ALGEBRAIC FUNCTION APPROXIMATION IN EIGENVALUE PROBLEMS OF LOSSLESS METALLIC WAVEGUIDES (REVISITED)

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Abstract—The problem of studying modal characteristics of metallic waveguides filled with lossless inhomogeneous and/or anisotropic media, is one of studying properties of the propagation constant of the guiding structure. It is shown that modal behavior in the neighborhood of critical frequencies such as cutoff frequencies and frequencies marking the onset of complex wave mode intervals, can be modeled through approximation of the propagation constant by a root of an algebraic equation. The particular form of the algebraic function approximating the propagation constant is discussed in the neighborhood of a singularity. A numerical example is included to stress the viability of the technique.

5 Conclusions

Appendix A. Constraints on Properties of an Algebraic Branch Point Where Propagation Constant Is Finite

Appendix B. Preclusion of an Algebraic Branch Point Where the Propagation Constant Is Infinite and Some Properties of the Propagation Constant with a Pole

Appendix C. Glossary

References

1. INTRODUCTION

Backward wave systems are used as generators and amplifiers in microwaves. In a paper published recently [1] an attempt was made to apply algebraic function theory to backward wave problems in metallic waveguides.

The purpose of this work is to continue to develop an approach for employing algebraic function theory concepts in eigenvalue problems of metallic lossless waveguides loaded inhomogeneously and/or anisotropically. The first part of this effort was previously published in [1], where the algebraic function approximation approach was established for backward wave modes of the above mentioned type of closed waveguides. In this work we shall treat the complex wave mode problem, along with exposing the modal behavior in the neighborhood of a cutoff frequency, both for the above mentioned guiding structures and both with a view to apply algebraic function theory concepts to assess the propagation constant function.

Application of operator theory to electromagnetic field problems of waveguides is extensively covered in the literature, e.g., in [2]. But to the author's knowledge, properties of the eigenvalues, i.e., the propagation constants as functions, are not discussed in a formalism reduced for electromagnetics. In this respect, the proposed method is original.

In this work, as the tool to obtain an approximate solution of Maxwell's equations, and to apply our method on the results thereof, we use the Method of Moments. This converts Maxwell's partial differential equations into a linear algebraic equation system of finite order. In particular, as expansion functions utilizing the modes of the same investigated closed waveguide when its loading is removed,

one arrives at the transmission line representation, wherein the entries of the coefficient matrices of the linear algebraic equations can be interpreted as the parameters of a system of coupled transmission lines [1, 3, 4].

Restricting ourselves to the class of inhomogeneous loadings which do not induce coupling between transverse and longitudinal field components, the system of linear algebraic equations which is the product of the Moment Method, can be expressed as [1, 3, 4]:

$$
\begin{bmatrix} \gamma(p)v(p) \\ \gamma(p)i(p) \end{bmatrix} = \begin{bmatrix} 0 & Z(p) \\ Y(p) & 0 \end{bmatrix} \begin{bmatrix} v(p) \\ i(p) \end{bmatrix}. \tag{1}
$$

Here $p = \sigma + j\omega$ is the complex frequency, $\gamma(p)$ is the propagation constant, and v and i are the vectors of transmission line voltages and currents. $Z(p)$ and $Y(p)$ are the series impedance and the shunt admittance coupling matrices, per unit length.

For the lossless guides with the above described type of loading, and when constituent $\hat{\varepsilon}$ and $\hat{\mu}$ matrices are rational functions of p, $Z(p)$ and $Y(p)$ are positive real, lossless and rational matrices and hence they are Foster matrices. Foster matrices are analytic in $\text{Re}\{p\} > 0$ real where p is real, and paraskew Hermitian, i.e., $-Z^T(-j\omega) = Z(j\omega)$, where $(\cdot)^T$ denotes the transpose [1, 3].

For the actual physical problem, corresponding to the closed guide with the above prescribed loading, $Z(p)$ and $Y(p)$ are infinite in order. However in line with the Moment Method, we use only finite truncations of the matrices $Z(p)$ and $Y(p)$ in our approximation of the physical problem. Therefore, by (1), $\gamma^2(p)$ in our notation stands for the eigenvalue of the matrix product $Z(p)Y(p)$ or of $Y(p)Z(p)$ which share the same eigenvalues and where $Z(p)$ and $Y(p)$ are above defined, truncated, finite dimensional $(m \times m)$ square matrices. $v(p)$ and $i(p)$ are the associated eigenvectors of $Z(p)Y(p)$ and $Y(p)Z(p)$ respectively.

It has been shown that for each p and for each eigenvalue $\gamma^2_{phy}(p)$ of the physical system (i.e., of $Z(p)Y(p)$ where $Z(p)$ and $Y(p)$ are not truncated), there always exist finite truncations of $Z(p)$ and $Y(p)$ such that $Z(p)Y(p)$ has at least one eigenvalue arbitrarily close to $\gamma^2_{phy}(p)$ [4]. This fact is the motivation to use Moment Method as the background setting for the derivation of properties of the propagation constant function.

In this context we notice that the characteristic equation for the matrix $Z(p)Y(p)$ in the finite $(m \times m)$ system, which the square of the (approximate) propagation constant must satisfy because of (1), transforms into an algebraic equation of degree m in $\gamma^2(p)$ with coefficients that are polynomials in p. The emergence of the algebraic

equation from this characteristic equation can be seen as follows. Expand the expression,

$$\det[\gamma^2(p)I - Z(p)Y(p)] = g(\gamma^2, p), \qquad (2)$$

where det denotes the determinant, and I the identity matrix. When the monic polynomial in γ^2 thus obtained is set equal to zero, the characteristic equation of $Z(p)Y(p)$ is found. If we then multiply both sides of this equation by the common denominator of the coefficients on the left hand side, we shall obtain a new form of the characteristic equation, in powers of γ^2 again, but with coefficients that are polynomials of p. Call the left hand side of this last equation $G(\gamma^2, p)$, and the coefficient of the ith power of γ^2 in it, $a_{m-i}(p)$. We then have

$$G(\gamma^2, p) = a_0(p)\gamma^{2m} + a_1(p)\gamma^{2m-2} + \cdots + a_m(p) = 0, \qquad (3)$$

which is the sought after algebraic equation.

The correspondence between the characteristic equation of $Z(p)Y(p)$ resulting from the Moment Method and an algebraic equation $G(\gamma^2, p) = 0$, is the crux of the proposed idea of implementing algebraic function theory in our eigenvalue problems. Hence our unknown, the square of the propagation constant, is a solution of an algebraic equation. One then simply recovers the singular points of this solution from poles of $Z(p)Y(p)$ and zeroes of the discriminant of $G(\gamma^2, p) = 0$. Because as discussed in [1], according to the theory of algebraic functions, singular points of the roots of $G(\gamma^2, p) = 0$ are either zeroes of $a_0(p)$ in (3), which must be poles of $Z(p)Y(p)$ due to (2), or zeroes of the discriminant of (3). This is a major gain, which may shed light on any design problem that involves these singular points of the dispersion characteristics.

The correspondence between the characteristic equation of truncated $Z(p)Y(p)$ of a guiding system and an algebraic equation was recognized early in [4]. But no attempt was made to exploit it in a function theory sense. Even though [5] refers to the concept of an algebraic function, the term algebraic singularity is not mentioned in it. Hence what the sources of such singularities are, e.g., that they can be traced to poles of $Z(p)Y(p)$ and zeroes of discriminant of the associated algebraic equation, is not discussed. I.e., some important features of algebraic function theory applied to guiding systems are not observed.

Throughout this work a reference to "the algebraic equation associated with $Z(p)Y(p)$" will be meant to imply a reference to "the mth degree algebraic equation in $\gamma^2(p)$ obtained as above from the characteristic equation of $Z(p)Y(p)$". This equation will be denoted by

$G(\gamma^2, p) = 0$. In this work we shall assume that $G(\gamma^2, p)$ is irreducible. After this point by the term *propagation constant* we shall mean the approximate propagation constant of the truncated system.

The organization of the paper is as follows. In Section 2 properties of critical points, namely end points of a frequency interval in which a mode has a complex propagation constant $\gamma(j\omega)$, are discussed. First it is shown that transition from a complex to a non-complex (i.e., pure real or pure imaginary) propagation constant can not take place at a point $j\omega_B$ where $\gamma^2(p)$ is regular. In fact this property applies to any functional dependence of $\gamma^2(p)$ on p. I.e., $\gamma^2(p)$ may or may not be the root of an algebraic equation. This proposition is analogous to the similar fact touched upon in [1], for the transition between backward and forward wave modes. In fact the statement in [1] is also a result mentioned in [2] and it prohibits backward wave-forward wave transition at a point $j\omega_B$, at which $\gamma^2(p)$ is regular. Again because the underlying proof uses only conditions of existence for a Taylor expansion of $\gamma^2(p)$ about $p = j\omega_B$, this statement is applicable for any functional dependence of $\gamma^2(p)$ on p, as well.

