# RELATIVISTIC ELECTRODYNAMICS: VARIOUS POSTULATE AND RATIOCINATION FRAMEWORKS

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**Abstract**—Presently various models consistent with Einstein's Special Relativity theory are explored. Some of these models have been introduced previously, but additional models are possible, as shown here. The topsy-turvy model changes the order of postulates and conclusions of Einstein's original theory. Another model is given in the spectral domain, with the relativistic Doppler Effect formulas replacing the Lorentz transformation. In this model a new principle tantamount to the constancy of the speed of light in vacuum is stated and analyzed, dubbed as the constancy of light slowness in vacuum. Because the slowness is derived in the spectral domain from the Doppler Effect formulas, this result is not trivially semantic. It is shown that potentials and equations of continuity can replace the Maxwell Equations used by Einstein for his "Principle of Relativity" in electrodynamics. It is also shown that defining convection currents and assuming the current-charge densities transformations can replace the Lorentz transformation. The list of feasible models representative rather than exhaustive, since parts of the models presented here can be combined to yield additional models. The two underlying elements of Einstein's original Special Relativity theory are always present: (1) the theory requires a kinematical element (e.g., the constancy of the speed of light in vacuum in Einstein's original model), and (2), a dynamical element (e.g., the form-invariance of the Maxwell Equations in all inertial systems of reference in Einsteins original model).

## 1. INTRODUCTION: EINSTEIN'S THEORY AND PRESENT NOTATION

The original approach by Einstein, in his monumental 1905 article [1], was to postulate the form-invariance of the Maxwell Equations, and the constancy of the speed of light relative to all observers in inertial (non-accelerated) frames of reference:

> "··· Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell's theory for stationary bodies ···"

We can quibble here about the sufficiency of these postulates, pointing out that "in the midst of the race" Einstein heuristically

introduces a few times the phrase "$\cdots$ from reasons of symmetry $\cdots$", which raises the question whether the assumed symmetry considerations should be considered to be additional postulates, but the two postulates above are the paramount basis of Relativistic Electrodynamics. Essentially Einstein's theory involves two elements: The first is a kinematical part, in the final form this is the Lorentz transformation, derived from the constancy of light speed postulate, expressed above. The second element is the dynamical part represented by the form-invariance of Maxwells Equations of electrodynamics. In what will be developed subsequently, these two elements must always be represented in one form or another, in order to provide a consistent framework for any equivalent Special Relativity theory.

Einstein [1] derives the transformation of the spatiotemporal coordinates, since then called the Lorentz transformation because H. A. Lorentz first introduced it in 1904. For historical background the reader is referred to Whittaker [2], and the many comments and references given by Pauli [3]. From the principle of relativity, Einstein derived transformation formulas for the various electromagnetic fields appearing in the Maxwell Equations. In spite of the fact that Einstein restricted his analysis to free space (vacuum), his results hold, with the appropriate modifications, for the general case of the macroscopic Maxwell Equations as used here.

For the mathematical structure of Einstein's theory see for example Stratton [4], who follows the technique of using the electromagnetic tensor. The present compact notation was used before in [5]. The macroscopic Maxwell Equations for the electromagnetic field (in the "unprimed" frame of reference denoted by $\Gamma$) are given by

$$
\begin{aligned}
\partial_{\mathbf{x}} \times \mathbf{E} &= -\partial_t \mathbf{B} - \mathbf{j}_m \\
\partial_{\mathbf{x}} \times \mathbf{H} &= \partial_t \mathbf{D} + \mathbf{j}_e \\
\partial_{\mathbf{x}} \cdot \mathbf{D} &= \rho_e \\
\partial_{\mathbf{x}} \cdot \mathbf{B} &= \rho_m
\end{aligned}
\tag{1}
$$

where $\partial_{\mathbf{x}}$ (often symbolized by $\nabla$ and called "Nabla", or sometimes "Del") and $\partial_t$ denote the space and time differential operators, respectively. In general all the fields are space and time dependent, e.g., $\mathbf{E} = \mathbf{E}(\mathbf{X})$. Here

$$
\mathbf{X} = (\mathbf{x}, ict)
\tag{2}
$$

denotes the spatiotemporal coordinate quadruplet, in terms of the Minkowski four-space notation [6], with $i$ the unit imaginary complex number. We do not subscribe to the mathematical properties of the Minkowski four-space, we only use the notation. Going beyond this

point of mere notation already implies the Lorentz transformation, a step which we do not wish to take at this point.

