave. Finally there exists the "shadow zone", where the excitation wave (31), (32) is annihilated by the transmitted wave

$$\boldsymbol{E}'_{tr} = -\hat{\boldsymbol{E}}' e' \delta(\tau'),\ \tau' = t' - \hat{\boldsymbol{k}}' \cdot \boldsymbol{r}'/c,\ \alpha' < \varphi' < 2\pi,\ \alpha' = \pi + \alpha'_0 \tag{56}$$

This terminates the discussion of broadband scattering from a half-plane at rest. In the next section, in a similar manner used above for circular-cylinders, the question of scattering form a half-plane in motion will be addressed.

8. BROADBAND SCATTERING BY A MOVING HALF-SPACE

The above discussion facilitates the extension of the results to the case of a moving half-plane. Investigation of a similar geometry, with monochromatic excitation, has been considered before by De Cupis et al. [16].

In addition to the convenience of using spatiotemporal operators to describe velocity-dependent scattering, what makes this problem

especially interesting is the fact that here we will have to deal with two substantially different kinds of angles: Angles appearing in the phase of a plane wave are usually transformed by invoking the phase-invariance principle (14) and the ensuing Doppler-effect (15). On the other hand, angles referring to purely geometrical properties in one reference-frame will have to be transformed using the Lorentz-transformation (11), and are expected to be spatiotemporally-dependent in another reference-frame.

First consider the excitation wave (29), (30) in Γ, which is transformed into reference-frame Γ' yielding once again a plane, delta-function impulsive excitation-wave (31), (32). From this point on we deal with scattering by a half-plane at rest in Γ', as discussed above. Accordingly, the diffractive contribution for the monochromatic case, is given by (49)–(52).

In view of the identical structure of (52), and the corresponding (34) for the case of the cylinder, the same conclusions apply: Thus in the far field and for a limited-broadband spectrum, the pulse is characterized by the equation of motion (41) for the present case as well. The transformation back to the reference-frame Γ, yielding $\boldsymbol{E}_{sc}(\boldsymbol{R}')$, is effected by the application of the operator as in (42), (43). We conclude that the scattering coefficients are affected by the velocity-dependent terms, but there are no major differences between $\boldsymbol{E}'_{sc}(\boldsymbol{R}')$, $\boldsymbol{E}_{sc}(\boldsymbol{R}')$, i.e., no new poles will appear in the solution. Once again, we have here the interesting effect of the equation of motion for the impulsive cylindrical-wave as given by (44), and its first-order approximation (45), (46).

In the present case we have the additional geometrical-optics terms, due to the pole in $g(\psi')$, leading to the reflected wave (54), and the transmitted wave (56) in Γ'. The transmitted wave is equal in magnitude and opposite in sign in relation to the excitation-wave (31), (32).

Therefore the application of the spatiotemporal operator to the transmitted wave (56) recovers in Γ the original excitation wave (31), (32), with an opposite sign. It follows that in the shadow zone the annihilation observed in Γ' persists also in the corresponding shadow-zone in Γ.

The reflected wave (54) transforms according to (cf. (32), (33))

$$\begin{aligned}
\boldsymbol{E}_{ref} &= \tilde{\boldsymbol{W}}'_{\boldsymbol{K}'_{ref}} \cdot \boldsymbol{E}'_{ref} = -\tilde{\boldsymbol{W}}'_{\boldsymbol{K}'_{ref}} \cdot \hat{\boldsymbol{E}}' e' \delta(\tau'_{ref}) = \hat{\boldsymbol{E}}_{ref} e' \delta(\tau_{ref}) \\
\hat{\boldsymbol{E}}_{ref} &= -p'_{ref} \tilde{\boldsymbol{W}}'_{\boldsymbol{K}'_{ref}} \cdot \hat{\boldsymbol{E}}', \ \tau_{ref} = t - \hat{\boldsymbol{k}}_{ref} \cdot \boldsymbol{r}/c, \ \omega' \tau'_{ref} = \omega_{ref} \tau_{ref} \\
\tilde{\boldsymbol{W}}'_{\boldsymbol{K}'_{ref}} &= \gamma \left(1 + \beta \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}}'_{ref}\right) \tilde{\boldsymbol{Z}}, \quad \tilde{\boldsymbol{Z}} = \hat{\boldsymbol{z}} \hat{\boldsymbol{z}}
\end{aligned} \tag{57}$$

$$\delta(\tau'_{ref}) = \int_{-\infty}^{\infty} e^{-i\omega'\tau'_{ref}} d\omega'/2\pi = \int_{-\infty}^{\infty} e^{-i\omega_{ref}\tau_{ref}} p'_{ref} d\omega_{ref}/2\pi$$
$$p'_{ref} = d\omega'/d\omega_{ref} = 1/\left(\gamma(1+\beta\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}'_{ref})\right)$$

where in (57) the transformation of $\hat{\boldsymbol{k}}'_{ref}$ in Γ', to $\hat{\boldsymbol{k}}_{ref}$ in Γ, is governed by (14), (15), and for the $\boldsymbol{E}$-field we find once again the cancellation of the factor $\gamma(1+\beta\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}'_{ref})$.