Also in Section 2 the particular form of the eigenvalue of $Z(j\omega)Y(j\omega)$ (square of the propagation constant) is given in the light of Appendix A.

In Section 3 which is on application of algebraic function theory concepts to cutoff frequencies of modes of our class of guiding structures mentioned above, it is found that a cutoff frequency on the $j\omega$ axis, is always an algebraic branch point of the solution $\gamma(p)$ of the algebraic equation associated with $Z(p)Y(p)$. About this singular point the Puiseux series has no negative power terms. It is further shown that $\gamma^2(p)$ is always regular at $j\omega_C$, if ω_C is the cutoff frequency of the mode represented by $\gamma^2(j\omega)$. In this section it is also found that at end points of complex wave mode frequency intervals, both real and imaginary parts of the propagation constant $\gamma(j\omega)$ can not vanish at the same frequency point.

Section 4 includes comments on some facts used in [1] and gives corrections on several misstatements that took place in [1].

Appendices A and B contain a development which constrains the nature of the square of the propagation constant $\gamma^2(p)$, at an algebraic singularity.

The Glossary at the end is intended to serve to making the article complete in itself.

2. COMPLEX WAVE MODES AND SOME CONCEPTS OF ALGEBRAIC FUNCTION THEORY

In the beginning of this section the reader's attention is called for the equivalence of the two statements;

1) $\gamma^2(j\omega)$ is non-real (i.e., $\gamma^2(j\omega)$ is complex or pure imaginary).
2) $\gamma(j\omega)$ is complex.

This equivalence will be used in the development.

Another convention in the paper is that reference to the range $\omega > \omega_B$ and $\omega > \omega_C$ or $\omega < \omega_B$ and $\omega < \omega_C$ is meant to imply a reference to the immediate vicinity of the point ω_B or ω_C on the side of the point indicated by the inequality sign. This reference never implies the whole interval (ω_C, ∞) or $(-\infty, \omega_C)$, for example.

2.1. Transition between Complex and Non-Complex Modes

We start this section by showing that a complex wave mode frequency interval on the frequency axis can not end on a point ω_B, if $\gamma^2(p)$ is regular at $j\omega_B$. Suppose it can, i.e., there exists a neighborhood $|p - j\omega_B| < R$ of $j\omega_B$ such that $\gamma^2(p)$ is regular in it, but $\gamma^2(j\omega)$ is non-real for one of the regions $\omega > \omega_B$, or $\omega < \omega_B$, while it is real for the other. Assume $0 < \rho < R$. Then one has,

$$\gamma^2(p) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n(\gamma^2(j\omega_B))}{dp^n}(p - j\omega_B)^n \tag{4}$$

uniformly in $|p - j\omega_B| \leq \rho$. I.e., $\gamma^2(p)$ is represented uniformly by its Taylor series in $|p - j\omega_B| \leq \rho$. On the $p = j\omega$ axis this series transforms into

$$\gamma^2(j\omega) = \gamma^2(j\omega_B) + \frac{d\gamma^2(j\omega_B)}{d\omega}(\omega - \omega_B) + \frac{1}{2}\frac{d^2\gamma^2(j\omega_B)}{d\omega^2}(\omega - \omega_B)^2 + \cdots. \tag{5}$$

Suppose $\gamma^2(j\omega)$ is real for $\omega > \omega_B$ as per our hypothesis. The series in (5) can be real for all ω such that $\omega_B < \omega \leq \rho$ if and only if $(\frac{d^n(\gamma^2(j\omega))}{d\omega^n})_{\omega=\omega_B}$ are real for $n = 0, 1, 2, \ldots$, since the $(\omega - \omega_B)^n$ factors are linearly independent. But this condition prohibits the mode for which $\gamma^2(j\omega)$ is determined to be real for $\omega > \omega_B$ to have complex values for $\omega < \omega_B$, as can be seen by inspection of the series in (5). Therefore transition from a complex to non-complex (i.e., pure real or pure imaginary) propagation constant can not take place at a point $j\omega_B$ where $\gamma^2(p)$ is regular. Furthermore since at $j\omega_B$, $\gamma^2(j\omega_B)$ is a finite and multiple root (in fact a double root by Appendix A) of the

algebraic equation $G(\gamma^2, p) = 0$, $j\omega_B$ is a root of the discriminant of $G(\gamma^2, p) = 0$.

2.2. Some Properties of Complex Wave Frequency Interval End Points

In Appendix A we have examined the conditions for the termination of complex wave mode intervals on algebraic branch points of the eigenvalues (or of the squares of the propagation constants) which have no negative power terms in their Puiseux expansion about the algebraic branch point. One result of Appendix A is that, if $j\omega_B$ is an algebraic branch point of $\gamma^2(p)$ at which $\gamma^2(p)$ is finite, $\gamma^2(j\omega_B)$ is a defective eigenvalue, implying that the derivative of $\gamma^2(j\omega)$ at ω_B is infinite. Furthermore for at least one side of $j\omega_B$ on the $j\omega$ axis, i.e., either for $\omega > \omega_B$, or for $\omega < \omega_B$ there always exist two real solutions $\gamma^2(j\omega)$. This last statement follows from the bifurcation property introduced in Appendix A and which holds for a non-zero defective eigenvalue. These two real solutions converge to a common real eigenvalue $\gamma^2(j\omega_B)$ of multiplicity $q = 2$, as ω approaches ω_B. If at ω_B, the upper end point of the complex propagation constant frequency interval,

$$\gamma^2(j\omega_B) = -\beta^2(j\omega_B) \neq 0, \tag{6}$$

holds, where $\beta(\omega_B)$ is the imaginary part of $\gamma(j\omega_B)$, the configuration of the dispersion curve in the neighborhood of ω_B must be as depicted in Figure 1.

An example to guiding structures with a branch point $j\omega_B$ at which the discriminant of $G(\gamma^2, p) = 0$ vanishes, and about which the propagation constant behaves as in Figure 1, is the cylindrical waveguide loaded with a coaxial, lossless dielectric rod and investigated in [3, 6].

Let us examine the behavior of the real and imaginary parts of the propagation constants at the terminal frequency point ω_B of the complex wave mode interval when $j\omega_B$ is such a branch point. Hence we have about this algebraic branch point [7]:

$$\begin{aligned} \gamma^2(j\omega) &= \gamma^2(j\omega_B) + jA_1\sqrt{\omega_B - \omega} - A_2(\omega_B - \omega) + \cdots \\ &\quad + (-1)^{n/2}A_n(\omega_B - \omega)^{n/2} + \cdots. \end{aligned} \tag{7}$$

Suppose the coefficients A_n are real for all n, to insure reality of $\gamma^2(j\omega)$ for $\omega > \omega_B$. In this case we have complex wave modes for $\omega < \omega_B$. Separating real and imaginary parts of $\gamma(j\omega)$ as $\text{Re}\{\gamma(j\omega)\} = \alpha(j\omega)$ and $\text{Im}\{\gamma(j\omega)\} = \beta(j\omega)$, and differentiating them with respect to
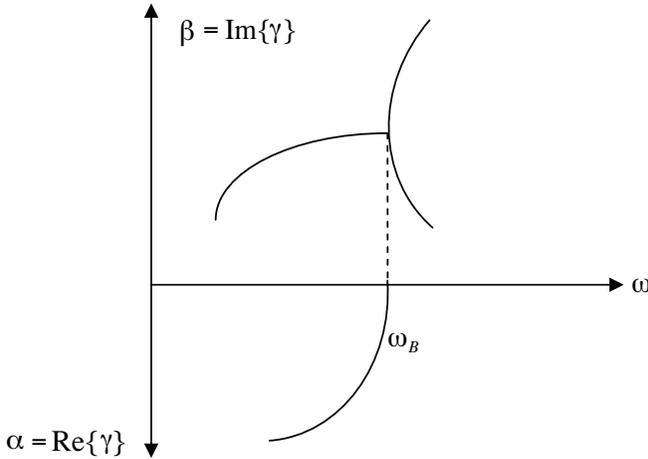
Figure 1. Dispersion characteristics about an algebraic branch point $j\omega_B$ at which $\gamma^2(j\omega_B)$ is finite.