For symmetry and completeness, in the present representation the Maxwell Equations include the usual electric (index $e$), as well as the fictitious magnetic (index $m$), current and charge density sources. To date, the existence of the magnetic current and charge densities in (1) has not been empirically established. Therefore at this time they should be considered as fictitious, in the sense that they are auxiliary and not intrinsic physical entities. The original set of Equations (1) can be split into two sets of fields one driven by $\mathbf{j}_e$, $\rho_e$ the other by $\mathbf{j}_m$, $\rho_m$. This yields

$$
\begin{array}{c|c}
\partial_{\mathbf{x}} \times \mathbf{E}_e = -\partial_t \mathbf{B}_e & \partial_{\mathbf{x}} \times \mathbf{E}_m = -\partial_t \mathbf{B}_m - \mathbf{j}_m \\[4pt]
\partial_{\mathbf{x}} \times \mathbf{H}_e = \partial_t \mathbf{D}_e + \mathbf{j}_e & \partial_{\mathbf{x}} \times \mathbf{H}_m = \partial_t \mathbf{D}_m \\[4pt]
\partial_{\mathbf{x}} \cdot \mathbf{D}_e = \rho_e & \partial_{\mathbf{x}} \cdot \mathbf{D}_m = 0 \\[4pt]
\partial_{\mathbf{x}} \cdot \mathbf{B}_e = 0 & \partial_{\mathbf{x}} \cdot \mathbf{B}_m = \rho_m
\end{array}
\tag{3}
$$

By adding the two sets (3), we obtain (1) once again, i.e., $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m$, etc.

The formal similarity between the two sets (3) leads to the following duality "dictionary"

$$
\begin{aligned}
\mathbf{j}_e &\Leftrightarrow -\mathbf{j}_m \\
\rho_e &\Leftrightarrow -\rho_m \\
\mathbf{E}_e &\Leftrightarrow \mathbf{H}_m \\
\mathbf{H}_e &\Leftrightarrow \mathbf{E}_m \\
\mathbf{B}_e &\Leftrightarrow -\mathbf{D}_m \\
\mathbf{D}_e &\Leftrightarrow -\mathbf{B}_m
\end{aligned}
\tag{4}
$$

By substitution according to this dictionary we obtain the $e$-indexed set of Maxwell Equations from the $m$-indexed one, and *vice-versa*.

The principle of relativity asserts that the form-invariance of the Maxwell Equations (1) applies to all inertial systems. Accordingly in another inertial frame (the "primed" frame of reference $\Gamma'$), the Maxwell Equations have the form

$$
\begin{aligned}
\partial_{\mathbf{x}'} \times \mathbf{E}' &= -\partial_{t'} \mathbf{B}' - \mathbf{j}'_m \\
\partial_{\mathbf{x}'} \times \mathbf{H}' &= \partial_{t'} \mathbf{D}' + \mathbf{j}'_e \\
\partial_{\mathbf{x}'} \cdot \mathbf{D}' &= \rho'_e \\
\partial_{\mathbf{x}'} \cdot \mathbf{B}' &= \rho'_m
\end{aligned}
\tag{5}
$$

where now $\mathbf{E}' = \mathbf{E}'(\mathbf{X}')$ etc., and the native, or proper, space-time coordinates in the $\Gamma'$ system are denoted by $\mathbf{X}' = (\mathbf{x}', ict')$, using once again the Minkowski four-vector notation. Corresponding to (5), there also exists an analog of (3), (4), with all the relevant fields and coordinates now denoted by primes.

The kinematical part in Einstein's theory starts with the postulate of the constancy of $c$ in all inertial frames of reference and culminates in the Lorentz transformation, mediating between spatiotemporal coordinates in $\Gamma$ and $\Gamma'$, which can be written in the form

$$
\begin{aligned}
\mathbf{x}' &= \tilde{\mathbf{U}} \cdot (\mathbf{x} - \mathbf{v}t), & t' &= \gamma(t - \mathbf{v} \cdot \mathbf{x}/c^2) \\
\gamma &= (1 - \beta^2)^{-1/2}, & \beta &= v/c, & v &= |\mathbf{v}|, \\
\tilde{\mathbf{U}} &= \tilde{\mathbf{I}} + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}}, & \hat{\mathbf{v}} &= \mathbf{v}/v
\end{aligned}
\tag{6}
$$

where $\mathbf{v}$ is the velocity by which the origin of $\Gamma'$ is moving, as observed from $\Gamma$, and the tilde denotes dyadics, $\tilde{\mathbf{I}}$ is the idemfactor or unit dyadic (same as unit matrix). The Lorentz transformation (6) can be symbolized in the form $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$. The role of $\tilde{\mathbf{U}}$ is to multiply the component along the velocity by $\gamma$. It is a simple matter of inverting a system of equations, in order to show that the inverse of (6), $\mathbf{X} = \mathbf{X}[\mathbf{X}']$, is obtained from (6) by interchanging primed and unprimed coordinates and inverting the sign of $\mathbf{v}$. This property is a key element of the theory, as it shows that the same transformation works in both directions, and thus there is no single preferred inertial frame of reference.