It remains to discuss the various regions valid for the above waves, as observed from the reference-frame Γ. This is a purely geometrical property, and therefore governed by the Lorentz-transformation (11). Consequently, regions that are purely spatial in Γ', i.e., independent of time t', are expected to become spatiotemporally-dependent in Γ, depending on both $\boldsymbol{r}$ and t.

The half-plane edge is defined in Γ' by the origin $\boldsymbol{r}' = 0$, hence in Γ (11) prescribes $\boldsymbol{r} = \boldsymbol{v}t$. By (11) the unit-vector $\hat{\boldsymbol{r}}'$ transforms according to

$$\begin{aligned}\hat{\boldsymbol{r}}' &= \boldsymbol{r}'/(\boldsymbol{r}'\cdot\boldsymbol{r}')^{1/2} = \tilde{\boldsymbol{U}}\cdot(\boldsymbol{r}-\boldsymbol{v}t)/\left(\gamma^2\left(r_\parallel^2 - 2vr_\parallel t + v^2t^2\right) + r_\perp^2\right)^{1/2} \\ &\approx (\boldsymbol{r}-\boldsymbol{v}t)/\left(r^2 - 2vrC_\xi t + v^2t^2\right)^{1/2}, \quad C_\xi = r/r_\parallel \end{aligned} \tag{58}$$

From (58), together with a choice of parallel x-axes in the two reference-systems, it follows that arbitrary angles obey

$$\begin{aligned} C_{\varphi'} &= \hat{\boldsymbol{x}}'\cdot\hat{\boldsymbol{r}}' = \hat{\boldsymbol{x}}\cdot\tilde{\boldsymbol{U}}\cdot(\boldsymbol{r}-\boldsymbol{v}t)/\left(\gamma^2\left(r_\parallel^2 - 2vr_\parallel t + v^2t^2\right) + r_\perp^2\right)^{1/2} \\ &\approx \hat{\boldsymbol{x}}\cdot(\boldsymbol{r}-\boldsymbol{v}t)/T = (rC_\varphi - vC_\xi t)/T \\ T &= \left(r^2 - 2vrC_\xi t + v^2t^2\right)^{1/2} \end{aligned} \tag{59}$$

It follows from (59) that the half-plane can be defined as $C_{\varphi'} = 1$. Applying (59) to the above defined angles for the illumination, reflection, and shadow zones, displays their spatiotemporal behavior as observed in Γ. With these remarks the problem can be considered as fully solved.

9. CONCLUDING REMARKS

The present study demonstrates how free-space velocity-dependent scattering of electromagnetic waves can be efficiently handled by using the new spatiotemporal differential-operators. One of the advantages of using these tools is the fact that broadband scattering is also easily handled.

In particular, two two-dimensional problems are tackled: Scattering by cylinders, which has been considered before in circular and elliptical coordinate systems, is investigated in the present context. The other example is scattering by a moving half-plane, for which exact and approximate results exist for the monochromatic case.

Intuitively one expects thin cylinders at-rest, excited by sharp plane pulses to re-radiate pulses located on concentric circles. When the cylinder is moving, intuition suggests that the pulses be located on a series of eccentric circles, whose centers are prescribed by the velocity-dependent location of the cylinder at the time the excitation pulse arrives and is scattered. A careful analysis proves that such notions are valid in the far field and only to the first order in v/c.

The field scattered by a half-plane contains diffractive contributions created by the edge, and additional geometrical-optics plane waves which are due to the pole in the pertinent scattering amplitude. The transformation of the latter waves from the co-moving reference-frame Γ' back to the initial reference-frame Γ, in which the excitation wave was defined, is also discussed. Although intuitively plausible, it is rigorously shown that the regions corresponding to the reflection and shadow zones can be identified also in Γ, where the half-plane is observed to be moving. In Γ the regions are spatiotemporally-dependent, subject to the Lorentz-transformation.

Numerical simulations are left for future study, and hopefully will help in visualizing the various characteristics of the present class of problems.

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