ω we find the following results. 1) $\gamma^2(j\omega_B) < 0$ case. In this case $\left|\lim\limits_{\omega\to\omega_B-}(\frac{d\alpha(j\omega)}{d\omega})\right| = \infty$, whereas $\left|\lim\limits_{\omega\to\omega_B-}(\frac{d\beta(j\omega)}{d\omega})\right| < \infty$. Here the limit $\omega \to \omega_B$ is taken from the left side, i.e., from the side of ω_B where $\gamma(j\omega)$ is complex. This direction is denoted by the minus sign in $\omega \to \omega_B-$. 2) $\gamma^2(j\omega_B) > 0$ case. In this case $\left|\lim\limits_{\omega\to\omega_B-}(\frac{d\alpha(j\omega)}{d\omega})\right| < \infty$, whereas $\left|\lim\limits_{\omega\to\omega_B-}(\frac{d\beta(j\omega)}{d\omega})\right| = \infty$. Again ω approaches ω_B from the side where $\gamma(j\omega)$ is complex. 3) $\gamma^2(j\omega_B) = 0$ case. This is when ω_B is a cutoff frequency at the same time. In this case both derivatives approach infinity. But as concluded in Section 3 this is not a permissible case.

Examples to guiding structures with the $\gamma(j\omega)$ vs. frequency characteristics of the first two cases are provided by the biaxial waveguide. This is a rectangular wave guide filled with a dielectric whose permittivity matrix is given by

$$\hat{\varepsilon} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}.$$

Structures for the two cases can be obtained when the relationship between ε_x, ε_y, ε_z is changed [4].

3. PROPAGATION CONSTANT BEHAVIOR NEAR A CUTOFF FREQUENCY USING ALGEBRAIC FUNCTION THEORY CONCEPTS

If ω_C is a cutoff frequency, we have $\gamma(j\omega_C) = \gamma^2(j\omega_C) = 0$ by definition. Consider additionally the following two possibilities which will go along and which we will examine. 1) $\gamma^2(p)$ is regular at $j\omega_C$. 2) $\gamma^2(p)$ is not regular at $j\omega_C$.

3.1. Case When $\gamma^2(p)$ Is Regular at $p = j\omega_C$ (ω_C: Cutoff Frequency)

Suppose $\gamma^2(p)$ is regular at $j\omega_C$. Then we have the following expansion in a neighborhood of $j\omega_C$.

$$\gamma^2(p) = C_1(p - j\omega_C) + C_2(p - j\omega_C)^2 + \cdots C_n(p - j\omega_C)^n + \cdots \quad (8)$$

This expansion for a propagating mode can have $C_1 = 0$ but not $C_2 = 0$ at the same time. Because, then we have $(\frac{d\beta}{d\omega})_{\omega=\omega_C} = 0$ and this means infinite group velocity, which is not physical.

On the other hand $\gamma(p)$ has a branch point at $j\omega_C$, because from (4) we have

$$\gamma(p) = \mp\sqrt{C_1(p - j\omega_C) + C_2(p - j\omega_C)^2 + \cdots + C_n(p - j\omega_C)^n + \cdots}, \quad (9)$$

and $(\frac{d\gamma}{dp})_{p=j\omega_C} = \infty$ follows. Now since $\gamma(j\omega_C) = 0 < \infty$, hence $\gamma(j\omega_C)$ is finite, $j\omega_C$ is an algebraic branch point for $\gamma(p)$, which sets the discriminant of the algebraic equation associated with the matrix product $Z(p)Y(p)$, and denoted by $G(\gamma^2, p) = 0$, equal to zero. Hence we have the following expansion for $\gamma(p)$ in the neighborhood of $j\omega_C$ [7].

$$\gamma(p) = \overline{C}_1(p - j\omega_C)^{1/q} + \overline{C}_2(p - j\omega_C)^{2/q} + \cdots + \overline{C}_n(p - j\omega_C)^{n/q} + \cdots. \quad (10)$$

Now from (9) we have the two γ, equal to negative of each other. Therefore the expansion (10) must also yield these same two solutions for γ. This is possible only if i) $\overline{C}_k$ for even k, vanish in (10), and ii) $q = 2$.

Hence in the neighborhood of a cutoff frequency ω_C, we must have the following expansion for the propagation constant:

$$\gamma(p) = \sum_{i=1}^{\infty} \overline{C}_{(2i-1)}[p - j\omega_C]^{(2i-1)/2}. \quad (11)$$

Note that even though $\gamma^2(p)$ is not singular at $j\omega_C$, $\gamma(p)$, is singular.

Setting $p = j\omega$ in (8) we have

$$\gamma^2(j\omega) = C_1 j(\omega - \omega_C) + C_2(j)^2(\omega - \omega_C)^2 + \cdots + C_n(j)^n(\omega - \omega_C)^n + \cdots. \tag{12}$$

First consider the $\omega < \omega_C$ side of ω_C, on the $j\omega$ axis. Suppose there exist terms for which powers of $(\omega - \omega_C)$ have non-real coefficients. I.e., the two branches of $\gamma(p)$ yield complex $\gamma(j\omega)$ on the $j\omega$ axis for $\omega < \omega_C$. $\gamma^2(p)$ is regular at $p = j\omega_C$. According to the result of Section 2, transition between complex and non-complex modes can not take place at regular points of $\gamma^2(p)$. Therefore in this case we shall continue to have complex wave modes for $\omega > \omega_C$.

If $\gamma^2(j\omega)$ is real for $\omega < \omega_C$ (or for $\omega > \omega_C$), then by the same reasoning, it is real for $\omega > \omega_C$ (or for $\omega < \omega_C$).

Let us concentrate on the case when $\gamma^2(j\omega)$ is real and negative in a neighborhood of ω_C. One possibility is $C_1 = 0$. Since $\gamma^2(j\omega)$ is additionally real, this corresponds to degenerate cutoff and the dispersion curve in the neighborhood of ω_C, is as in Figure 2a. In this case $(\frac{d\beta(j\omega)}{d\omega})_{\omega=\omega_C}$ is finite as can be seen by differentiating (12). If $C_1 \neq 0$, while $\gamma^2(j\omega)$ is real, we have an 'ordinary' cutoff frequency either for a single forward wave mode or a single backward wave mode as in Figure 2b.
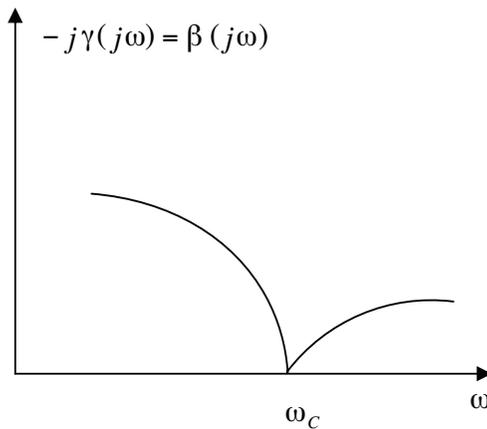
Figure 2a. Dispersion characteristics about a degenerate cutoff frequency.
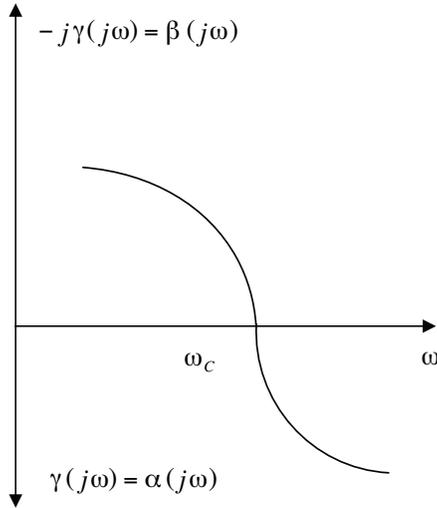
Figure 2b. Dispersion characteristics about an 'ordinary' cutoff frequency.

3.2. Proof of Regularity of $\gamma^2(p)$ at $p = j\omega_C$ (ω_C: Cutoff Frequency)

Suppose $\gamma^2(p)$ is not regular at $j\omega_C$. This may be because i) $a_0(j\omega_C) = 0$. Here $a_0(p)$ is the coefficient of the leading term in the algebraic equation associated with $Z(p)Y(p)$. Please see (3). This implies infinite $\gamma^2(j\omega_C)$ value [7] and is ruled out because $\gamma^2(j\omega_C) = 0$. ii) At $p = j\omega_C$ discriminant of the equation $G(\gamma^2, p) = 0$ vanishes [7].