By taking differentials in (6), the differential Lorentz transformation is obtained

$$
d\mathbf{x}' = \tilde{\mathbf{U}} \cdot (d\mathbf{x} - \mathbf{v}dt), \qquad dt' = \gamma(dt - \mathbf{v} \cdot d\mathbf{x}/c^2)
\tag{7}
$$

By applying the chain-rule of calculus to (7), the relations between derivatives in $\Gamma$ and $\Gamma'$ are established

$$
\partial_{\mathbf{x}'} = \tilde{\mathbf{U}} \cdot (\partial_{\mathbf{x}} + \mathbf{v}\partial_t/c^2), \qquad \partial_{t'} = \gamma(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}})
\tag{8}
$$

where the new transformation (8) is fully equivalent to the original Lorentz transformation (6), and thus could have provided a starting point for Einstein's model. This is an example of the many alternatives for possible "games", i.e., consistent models of the theory. Similarly to the notation $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$ above, we can compactly denote (8) by $\partial_{\mathbf{x}'} = \partial_{\mathbf{x}'}[\partial_{\mathbf{x}}]$, to which also corresponds an inverse transformation $\partial_{\mathbf{x}} = \partial_{\mathbf{x}}[\partial_{\mathbf{x}'}]$.

The four-gradient $\partial_{\mathbf{x}}$ can be defined as a Minkowski four-vector

$$
\partial_{\mathbf{x}} = (\partial_{\mathbf{x}}, -\frac{i}{c}\partial_t)
\tag{9}
$$

Essentially by exploiting (8) in (5), and comparing to (1), Einstein derives the transformation formulas for the various fields

$$
\begin{aligned}
\mathbf{E}' &= \tilde{\mathbf{V}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}), & \mathbf{B}' &= \tilde{\mathbf{V}} \cdot (\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2) \\
\mathbf{D}' &= \tilde{\mathbf{V}} \cdot (\mathbf{D} + \mathbf{v} \times \mathbf{H}/c^2), & \mathbf{H}' &= \tilde{\mathbf{V}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \\
\tilde{\mathbf{V}} &= \gamma \tilde{\mathbf{I}} + (1 - \gamma) \hat{\mathbf{v}} \hat{\mathbf{v}} & & \quad\quad (10)
\end{aligned}
$$

Unlike (6)–(8), the role of $\tilde{\mathbf{V}}$ in (10) is to multiply the components perpendicular to the velocity by $\gamma$. In addition to (10), we find for the sources

$$
\mathbf{j}' = \tilde{\mathbf{U}} \cdot (\mathbf{j} - \mathbf{v}\rho), \qquad \rho' = \gamma(\rho - \mathbf{v} \cdot \mathbf{j}/c^2) \tag{11}
$$

where (10), (11) are applicable to both $e$-indexed and $m$-indexed fields and sources, as well as the sum fields.

The equations (10), (11) look deceptively simple, but it must be noted that the two sides of each equation depend on different coordinates, e.g., the first expression in (10) reads

$$
\mathbf{E}'(\mathbf{X}') = \tilde{\mathbf{V}} \cdot (\mathbf{E}(\mathbf{X}) + \mathbf{v} \times \mathbf{B}(\mathbf{X})) \tag{12}
$$

consequently (10), (11) are meaningless for comparing measurements in two inertial frames of reference unless we have at our disposal $\mathbf{X}' = \mathbf{X}'[\mathbf{X}]$ in (6) to mediate between the two sets of spatiotemporal coordinates.

This, in a nutshell, summarizes the Special Relativity theory. The question posed in this study is whether alternative yet compatible theories can be stated. For example, the functional similarity of (6) and (11) is obvious and intriguing. This immediately suggests that (11) can be used to state an alternative set of equations for the theory. Can we assume (11) and derive (6)? Are the Maxwell Equations (1), (5) indispensable for the statement of the theory, or can equivalent forms be identified? These and other questions will concern us subsequently.

## 2. THE TOPSY-TURVY MODEL

The topsy-turvy model has been devised mainly for didactic reasons, in order to facilitate a straight-forward and compact analysis [5, 7] for an application-oriented audience. Accordingly the Lorentz transformation (6) has been taken as a postulate, replacing the kinematical postulate of the constancy of $c$.