In investigating case ii) we shall first consider the condition when there exist branches of $\gamma^2(p)$ with both real and non-real $\gamma^2(j\omega)$, at least on one side of ω_C. We shall prove that this is not possible and next go on to prove that the case when all branches yield non-real $\gamma^2(j\omega)$ on both sides of $j\omega_C$, is not possible either. In this way we shall have proved that $\gamma^2(p)$ can not be singular at a cutoff frequency and hence case ii) is not permitted either.

So, first assume there exist branches of $\gamma^2(p)$ with both real and non-real $\gamma^2(j\omega)$, at least on one side of ω_C. This requires transition from a complex wave mode to a non-complex wave mode at $j\omega_C$. This is because, if there exists a branch of $\gamma^2(p)$, real on $j\omega$, for $\omega > \omega_C$ (or $\omega < \omega_C$) and if $j\omega_C$ is a zero of the discriminant of $G(\gamma^2, p) = 0$, and hence $\gamma^2(p)$ is multiple at $j\omega_C$, then from the series development of $\gamma^2(p)$ about $j\omega_C$ we know the same branch yields necessarily non-real $\gamma^2(j\omega)$ on $j\omega$ axis on the other side of ω_C. Proof of this fact is not

given for brevity.

On the other hand by Section 2, transition from a complex wave mode to a non-complex wave mode can occur only at singular points of $\gamma^2(p)$ on the $j\omega$ axis. Because of this, together with the hypothesis that $\gamma^2(p)$ is singular at $j\omega_C$, $\gamma^2(p)$, admits an expansion in the neighborhood of $j\omega_C$, in the form [7]:

$$\gamma^2(j\omega) = A_1(j\omega - j\omega_C)^{1/q} + A_2(j\omega - j\omega_C)^{2/q} + \cdots + A_n(j\omega - j\omega_C)^{n/q} + \cdots. \tag{13}$$

We are dealing with the case in which at least on one side of $j\omega_C$ on the $j\omega$ axis, we have at least one branch of $\gamma^2(p)$ that is real. Choose the range $\omega > \omega_C$ as the side where $\gamma^2(p)$ has this property. We shall prove that when $q \geq 2$, $j\omega_C$ can not be a branch point with this property.

Suppose q is odd. The q roots occur as complex conjugate pairs [4] due to Foster matrix properties of $Z(p)$ and $Y(p)$. Because at the branch point $j\omega_C$, $\gamma^2(j\omega_C) = 0$ is true, in the proximity of ω_C, one of the roots $\gamma^2(j\omega)$ must be real when q is odd. Otherwise requirement of conjugacy of complex roots will not be satisfied.

When q is odd, having chosen $\omega > \omega_C$ as the region for the real branch of $\gamma^2(p)$ on the $j\omega$ axis, we conclude this branch yields non-real $\gamma^2(j\omega)$ for $\omega < \omega_C$, as we stated above without proof for brevity.

The dispersion curve for the transition of this mode from complex to non-complex character at ω_C can either be as in Figure 3a or Figure 3b, depending on whether the real value assumed by the considered $\gamma^2(j\omega)$ for $\omega > \omega_C$ is negative or positive respectively.

Suppose q is even. In this case, according to one of the side results of Appendix A, the number of real branches of $\gamma^2(p)$ on either side of ω_C on the $j\omega$ axis, must be two, if there are any real branches, which is the case for us as per our assumption above for $\omega > \omega_C$. Each of these two real branches must be distinct from the other. Furthermore as noted above again, these non-complex wave modes traverse ω_C to become complex wave modes on the $\omega < \omega_C$ side. Each of these transitions will be as one of Figure 3a and Figure 3b. If for $\omega > \omega_C$ we have $\gamma^2(j\omega) < 0$, Figure 3a, otherwise Figure 3b will be applicable. In Figures 3a, 3b, and 3c, the symbol $>$ has been affixed for quantities on $\omega > \omega_C$ side, whereas the symbol $<$ has been affixed for those on the opposite side.

Inspection of Figures 3a and 3b reveals that in fact for both q odd and q even cases, at $j\omega_C$, the complex wave mode propagation constant real and imaginary parts which exist for $\omega < \omega_C$, have bifurcated at ω_C in the transition from $\omega < \omega_C$ side to the $\omega > \omega_C$ side. If bifurcation at a defective eigenvalue where eigenvalue is also zero, is not allowed, then the configurations of Figures 3a and 3b are not permitted. This will exclude both q odd and q even conditions of the case ii when
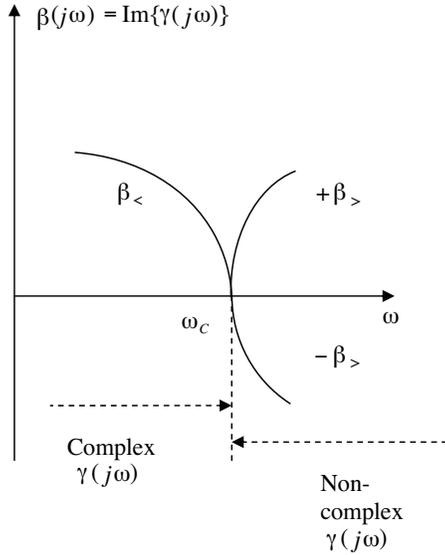
Figure 3a. Phase coefficient β vs. frequency about ω_C, with real $\gamma^2(j\omega)$ on at least one side.

Figure 3b. Attenuation coefficient α vs. frequency about ω_C, with real $\gamma^2(j\omega)$ on at least one side.

Figure 3c. Phase coefficient $-\beta$ vs. frequency about ω_C, with real $\gamma^2(j\omega)$ on at least one side. Here phase coefficient is negative of β in Figure 3a.

investigated for the requirement of at least one real $\gamma^2(j\omega)$ on at least one side of ω_C.

Suppose bifurcation at a defective eigenvalue where eigenvalue is also zero is allowed. Then the dispersion curves of a complex wave mode (i.e., real and imaginary part characteristics of its propagation constant) on $\omega < \omega_C$ side, bifurcate. In this case the overall non-complex roots on $\omega > \omega_C$ side will be as pairs of identical $\gamma(j\omega)$. This can be seen by noting that for the imaginary part of the complex conjugate of the complex propagation constant in Figure 3a depicted on $\omega < \omega_C$ side, one has the bifurcation diagram in Figure 3c which is identical with Figure 3a in the $\omega > \omega_C$ part. This implies identical dispersion curves for pairs of pure imaginary $\gamma(j\omega)$ in the $\omega > \omega_C$ range. This is not admissible because $G(\gamma^2, p)$ is irreducible and hence the discriminant of $G(\gamma^2, p) = 0$ can not vanish identically, which it would have to, when propagation constant square of a mode is multiple over an interval on the frequency axis. Independently from this argument on bifurcation of the imaginary part curves, we can state in the same way that at ω_C, bifurcation of the real part curves of $\gamma(j\omega)$ and its conjugate ($\gamma(j\omega)$ is complex on $\omega < \omega_C$ side), yields a similar pair of identical pure real $\gamma(j\omega)$ on the $\omega > \omega_C$ side, thus proving

unfeasibility in like manner.

Therefore regardless of whether bifurcation takes place or not when $\gamma(j\omega_C) = 0$, for both q odd and q even all cutoff frequencies ω_C, on at least one side of which there exists one branch of $\gamma^2(p)$ which is real, while $\gamma^2(p)$ has an algebraic branch point at $j\omega_C$ of order greater than or equal to 1, are excluded.

The only remaining case to be discussed is when there are only complex wave modes on both sides of the singular point $j\omega_C$, while the propagation constant vanishes at ω_C. We shall prove below that this is not permissible either.

For this goal we shall first rule out odd values for the multiplicity q of $\gamma^2(p)$ at $j\omega_C$ by the same complex conjugate pairs requirement argument as above. Because an odd q value entails one non-complex wave mode on both sides of ω_C, and this is a violation of our assumption.

Consider then the q even condition. In this case we have the following series expansion on $j\omega$ axis for $\omega > \omega_C$ [7]:

$$\gamma_i^2(j\omega) = \sum_{n=1}^{\infty} C_n \exp[j(\pi/2 + 2\pi s_i)n/q](\omega - \omega_C)^{n/q},$$

$$s_i = 0, \mp 1, \mp 2, \ldots, \mp(q-1). \qquad (14)$$

Here s_i stands for the index determining the ith branch. If we denote the argument of C_n by φ_n, and the overall argument of each term in the series by ψ_{in}, then

$$\psi_{in} = \varphi_n + \left(\mp\frac{\pi}{2} + 2\pi s_i\right)n/q. \qquad (15)$$

The $(+)$ in front of $\frac{\pi}{2}$ refers to $\omega > \omega_C$ side of ω_C whereas $(-)$ refers to the opposite side.

Whether there exists an ith branch $\gamma_i^2(p)$, which is real on $j\omega$ axis for $\omega > \omega_C$ or $\omega < \omega_C$, is a problem equivalent to existence of an integer s with $0 \leq s \leq q - 1$ such that when it is substituted for s_i in (15), one gets ψ_{in} which are zero or multiples of π for $n = 1, 2 \ldots$.

Complex conjugacy requirement between $i = 1$ and $i = 2$, for $\omega > \omega_C$ on the $j\omega$ axis implies

$$\varphi_n = -\frac{1}{2}[2(s_1 + s_2) + 1](\pi n/q). \qquad (16)$$

To insure (16) to yield same φ_n for all complex conjugate pair choices, sum of indices $s_u + s_v$ for conjugate pairs $\gamma_u^2(j\omega)$ and $\gamma_v^2(j\omega)$ must remain constant for all such u and v. This implies $s_u + s_v = q - 1$,

because the indices of conjugate pairs must then be as $(0, q-1), (1, q-2), (3, q-2), \ldots$. Since $s_u + s_v$ is odd, it can be shown that there exists s equal to or different from s_u and s_v, such that $-1 + [2s - (s_u + s_v)] = 0$ holds and for $\omega < \omega_C$, (15) vanishes for all n when this s is substituted for s_i in it.

Together with the same type of arguments for s_u and s_v which this time stand for a complex conjugate pair for $\omega < \omega_C$ on the $j\omega$ axis, we are led to derive that in general there always exists at least one real $\gamma_i^2(j\omega)$ for either $\omega > \omega_C$ or $\omega < \omega_C$, if $\gamma_i^2(j\omega_C) = 0$.

Hence we conclude: if ω_C is a cutoff frequency, there can not exist only complex wave modes on both sides of the singular point $j\omega_C$, while the propagation constant vanishes at ω_C.

As the result we exclude all cutoff frequencies ω_C such that $\gamma^2(p)$ are singular at $j\omega_C$. This follows, because a) we first showed in the above paragraphs, that we must exclude cutoff frequencies, on at least one side of which there always exists a branch of $\gamma^2(p)$ which is real there, while $\gamma^2(p)$ is singular at $j\omega_C$. And now b) additionally we have shown impossibility of a cutoff frequency which is a singularity on the $j\omega$ axis on both sides of which there exist only complex wave modes.

3.3. A Numerical Example

By means of a numerical example we shall illustrate some of the ideas developed in this section and highlight the general algebraic function theory approach to the understanding of singularities of dispersion characteristics of the type of guiding structures considered.

We chose a cylindrical waveguide loaded with a lossless coaxial dielectric rod as the guiding structure (inset of Figure 4) for the numeric computations. Exact solutions for this problem are known [6]. Parameter values are $\varepsilon_2 = 15\varepsilon_0$, $\varepsilon_1 = \varepsilon_0$, $r_1 = 0.25''$, $r_2 = 0.67r_1$. A personal computer was used in the computations. As the frequency interval for verification of the method we picked up the neighborhood of the cutoff frequency of the backward wave mode in the dispersion characteristics. These characteristics are given in [3, 6] for the two lowest order modes. We defined a normalized frequency as $V = \omega\sqrt{\varepsilon_0\mu_0}r_1$. The exact value of the normalized cutoff frequency found from the exact transcendental characteristic equation was $V_C = 1.020323$. Two different orders of truncation were run in application of the Moment Method. Using 150 TE and 150 TM modes of the empty guide, the cutoff frequency found was $V_C = 1.01806509$. For 500 TE and 500 TM modes, the cutoff frequency found was $V_C = 1.0196330$. The relative error dropped to 0.0676% from 0.221%. The convergence of numerical results with the increase of order of truncation, indicates
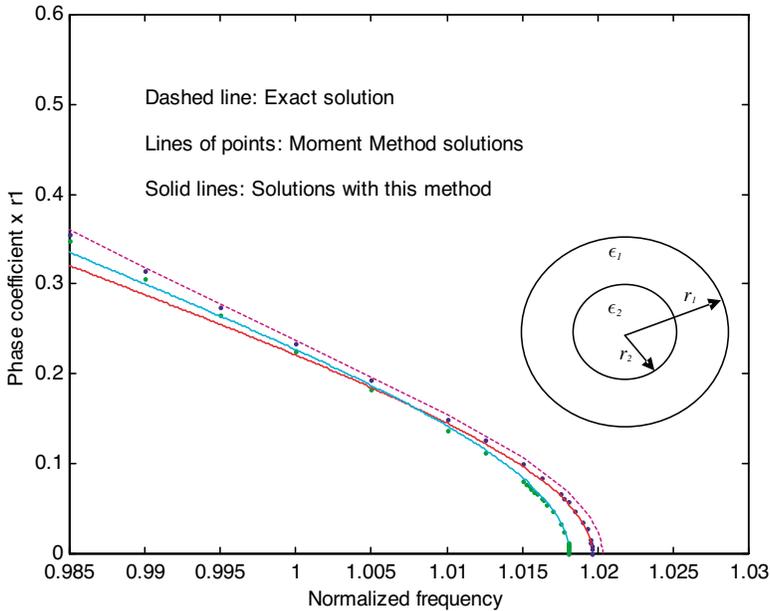
Figure 4. Dispersion characteristics about the cutoff frequency of the backward mode in the numerical example. Inset: The guiding structure of the example.

that with a more powerful computing capability errors can be reduced further.

The first two terms of the expansion (11) were considered. Formulas for the coefficients $\overline{C}_1$ and $\overline{C}_3$ were determined using their definitions in terms of derivatives of $\gamma^2(p)$ with respect to p at the point $p = j\omega_C$, which can in turn be found from consideration of $\gamma^2(p)$ as an implicit function in the algebraic equation $G(\gamma^2, p) = 0$, and from differentiation of this equation. Values for $\overline{C}_1$ and $\overline{C}_3$ were numerically found for both orders of truncation in the Moment Method (see Table 1). Lower and upper lines of points in Figure 4 are Moment Method results with 150 TE and TM and 500 TE and TM mode

Table 1. Computed $\overline{C}_1$ and $\overline{C}_3$ coefficient values.

	150 TE, 150 TM modes	500 TE, 500 TM modes
$\overline{C}_1$	50.9887565 $(1 - j)$	47.022203 $(1 - j)$
$\overline{C}_3$	350.9576625 $(1 + j)$	350.745063 $(1 + j)$

truncations respectively. Perfect agreement of these with results of the method in this paper, especially in the neighborhood of the cutoff frequency, indicates applicability of the approach presented. The fact that this agreement is achieved taking only two terms of the series (11) and generating each of the solid line curves from computation of only two coefficients $\overline{C}_1$ and $\overline{C}_3$, further strengthens the point made. Indeed the curves for the Moment Method involve solution of an eigenvalue problem at each point, which requires many more computations.

4. SOME REMARKS ON REFERENCE [1]

In this section we substantiate some of the propositions made in Section 2 of [1] and give corrections to some of the misstatements in the same paper. Several of these corrections follow from the new results of the development of the present paper.

In Section 2 of [1], it is stated without proof that singularities of $\gamma^2(p)$ imply infinite derivatives, at $p = j\omega_B$, if $j\omega_B$ is a singular point of $\gamma^2(p)$. For the case of an algebraic branch point where the propagation constant is finite, this can be seen if one considers Appendix A, where the order of an algebraic branch point $j\omega_B$, to be encountered in guiding structures considered in this paper, is shown to be always 1, and where additionally, it is shown that the multiple eigenvalue $\gamma^2(j\omega_B)$ must be defective. These conditions can be expressed by expanding $\gamma^2(p)$ as [7],

$$\gamma^2(p) = \gamma^2(j\omega_B) + \sum_{n=1}^{\infty} C_n[p - j\omega_B]^{n/2}. \tag{17}$$

where C_1 is non-zero, to insure that $\gamma^2(p)$ is not differentiable at $p = j\omega_B$. Then,

$$\left| \frac{d\gamma^2(j\omega_B)}{dp} \right| = \infty. \tag{18}$$

From this, we have $(\frac{d\gamma}{dp})_{p=j\omega_B} = \infty$ because $\frac{d\gamma^2}{dp} = 2\gamma\frac{d\gamma}{dp}$ and $\gamma(j\omega_B)$ is finite.