By dividing the two Equations (7), velocities are defined, thus leading to the relativistic formula for addition of velocities

$$
\mathbf{u}' = \mathbf{u}'_\perp + \mathbf{u}'_\parallel = \tilde{\mathbf{U}} \cdot (\mathbf{u} - \mathbf{v})/(\gamma L) = (\mathbf{u}_\perp + \gamma(\mathbf{u}_\parallel - \mathbf{v}))/(\gamma L)
$$

$$u'_\perp = u_\perp/(\gamma L), \qquad u'_\parallel = (u_\parallel - v)/L$$
$$L = 1 - \mathbf{v} \cdot \mathbf{u}/c^2 = 1 - vu_\parallel/c^2, \quad \mathbf{u} = d\mathbf{x}/dt, \quad \mathbf{u}' = d\mathbf{x}'/dt' \tag{13}$$

where in (13) the components of the velocities parallel and perpendicular to the relative velocity between the inertial frames, $\mathbf{v}$, are indicated. Some arithmetic manipulation shows that if we assume

$$u'^2_\perp + u'^2_\parallel = c^2 \tag{14}$$

then

$$u^2_\perp + u^2_\parallel = c^2 \tag{15}$$

follows, hence the constancy of $c$, the speed of light in free space, in all inertial frames, is established. From the postulated Lorentz transformation (6), the transformation of space and time derivatives (8) follows. This terminates the kinematical part.

Like Einstein [1], the Maxwell Equations (1), or in the form (3) are assumed for the topsy-turvy model, but instead of postulating the form-invariance of (1), (5), the model postulates (10), (11). By substitution of (8), (10), (11) into (5), the Maxwell Equations (1) are derived, so the form-invariance is here a consequence.

Thus the presentation of Special Relativity in terms of Einstein's original model and according to the topsy-turvy model are equivalent.

## 3. LORENTZ TRANSFORMATION, PHASE INVARIANCE, AND DOPPLER EFFECT

We are already familiar with the Lorentz transformation (6) as the kinematical element of the Special Relativity theory. Is it an indispensable element of the theory, or (in addition to the obvious (8) derived from (6)) can it be replaced by other postulates? In the present section we introduce the Phase Invariance and Doppler Effect concepts, and show that any two of these three elements implies the third one.

Einstein's "Principle of Relativity" [1], i.e., the form-invariance of the Maxwell Equations, *does not* imply that the solutions of these equations are also form-invariant. In case such an assumption is made for any solution, it must be properly stipulated as a postulate. In spite of this, when Einstein [1] discusses the relativistic Doppler Effect, he tacitly assumes that a plane wave in one inertial frame appears as a plane wave in another inertial frame too. Without this lesser postulate, he could not have derived the formulas for the relativistic Doppler

Effect. However, "cosi fan tutti", many authors do the same and do not emphasize this point. See for example Kong [8].

In addition to the location four-vector (2) we define now a quadruplet involving the wave-vector and (angular) frequency in the Minkowski notation

$$\mathbf{K} = (\mathbf{k}, i\omega/c) \tag{16}$$

The inner product for such Minkowski quadruplets is formally defined as

$$\mathbf{K} \cdot \mathbf{X} = \mathbf{k} \cdot \mathbf{x} + (-i\omega/c)ict = \mathbf{k} \cdot \mathbf{x} - \omega t \tag{17}$$

which is the phase of a plane wave. Note that we are still using Minkowski four-vectors only in the sense of a convenient compact notation. If we assume that plane waves are plane waves in all inertial systems, we essentially postulate the Phase Invariance, i.e.,

$$\mathbf{K} \cdot \mathbf{X} = \mathbf{k} \cdot \mathbf{x} - \omega t = \mathbf{K}' \cdot \mathbf{X}' = \mathbf{k}' \cdot \mathbf{x}' - \omega't' \tag{18}$$

From (6) and (18) the transformation

$$\mathbf{k}' = \tilde{\mathbf{U}} \cdot (\mathbf{k} - \mathbf{v}\omega/c^2), \qquad \omega' = \gamma(\omega - \mathbf{v} \cdot \mathbf{k}) \tag{19}$$

is derived. This is referred to as the relativistic Doppler effect, first announced by Einstein [1]. It is worthwhile to mention that before the advent of Einstein's theory, the relativistically correct Doppler Effect for reflection from a moving mirror was worked out by Abraham [9], see also [3]. The second expression (19) looks quite familiar as being the Doppler frequency-shift formula. The first expression (19) is akin to the phase velocity in moving media, in fact, to the first order it is a statement of the Fresnel Drag Effect, related to the celebrated Fizeau experiment [3, 8, 10]. It is now clear that (6) and (19) satisfy (18), or any two out of (6), (18), (19) satisfy the remaining formula.