If the singularity $j\omega_B$ is due to a zero of $a_0(p)$, which is the coefficient of the leading term in the algebraic equation $G(\gamma^2, p) = 0$ associated with $Z(p)Y(p)$, then according to Appendices A and B, $\gamma^2(p)$ admits an expansion of the form

$$\gamma^2(p) = \gamma^2(j\omega_B) + \sum_{n=-n_1}^{\infty} C_n[p - j\omega_B]^n. \tag{19}$$

Here $n_1 > 0$. If one considers the dominant term in this series, for $\frac{d\gamma}{dp}$ one can obtain

$$\frac{d\gamma}{dp} \approx \sqrt{C_{-n_1}}(p - j\omega_B)^{-(\frac{n_1}{2}+1)} \cdot (-n_1/2). \tag{20}$$

This derivative approaches infinity as $p \to j\omega_B$.

In summary, all possible types of algebraic singularities of $\gamma^2(p)$ that can exist for the class of structures considered, imply infinite $\frac{d\gamma}{dp}$ as the singularity.

Cases i) and ii) in Section 2 of [1] are ruled out, because as shown in Section 3 of the present paper $j\omega_B$ can not be a singular point of $\gamma^2(p)$, since $\gamma^2(j\omega_B) = 0$.

In case iii), the right hand limit for the evaluation of the derivative $\left|(\frac{d\beta}{d\omega})_{\omega=\omega_B}\right|$ has been assumed to possess two different finite values. This configuration depicted in Figure 3 of [1] is not permitted. This follows if one observes the two results of Appendix A of this paper, namely 1) order of an algebraic branch point where the propagation constant is finite, can only be 1 and 2) such an algebraic branch point must be defective (i.e., $\left|(\frac{d\beta}{d\omega})_{\omega=\omega_B}\right| = \infty$, if $j\omega_B$ is the branch point).

In case iv) of Section 2 in [1], we noted that when ω_B is the end point of a complex wave mode frequency interval, and $\beta(\omega_B) \neq 0$, the only possible dispersion curves for $\omega > \omega_B$ are those depicted in Figure 4 in [1]. It was mentioned in [1] that the necessity of this nature of the dispersion characteristics was to be supplied with a proof in a separate publication.

Indeed by Appendix A, we claim that $q = 2$ is the only possible value for q, the multiplicity of an eigenvalue at the branch point. Furthermore if there is a region of complex wave modes for $\omega < \omega_B$ ending at ω_B, then the frequency region immediately above it can not support complex wave modes, and the two modes must necessarily have pure imaginary propagation constants in that region. Again by Appendix A this multiple eigenvalue must be defective. If one also makes use of the bifurcation property [2] the dispersion characteristics in Figure 4 of [1], can be obtained.

Four corrections are in the order for some misstatements in [1]. On p. 1401 in [1], $a_0(p)$ is defined to be the polynomial equal to the common denominator of the entries of the matrix $Z(p)Y(p)$. The definition for $a_0(p)$ must be corrected as 'the lowest common denominator of the expression $g(\gamma^2, p) = \det[\gamma^2(p)I - Z(p)Y(p)]$' whose members are defined in [1].

On p. 1404 the $(-1)^{1/q}$ real case is addressed as part of the proof of the existence of a complex wave mode frequency interval below a

backward wave mode interval. This problem was originally taken up in [8]. By Appendix A of the present paper the q odd case is precluded for the multiplicity of an eigenvalue at a branch point. Since $(-1)^{1/q}$ can not be real for q even, the part of the proof appearing in [1] becomes unnecessary as a whole. However one should note that $(-1)^{i/q}$ factor in [8] is always complex when q is even. Only if i attains only even values could this fact be untrue. But then, one has a cancellation by 2 in i/q between all indices i and q. This in turn implies reducible $G(\gamma^2, p)$ which is incompatible with our hypothesis and is disregarded.

On p. 1407 in [1], a necessary and sufficient condition for the existence of a backward wave mode for $\omega \geq \omega_B$ is given when ω_B is the upper end point of a complex frequency region. By Appendix A, $\gamma^2(j\omega_B)$ is a defective double eigenvalue. Using the notation in Section 1, if $i(j\omega_B)$ and $v(j\omega_B)$ are the transmission line currents and voltages corresponding to eigenvalue $\gamma^2(j\omega_B)$, $v^+(j\omega_B)i(j\omega_B) = 0$ follows. To see this, write this product of eigenvectors of $Z(j\omega_B)Y(j\omega_B)$ and $Y(j\omega_B)Z(j\omega_B)$, in terms of generalized eigenvector of $Z(j\omega_B)Y(j\omega_B)$ and eigenvector of $Y(j\omega_B)Z(j\omega_B)$. Then the necessary and sufficient conditions for the existence of a backward wave mode for $\omega > \omega_B$, when ω_B is the upper end point of the complex frequency region is now modified as follows.

$$v^+i = \begin{cases} 0 & \text{for } \omega \leq \omega_B \\ x \neq 0 \text{ and } x \text{ pure real} & \text{for } \omega > \omega_B. \end{cases}$$

The last correction is in the second paragraph from the bottom on p. 1413. Here the algebraic branch point at $j\omega_B$ due to a pole of $Z(p)Y(p)$ there, belongs to $\gamma^2(p)$ and not $Z(p)Y(p)$.

5. CONCLUSIONS

Complex wave modes and modal behavior in the neighborhood of cutoff frequencies have been assessed using algebraic function theory concepts with focus on metallic waveguides containing a class of lossless inhomogeneous and/or anisotropic media which in practice correspond to a broad range of applications.

It has been shown that the problem of studying propagation constant characteristics of complex wave modes in the vicinity of end points of frequency intervals supporting such modes, is one of considering zeroes of the discriminant of the algebraic equation associated with the matrix $Z(p)Y(p)$.

The particular form of the Puiseux series expansion of the propagation constant has been established in the neighborhood of

cutoff frequencies which are this time shown to be algebraic branch points of the square root of the solution of the same algebraic equation. In the assessment of behavior about cutoff, it is found that at a cutoff frequency point on the $j\omega$ axis, $\gamma^2(p)$ must necessarily be regular whereas $\gamma(p)$ is not.

It has also been proved that the order of an algebraic branch point at which the square of the propagation constant, i.e., the solution of $G(\gamma^2, p) = 0$ is finite is necessarily 1. It has further been proved that at such a point $j\omega_B$, the eigenvalue $\gamma^2(j\omega_B)$ of $Z(j\omega_B)Y(j\omega_B)$ must be defective. It has also been found that at end points of complex wave mode intervals real and imaginary parts of the propagation constant can not vanish simultaneously.

Additionally it has also been demonstrated that $\gamma^2(p)$ can not have a pole branch singularity and that a finite order pole singularity corresponds to a cutoff frequency of the system represented by shunt admittance and series impedance matrices per unit length, $Z^{-1}(p)$ and $Y^{-1}(p)$.

A numerical example has been given which illustrates that the method can be applied to efficiently approximate the propagation constant in the neighborhood of the cutoff frequency of a backward wave mode.

APPENDIX A. CONSTRAINTS ON PROPERTIES OF AN ALGEBRAIC BRANCH POINT WHERE PROPAGATION CONSTANT IS FINITE

A.1. Proof of Defective Property of a Finite Multiple Eigenvalue

First we prove two facts.

Fact 1. If eigenvalue $\gamma_1^2(j\omega_B)$ of $Z(j\omega_B)Y(j\omega_B)$ (or $Y(j\omega_B)Z(j\omega_B)$ since using Foster matrix properties of $Z(p)$ and $Y(p)$, these matrices can be shown to possess the same eigenvalues) is real and non-defective, no branch of $\gamma_1(p)$ can be complex on both sides of $j\omega_B$ on the $j\omega$ axis.

Proof. Suppose $\gamma_1^2(j\omega_B)$ is real and non-defective and there exists a branch of $\gamma_1(p)$ which is complex on both sides of $j\omega_B$. At those frequencies where $\gamma_1(j\omega)$ is complex, $v_1^+(j\omega)i_1(j\omega)$ vanishes, if additionally the eigenvalues are distinct at that frequency [1]. Definitions of the quantities $v_1(j\omega)$ and $i_1(j\omega)$ are the same as of $v(p)$ and $i(p)$ of Section 1. Recall that they are eigenvectors of $Z(j\omega)Y(j\omega)$ and $Y(j\omega)Z(j\omega)$ respectively, both corresponding to the common eigenvalue $\gamma_1^2(j\omega)$. Because $\gamma_1^2(j\omega_B)$ is non-defective $v_1(j\omega)$ and $i_1(j\omega)$ can be chosen with elements continuous at $j\omega_B$ [9].

Therefore we claim $v_1^+(j\omega)i_1(j\omega)$ is continuous at $j\omega_B$. Since $\gamma_1(j\omega)$ is complex on both sides of $j\omega_B$, the form $v_1^+(j\omega)i_1(j\omega)$ vanishes at these frequencies. But because of continuity at $j\omega_B$, the same form at $j\omega_B$, i.e., $v_1^+(j\omega_B)i_1(j\omega_B)$ must vanish. However it is possible to show that $v_1^+(j\omega_B)i_1(j\omega_B)$ is non-zero if $Z(j\omega_B)Y(j\omega_B)$ possesses a complete set of eigenvectors [4]. Since our eigenvalue is assumed to be non-defective the set of eigenvectors is complete. We have a contradiction. QED.

Fact 2. A multiple eigenvalue $\gamma_1^2(j\omega_B)$ can not be non-defective, if $\gamma_1(j\omega)$ is complex for $\omega < \omega_B$ (or for $\omega > \omega_B$) and non-complex for $\omega > \omega_B$ (or for $\omega < \omega_B$).

Proof. Suppose $\gamma_1^2(j\omega_B)$ is a non-defective multiple eigenvalue and additionally $\gamma_1(j\omega)$ is complex on one side of $j\omega_B$. Eigenvectors $v_1(j\omega)$ and $i_1(j\omega)$ can be chosen with elements continuous at $j\omega_B$ [9]. Hence $v_1^+(j\omega)i_1(j\omega)$ can be chosen to be continuous at $j\omega_B$. Coupled with the vanishing of this form for $\omega < \omega_B$ (or for $\omega > \omega_B$) [1], its continuity at $j\omega_B$ can be used to show that $v_1^+(j\omega_B)i_1(j\omega_B)$ must vanish. This is the same type of contradiction for a non-defective eigenvalue as in the proof of Fact 1. Therefore our hypothesis is wrong. QED.

We can now claim by these two facts proved that, a complex wave mode frequency interval can not end at a singular point at which the square of the propagation constant is a non-defective eigenvalue. The square of the propagation constant at a singularity where the discriminant of the equation $G(\gamma^2, p) = 0$ vanishes, and where a complex wave mode interval ends, is a defective eigenvalue.

A.2. Bifurcation Property and Some of Its Consequences

In our further investigation of the frequency points which mark the onset of complex wave mode intervals, we refer to the fact that unless $\gamma = 0$, singularities in the derivative of a dispersion characteristics bring about the bifurcation of a curve [2].

On the other hand a singularity can occur in the derivative of a dispersion characteristics where $\gamma_1^2(j\omega)$ is non-zero only if at the frequency point on the $j\omega$ axis, $\gamma_1^2(j\omega)$ is a defective eigenvalue. Let us assume below that the point $j\omega_B$ is such a point. Then we shall show that the multiplicity q, of $\gamma_1^2(j\omega_B)$ must be 2. This will be done by first proving that q can not be odd. Then q even case will be considered leading to a further restriction that q can only be equal to 2.

Before going on with this proof, in order to obtain two auxiliary facts that we need in the development, let us examine the argument of each term in the series for $\gamma_1^2(j\omega)$ in (A1) below, in view of determining the number of branches of $\gamma_1^2(p)$ that are real on the $j\omega$ axis. Reality of a branch on the $j\omega$ axis is equivalent to each term in this series having

arguments equal to zero or integer multiples of π. If [7]

$$\gamma_1^2(j\omega) = \gamma_1^2(j\omega_B) + \sum_{n=1}^{\infty} C_n[j\omega - j\omega_B]^{n/q}, \tag{A1}$$

for $\omega > \omega_B$, the argument of each term with $n \geq 1$ can be written as

$$\varphi_n + \left(\frac{\pi}{2} + 2\pi s\right)\frac{n}{q} \quad \text{with} \quad s = 0, \mp 1, \mp 2, \ldots, \mp(q-1), \tag{A2a}$$

where s is the index representing the branch of $\gamma_1^2(p)$, and φ_n stands for the argument of C_n. Alternatively for $\omega < \omega_B$ the same argument can be written as

$$\varphi_n + \left(-\frac{\pi}{2} + 2\pi s\right)\frac{n}{q} \quad \text{with} \quad s = 0, \mp 1, \mp 2, \ldots, \mp(q-1). \tag{A2b}$$

Using (A2), one can find these two results. i) If q is odd, there can at most be one branch of $\gamma_1^2(p)$ real on either side of the point $j\omega_B$ on the $j\omega$ axis. ii) If q is even, on either side of $j\omega_B$, either there are no real branches of $\gamma_1^2(p)$, or if there are real branches, their number must be two.

Now we can proceed with our proof. Consider that part of the curve, say on the left side of $j\omega_B$, on the dispersion characteristics for $\mathrm{Im}\{\gamma_1(j\omega)\}$ where we assume $\gamma_1(j\omega)$ is complex. Since $\gamma_1^2(j\omega_B)$ is a multiple non-zero eigenvalue with non-zero $\beta(j\omega_B)$ but zero $\alpha(j\omega_B)$ and by Section 2 and defective property of the eigenvalue, $(\frac{d\beta}{d\omega})_{\omega=\omega_B} = \infty$ holds, this part of the curve must bifurcate at $j\omega_B$. The two curves that appear on the right of $j\omega_B$ after bifurcation, must correspond to two real branches of $\gamma_1^2(p)$ on the $j\omega$ axis. Otherwise, because it can be shown that complex conjugate of $\gamma_1^2(j\omega)$ is also an eigenvalue of $Z(j\omega)Y(j\omega)$ due to Foster matrix properties of $Z(p)$ and $Y(p)$, the total number of eigenvalues will not have been conserved.

If q is odd, since the curve for $\mathrm{Im}\{\gamma_1(j\omega)\}$ bifurcates at $j\omega_B$, by the argument of the above paragraph there will exist at least two real branches of $\gamma_1^2(p)$ on the $j\omega$ axis for $\omega > \omega_B$. The bifurcation argument due to a degeneracy of the kind $(\frac{d\alpha}{d\omega})_{\omega=\omega_B} = \infty$, will also yield the same result. These two results do not conform with result i) above and lead to preclusion of odd q.

We shall consider the q even case in two steps. We shall exhaust possible even values first assuming $q \geq 6$. Bifurcation property of $\mathrm{Im}\{\gamma_1(j\omega)\}$ and $\mathrm{Re}\{\gamma_1(j\omega)\}$ curves in this case, yields real branches of $\gamma_1^2(p)$ on the $j\omega$ axis, always outnumbering the limit of two, set by result ii). Therefore q can be at most 4. In fact we shall see below that if $q > 1$, it must be equal to 2.

As the second step take q equal to 4 and start by assuming that two non-real branches of $\gamma_1^2(p)$ exist on $j\omega$ for $\omega < \omega_B$ only (or for $\omega > \omega_B$ only), the $\beta(j\omega) = \text{Im}\{\gamma_1(j\omega)\}$ or $\alpha(j\omega) = \text{Re}\{\gamma_1(j\omega)\}$ curves for which must split into a total of four real $\gamma_1^2(j\omega)$ branches in the region $\omega > \omega_B$ (or $\omega < \omega_B$). This condition is inconsistent with result ii) and when $q = 4$, existence of two non-real branches on only one side of ω_B is ruled out.

Continue with the $q = 4$ case considering next the existence of only two branches of $\gamma_1^2(p)$ non-real on one side of $j\omega_B$ on the $j\omega$ axis, while only two branches non-real on the other side, exist in distinction from existence of two non-real branches on only one side, and four real branches on the other side taken up above. But utilizing (A2), it can be seen that when $q = 4$, existence of only two branches of $\gamma_1^2(p)$ which yield complex $\gamma_1(j\omega)$ on one side of $j\omega_B$, while a set of only two branches consisting of different or same two branches, or still as another possibility, consisting of one different and one same branch, yields also complex $\gamma_1(j\omega)$ on the other side, is possible only if the index n of summation in (A1) attains even values.

By inspection of (A1) one can find that if n takes on only even values, the exponents of the terms in the summation then become $\frac{n}{q} = \frac{n}{4} = \frac{n'}{2}$, where n' may take the values $n' = 1, 2, 3, \ldots$. In other words $q = 4$ case reduces to the $q = 2$ case if there exist branches of $\gamma_1^2(p)$ that give complex $\gamma_1(j\omega)$ on both sides of $j\omega_B$ on the $j\omega$ axis. But this corresponds to existence of multiple roots $\gamma^2(p)$, which cause the discriminant of $G(\gamma^2, p) = 0$ to vanish identically. This in turn requires $G(\gamma^2, p)$ to be reducible which is incompatible with our hypothesis.