## 4. FOURIER TRANSFORM, MINKOWSKI SPACE, AND DOPPLER EFFECT

Thus far we avoided the full Minkowski-space mathematical structure. Forms like (2), (9), (16) were used as a compact notation only. An arbitrary quadruplet $\mathbf{Q}$ is a proper Minkowski four-vector if and only if

$$\mathbf{Q} \cdot \mathbf{X} = \mathbf{Q}' \cdot \mathbf{X}' \tag{20}$$

where in (20) $\mathbf{X}$ is a-priori considered as a Minkowski four-vector, i.e.,

$$\mathbf{X} \cdot \mathbf{X} = \mathbf{x} \cdot \mathbf{x} - c^2 t^2 = \mathbf{X}' \cdot \mathbf{X}' = \mathbf{x}' \cdot \mathbf{x}' - c^2 t'^2 \tag{21}$$

By substituting the Lorentz transformation (6) into (21) it is verified that (21) is identically satisfied. Therefore the assumption of $\mathbf{X}$ being a Minkowski four-vector is tantamount to postulating the Lorentz transformation, and *vice-versa*. Equation (21) is therefore also a statement of the constancy of $c$ postulate, because it asserts that if the equation of motion is $\mathbf{x} \cdot \mathbf{x} = c^2 t^2$ in $\Gamma$, it will be $\mathbf{x}' \cdot \mathbf{x}' = c^2 t'^2$ in $\Gamma'$, with the same factor $c$.

The general definition (20) facilitates the introduction of additional four-vectors. Thus (9) can be tested by evaluating

$$\partial_{\mathbf{x}} \cdot \mathbf{X} = \partial_{\mathbf{x}} \cdot \mathbf{x} + \partial_t t = \partial_{\mathbf{x}'} \cdot \mathbf{X}' = \partial_{\mathbf{x}'} \cdot \mathbf{x}' + \partial_{t'} t' = 4 \tag{22}$$

It follows that $\partial_{\mathbf{x}}$, (9), is a four-vector too. Of course, this does not come as a surprise, because (9) was derived from (2) using the chain-rule of calculus, but (22) provides the formal proof.

One could play the game a little differently, by starting with a statement that the four-gradient $\partial_{\mathbf{x}}$, (9), is a proper Minkowski four-vector, and then using (22) to *derive* $\mathbf{X}$ as a Minkowski vector.

Once a quadruplet $\mathbf{Q}$ has been established as a Minkowski four-vector, it can be used to test any new quadruplet $\mathbf{P}$. In general, any four-vector multiplied by itself is an invariant

$$\mathbf{Q} \cdot \mathbf{Q} = \mathbf{Q}' \cdot \mathbf{Q}' \tag{23}$$

and this can be used as a definition of a four-vector in the statement: the length of a four-vector in the Minkowski space is a scalar invariant under rotation. However, it must be remembered that this whole mathematical edifice rests on the original definition (21), i.e., when the physics comes into the game, the Lorentz transformation is already assumed here.

The use of (20) etc. provides a very convenient technique for dealing with various aspects of the Special Relativity theory, but it is not an essential part of it. In other words, anything that we can do using the Minkowski four-space we can also do without it.

As an illustration, let us demonstrate how (19) can be derived using the Minkowski four-vector concept. We start with the assumption that (2) is a four-vector proper. It follows that $\partial_{\mathbf{x}}$, (9), is a four-vector too, because (22) is satisfied.

Now consider the four-dimensional Fourier transformation

$$f(x, y, z, ict) = q \int f(k_x, k_y, k_z, \frac{i\omega}{c})$$
$$e^{i(k_x x + k_y y + k_z z - \omega t)} dk_x dk_y dk_z d\frac{i\omega}{c}$$
$$q = (2\pi)^{-4} \tag{24}$$

which can be compactly recast [5] in the form

$$f(\mathbf{X}) = q \int (d^4 \mathbf{K}) f(\mathbf{K}) e^{i\mathbf{K} \cdot \mathbf{X}} \tag{25}$$

with the corresponding inverse transformation

$$f(\mathbf{K}) = \int (d^4 \mathbf{X}) f(\mathbf{X}) e^{-i\mathbf{K} \cdot \mathbf{X}} \tag{26}$$

Applying the four-gradient operator $\partial_\mathbf{x}$ to (25) yields

$$\partial_\mathbf{x} f(\mathbf{X}) = q \int (d^4 \mathbf{K}) f(\mathbf{K}) i \mathbf{K} e^{i\mathbf{K} \cdot \mathbf{X}} \tag{27}$$

From (27) it follows that if $\partial_\mathbf{x}$ is a Minkowski four-vector, then so must $\mathbf{K}$ be too, and *vice-versa*. In turn, it follows that $\mathbf{K} \cdot \mathbf{X}$ is a Minkowski-space invariant, i.e., the Phase Invariance (18) appears as a result of (27), and therefore becomes a special case in such a context. Consequently (27) also prescribes (19), since the various coordinates in (16) must agree one by one with the corresponding coordinates in (9). In other words, we have here a "dictionary"

$$\partial_\mathbf{x} \Leftrightarrow i\mathbf{K}, \qquad \partial_\mathbf{x} \Leftrightarrow i\mathbf{k}, \qquad \partial_t \Leftrightarrow -i\omega \tag{28}$$

where in (28) the last two expressions are the three-space and time representation.

Direct application of the dictionary (28), by substitution of the components of $i\mathbf{K}$ into (9) yields the relativistic Doppler Effect (19) and *vice-versa*. The direct use of such dictionaries can save a lot of manipulation. It seems therefore advantageous to use the Minkowski-space and its associated four-vectors. However the game we play relative to the postulates and ratiocinations of the Special Relativity theory must be carefully stated. As another example, consider the identical functional structure of (6) and (11), already noticed above. From the similarity and the statement that (2) is a four-vector, it follows that

$$\mathbf{J} = (\mathbf{j}, ic\rho) \tag{29}$$

is a Minkowski four-vector as well, where (29) applies to both $e$-indexed and $m$-indexed sources. Using (11), it is easily verified that **J**, (29), satisfies (23). We have established that subject to the Lorentz transformation, which is embedded in (21), the quadruplet **J** in (29) is a Minkowski four-vector. Conversely, if we start with a statement that (29) constitutes a Minkowski four-vector, then the transformation (11) is concluded.

Originally (11) was presented as a result of the form-invariance (1), (5) of the Maxwell Equations, and in addition (8), which already assumes the Lorentz transformation. The question whether it is possible to postulate (11) in order to derive the Lorentz transformation (6) will be considered in a subsequent section.

## 5. DOPPLER EFFECT AND CONSTANCY OF SLOWNESS IN VACUUM

The process of using (7), which led to the relativistic velocity transformation (13) can be mimicked using for the Doppler Effect (19), thus yielding

$$
\begin{aligned}
d\mathbf{k}' &= \tilde{\mathbf{U}} \cdot (d\mathbf{k} - \mathbf{v}d\omega/c^2), \qquad \omega' = \gamma(\omega - \mathbf{v} \cdot \mathbf{k}) \\
\mathbf{s}' &= \mathbf{s}'_\perp + \mathbf{s}'_\parallel = \tilde{\mathbf{U}} \cdot (\mathbf{s} - \mathbf{v}/c^2)/(\gamma M) \\
&= (\mathbf{s}_\perp + \gamma(\mathbf{s}_\parallel - \mathbf{v}/c^2))/(\gamma M) \\
s'_\perp &= s_\perp/(\gamma M), \qquad s'_\parallel = (s_\parallel - v/c^2)/M \\
M &= 1 - \mathbf{v} \cdot \mathbf{s}, \qquad \mathbf{s} = d\mathbf{k}/d\omega, \qquad \mathbf{s}' = d\mathbf{k}'/d\omega'
\end{aligned}
\tag{30}
$$

In (30) the new function **s** is dubbed as "slowness" [11], due to its dimensions. Obviously velocity is not simply the inverse of slowness, because we have vectors in (13) and (30). Also it is noted that **s**, defined in the spectral domain, does not actually refer to motion of an object. Note carefully that the velocity in (13) did not require a special definition, because velocity as the derivative along the trajectory is already available from mechanics. On the other hand, slowness, which can be understood as the derivative along the trajectory in the spectral domain, is a new concept, and needs to be stated.

Analogously to (13) we assume in (30)

$$
s_\perp^{'2} + s_\parallel^{'2} = 1/c^2
\tag{31}
$$

After some manipulation (30), (31) yield, analogously to (15)

$$
s_\perp^2 + s_\parallel^2 = 1/c^2
\tag{32}
$$

This is a remarkable (novel, as far as this author can assess) result showing that the Doppler Effect has the same properties as the Lorentz transformation, namely, when the slowness in one inertial frame is the vacuum slowness $1/c$, then this is a constant slowness observed in all inertial systems. Therefore the kinematical element of the Special Relativity theory can be embodied in this new principle, or by invoking the Doppler Effect formulas (19).

## 6. EINSTEINIAN AND TOPSY-TURVY SPECTRAL MODELS

There is no apparent reason for starting with the Maxwell Equations in the spatiotemporal domain. The Fourier transform pair (25), (26) can be used to transform (1), (5) (or (3) and the corresponding Maxwell equations in $\Gamma'$) into the spectral domain. Conversely, we can start with the spectral representation and from there transform into the spatiotemporal domain.