Therefore when $q = 4$, a) the case of existence of four real $\gamma^2(j\omega)$ on one side of $j\omega_B$ violates result ii) above and must be disregarded. b) Existence of only two real and two non-real branches of $\gamma^2(j\omega)$ on both sides of $j\omega_B$ is ruled out too by the argument of irreducibility of $G(\gamma^2, p)$. c) Existence of only non-real $\gamma^2(j\omega)$ on at least one of the two sides of $j\omega_B$ is prohibited too. Because then a non-zero defective eigenvalue must bifurcate into real branches, and this means more than two real $\gamma^2(j\omega)$ on at least one side of $j\omega_B$, thus violating result ii) above.

Similar arguments hold when $\gamma^2(j\omega_B)$ is a defective multiple eigenvalue with $\beta(j\omega_B) = 0$, but $\alpha(j\omega_B) \neq 0$ and $(\frac{d\alpha}{d\omega})_{\omega=\omega_B} = \infty$ for when q is even as was the case for q odd above. We omit them for brevity.

Since none of the members of the above set of alternatives a) through c) which exhausts all possibilities is feasible, we infer $q = 4$ case is not permissible either.

We do not need to consider an algebraic branch point $j\omega_B$ at which both $\alpha(j\omega_B)$ and $\beta(j\omega_B)$ are nonzero. Because then we can not have pure real or pure imaginary $\gamma(j\omega)$ for either side of $j\omega_B$, given the continuity of the root of the algebraic equation at an algebraic branch point where it is finite. But we must have real $\gamma^2(j\omega)$, because due to the bifurcation argument, complex propagation constant real and imaginary parts which are nonzero at ω_B and have infinite derivatives on one side of ω_B, must split into real $\gamma^2(j\omega)$ at the algebraic branch point in order to accommodate the degeneracies due to the infinite derivatives. If these splits were into non-real $\gamma^2(j\omega)$, total number of eigenvalues would not be conserved since complex conjugates of $\gamma^2(j\omega)$ are also roots of the algebraic equation because of Foster matrix properties of $Z(p)$ and $Y(p)$.

In summary $q = 2$ necessarily if $\gamma_1^2(j\omega_B)$ is a finite non-zero root of the algebraic equation associated with $Z(p)Y(p)$ and with multiplicity q. Furthermore $\gamma_1^2(j\omega_B)$ is necessarily a defective eigenvalue.

APPENDIX B. PRECLUSION OF AN ALGEBRAIC BRANCH POINT WHERE THE PROPAGATION CONSTANT IS INFINITE AND SOME PROPERTIES OF THE PROPAGATION CONSTANT WITH A POLE

Consider a pole branch point $p = j\omega_B$ of eigenvalue $\gamma_1^2(p)$. By this we mean $\gamma_1^2(p)$ has a finite number of negative power terms of the $(p - j\omega_B)^{-n/q}$ type in its Puiseux expansion, where $(q-1)$ is the order of the branch point. Then at this point,

$$\gamma_1'^{\,2}(p)\Big|_{p=j\omega_B} = \frac{1}{\gamma_1^2(p)\Big|_{p=j\omega_B}} = 0 \qquad (B1)$$

is finite. Furthermore $\gamma_1'^{\,2}(p) = \frac{1}{\gamma_1^2(p)}$ is the root of a new algebraic equation $G'(\gamma'^2, p) = 0$ which is obtained from the characteristic equation of the matrix $Y^{-1}(p)Z^{-1}(p)$ in the same way $G(\gamma^2, p) = 0$ is obtained from the characteristic equation of the original matrix $Z(p)Y(p)$. Here $Y^{-1}(p)Z^{-1}(p)$ is a product of two other Foster matrices since the inverse of a Foster matrix is also a Foster matrix.

Because this matrix is the inverse of $Z(p)Y(p)$, its eigenvalues are reciprocals of those of $Z(p)Y(p)$. Therefore one would expect the pole branch point of the original eigenvalue problem to be an algebraic branch point in the new eigenvalue problem at which the new eigenvalue is finite (in fact zero). Hence the Puiseux expansion of

$\gamma_1'^2(p)$ in the neighborhood of $j\omega_B$ should contain no negative power terms.

On the other hand we have shown in Section 3 that a mode can not have a cutoff frequency ω_C, when $\gamma^2(p)$ is not regular at $j\omega_C$. Because now ω_B is a cutoff frequency for the system obtained by inverting $Z(p)Y(p)$ with $\gamma_1'^2(j\omega_B) = 0$ holding, $\gamma_1'^2(p)$ must be regular at $j\omega_B$. But this does not conform with the fact that $\gamma_1^2(p)$, hence $\gamma_1'^2(p)$ is multi-valued about $j\omega_B$. Therefore we must rule out the case of an algebraic branch point at which the propagation constant is infinite, for all guiding systems of the type considered throughout this paper.

Assume the same sequence of arguments are applied to a pole $p = j\omega_B$ of eigenvalue $\gamma_1^2(p)$. The reciprocal of $\gamma_1^2(j\omega_B)$, denoted by $\gamma_1'^2(j\omega_B)$ will again be zero. But this time $\gamma_1'^2(p)$ will be regular at $p = j\omega_B$. This can be seen by writing the Laurent series for $\gamma_1^2(p)$ which has a finite number of negative power terms, and taking the reciprocal of the expression for this series.

On the other hand by Section 3, ω_B can be a cutoff frequency for $\gamma_1'^2(j\omega)$, so long as $\gamma_1'^2(p)$ is regular at $p = j\omega_B$. Therefore a pole is a permissible singularity for $\gamma^2(p)$ which is an eigenvalue of $Z(p)Y(p)$, and it corresponds to a cutoff frequency for the corresponding mode with eigenvalue $\gamma'^2(p)$ obtained for the guiding structure represented by the shunt admittance and series impedance matrices per unit length, $Z^{-1}(p)$ and $Y^{-1}(p)$.

APPENDIX C. GLOSSARY

This section was compiled by referring mainly to [7, 10].

Algebraic equation; An equation of the form $G(w, p) = 0$, where G denotes an entire rational function of p and w. If we imagine G to be arranged in ascending powers of w, it can be written in the form

$$G(w, p) = g_0(p) + g_1(p)w + g_2(p)w^2 + \cdots + g_m(p)w^m = 0,$$

where the coefficients $g_v(p)$ represent polynomials in p alone.

Algebraic singularity; may be a *pole*, or an *algebraic branch point* as per the below definitions.

1) *Pole*; is a zero of $g_m(p)$, the coefficient of the leading term in the algebraic equation, in the neighborhood of which the Laurent expansion has a finite number of negative powers. It can be traced to be a pole of $Z(p)Y(p)$ in our formalism (see Section 1 and [1]).

2) *Algebraic branch point* which admits a Puiseux expansion with no negative power terms. This point is a zero of the discriminant

of $G(w, p) = 0$. The root of the algebraic quation is finite and continuous at this type of a singularity.

3) *Algebraic branch point* which admits an expansion containing fractional exponents with a finite number of negative power terms. This expansion can be considered as a Puiseux series with a finite number of negative powers. This singular point is also a zero of $g_m(p)$. This singularity is also referred to as a *pole branch point* [10]. It can be traced to be a pole of $Z(p)Y(p)$ in our formalism (see Section 1 and [1]).

Defective multiple eigenvalue; is a multiple eigenvalue for which the number of associated linearly independent eigenvectors is less than the multiplicity of the eigenvalue. Such an eigenvalue is not differentiable.

Discriminant of the irreducible algebraic equation $G(w, p) = 0$; is a polynomial in p the zeroes of which correspond to multiple roots of $G(w, p) = 0$.

Generalized eigenvector of index $k(\geq 1)$ of a matrix A with respect to eigenvalue λ; is a vector x which satisfies

$$(A - \lambda I)^r x = 0$$

if and only if $r \geq k$. Here r and k are integers, while I is the unit matrix.

Irreducible $G(w, p) = 0$; is an equation not expressible as the product of two polynomials of the same type as G. For the treatment of an equation of the form

$$G_1(w, p) \cdot G_2(w, p) = 0$$

can be replaced by the separate consideration of the equations $G_1 = 0$ and $G_2 = 0$.

Non-defective multiple eigenvalue; is a multiple eigenvalue for which the number of associated linearly independent eigenvectors is equal to the multiplicity of the eigenvalue. Such an eigenvalue is differentiable.

Puiseux series; (for our purpose) is a power series containing fractional exponents.

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