Thus a consistent Special Relativity model can be stated in the spectral domain. We start with the Fourier transformed Maxwell Equations in $\Gamma$, obtained by applying (25) to (1)

$$
\begin{aligned}
i\mathbf{k} \times \mathbf{E} &= i\omega\mathbf{B} - \mathbf{j}_m \\
i\mathbf{k} \times \mathbf{H} &= -i\omega\mathbf{D} + \mathbf{j}_e \\
i\mathbf{k} \cdot \mathbf{D} &= \rho_e \\
i\mathbf{k} \cdot \mathbf{B} &= \rho_m
\end{aligned}
\tag{33}
$$

where in (33) the transformed fields $\mathbf{E} = \mathbf{E}(\mathbf{K})$ etc. are understood. Following the Einstein approach, the principle of relativity now prescribes in $\Gamma'$ the corresponding set of the transformed Maxwell Equations

$$
\begin{aligned}
i\mathbf{k}' \times \mathbf{E}' &= i\omega'\mathbf{B}' - \mathbf{j}'_m \\
i\mathbf{k}' \times \mathbf{H}' &= -i\omega'\mathbf{D}' + \mathbf{j}'_e \\
i\mathbf{k}' \cdot \mathbf{D}' &= \rho'_e \\
i\mathbf{k}' \cdot \mathbf{B}' &= \rho'_m
\end{aligned}
\tag{34}
$$

where in (34) $\mathbf{E}' = \mathbf{E}'(\mathbf{K}')$, etc.

Consistent with Einsteins approach, we assume now the principle of constancy of slowness $1/c$, and derive the analog of the Lorentz transformation, namely the Doppler Effect formulas $\mathbf{K}' = \mathbf{K}'[\mathbf{K}]$ given in (19). This yields the analog of (10), (11), i.e., transformation formulas for the fields and sources, where the independent variables

are in the spectral domain, e.g., the analog of (12) is now

$$\mathbf{E}'(\mathbf{K}') = \tilde{\mathbf{V}} \cdot (\mathbf{E}(\mathbf{K}) + \mathbf{v} \times \mathbf{B}(\mathbf{K})) \tag{35}$$

The topsy-turvy analog follows in an almost trivial way: We start with the postulated Fourier transformed Maxwell Equations (33), the fields and sources transformation formulas, i.e., (10) and (11) modified according to prescription (35), and the Doppler Effect formulas (19). From these the Maxwell Equations set (34) is derived.

## 7. MAXWELL EQUATIONS, POTENTIALS, AND EQUATIONS OF CONTINUITY

The Maxwell Equations facilitate the statement of potentials. These are usually derived from the sourceless equations in each set of (3). In addition, by applying a divergence, $\partial_\mathbf{x}\cdot$, operator to the vector equations containing sources, or equivalently, applying a factor $i\mathbf{k}\cdot$ in the Fourier transformed equations, equations of continuity are derived. Thus for the set of Maxwell Equations (3) we have

$$
\begin{array}{l|l}
\mathbf{B}_e = \partial_\mathbf{x} \times \mathbf{A}_e & \mathbf{D}_m = \partial_\mathbf{x} \times \mathbf{A}_m \\
\mathbf{E}_e = -\partial_\mathbf{x}\phi_e - \partial_t\mathbf{A}_e & \mathbf{H}_m = \partial_\mathbf{x}\phi_m + \partial_t\mathbf{A}_m \\
\partial_\mathbf{X} \cdot \mathbf{J}_e = \partial_\mathbf{x} \cdot \mathbf{j}_e + \partial_t\rho_e = 0 & \partial_\mathbf{X} \cdot \mathbf{J}_m = \partial_\mathbf{x} \cdot \mathbf{j}_m + \partial_t\rho_m = 0
\end{array}
\tag{36}
$$

where in (36) $\mathbf{E} = \mathbf{E}(\mathbf{X})$ etc., and the equations of continuity are also expressed in terms of the four-vector notation. Equivalently, in the spectral domain

$$
\begin{array}{l|l}
\mathbf{B}_e = i\mathbf{k} \times \mathbf{A}_e & \mathbf{D}_m = i\mathbf{k} \times \mathbf{A}_m \\
\mathbf{E}_e = -i\mathbf{k}\phi_e + i\omega\mathbf{A}_e & \mathbf{H}_m = i\mathbf{k}\phi_m - i\omega\mathbf{A}_m \\
\mathbf{K} \cdot \mathbf{J}_e = \mathbf{k} \cdot \mathbf{j}_e - \omega\rho_e = 0 & \mathbf{K} \cdot \mathbf{J}_m = \mathbf{k} \cdot \mathbf{j}_m - \omega\rho_m = 0
\end{array}
\tag{37}
$$

where in (37) $\mathbf{E} = \mathbf{E}(\mathbf{K})$ etc. We can work backwards and derive the Maxwell Equations (3), or the corresponding spectral domain set obtained from (33), by exploiting (36), or (37), respectively. For example, in (36) perform $\partial_\mathbf{x} \times \mathbf{E}_e = -\partial_t\partial_\mathbf{x} \times \mathbf{A}_e = -\partial_t\mathbf{B}_e$ to get the first equation on the left, (3). Inasmuch as any scalar can be represented as a divergence of a vector, we can arbitrarily define $\partial_\mathbf{x} \cdot \mathbf{D}_e = \rho_e$. Consequently, from the equation of continuity in (36) we have $\partial_\mathbf{x} \cdot (\mathbf{j}_e + \partial_t\mathbf{D}_e) = 0$, and because $\partial_\mathbf{x} \cdot \partial_\mathbf{x} \times$ of any vector vanishes,

we conclude that $\mathbf{j}_e + \partial_t \mathbf{D}_e = \partial_{\mathbf{x}} \times \mathbf{H}_e$, and so on. A similar procedure applies to spectral set (37).

Hence we could apply Einstein's principle of relativity to (36) or (37) instead of the original Maxwell equations. Stipulating these expressions to be form-invariants in all inertial systems prescribes

$$
\begin{aligned}
\mathbf{B}'_e &= \partial_{\mathbf{x}'} \times \mathbf{A}'_e & \qquad \mathbf{D}'_m &= \partial_{\mathbf{x}'} \times \mathbf{A}'_m \\
\mathbf{E}'_e &= -\partial_{\mathbf{x}'} \phi'_e - \partial_{t'} \mathbf{A}'_e & \qquad \mathbf{H}'_m &= \partial_{\mathbf{x}'} \phi'_m + \partial_{t'} \mathbf{A}'_m \\
\partial_{\mathbf{x}'} \cdot \mathbf{j}'_e + \partial_{t'} \rho'_e &= 0 & \qquad \partial_{\mathbf{x}'} \cdot \mathbf{j}'_m + \partial_{t'} \rho'_m &= 0
\end{aligned}
\tag{38}
$$

where in (38) $\mathbf{E}' = \mathbf{E}'(\mathbf{X}')$ etc. Similarly, add primes in (37) and express the fields in terms of the appropriate quadruplet of spectral coordinates, e.g., $\mathbf{E}' = \mathbf{E}'(\mathbf{K}')$ etc.

Finally, in addition to the form-invariant (36), (38), include the Lorentz transformation (6) in the model. In the corresponding spectral domain model, in addition to (37) and the corresponding primed equations, include the Doppler Effect transformation (19). Thus consistent and complete Special Relativity models are obtained with the form-invariance of the Maxwell Equations appearing as a consequence, rather than a postulate. In a similar fashion the relevant topsy-turvy models can be stated. It is important to note that in addition to the potentials, the equations of continuity must be postulated, in order to achieve consistent and complete Special Relativity models.

The above argument can be presented using the Minkowski four-space formalism. But the need for new concepts, such as six-vectors and the electromagnetic tensor, and the associated loss of compactness, together with increasing departure from the original concepts, makes this a cumbersome game of dubious advantage. Inasmuch as this game is widely used, e.g., see [3, 4, 12], it is worthwhile to cursorily outline its origins here.

The potentials appearing in (36) can be regrouped into four-vectors

$$
\boldsymbol{\Phi} = (\mathbf{A}, i\phi/c)
\tag{39}
$$

where (39) applies to both $e$-indexed and $m$-indexed fields. From the formal similarity of the definition (16) and the Doppler

Effect transformations (19), it immediately follows that the potentials transform according to

$$
\begin{aligned}
\mathbf{A}' &= \tilde{\mathbf{U}} \cdot (\mathbf{A} - \mathbf{v}\phi/c^2) \\
\phi' &= \gamma(\phi - \mathbf{v} \cdot \mathbf{A})
\end{aligned}
\tag{40}
$$

where (40) applies to both $e$-indexed and $m$-indexed potentials. Therefore we have

$$\mathbf{\Phi}' \cdot \mathbf{\Phi}' = \mathbf{\Phi} \cdot \mathbf{\Phi} \tag{41}$$

i.e., Minkowski-space invariants for both $e$-indexed and $m$-indexed potentials.

A four-dimensional $\partial_{\mathbf{x}} \times \mathbf{\Phi}$ operation is now defined

$$\partial_{\mathbf{X}} \times \mathbf{\Phi} = (\partial_{X_i} \mathbf{\Phi}_j - \partial_{X_j} \mathbf{\Phi}_i), \qquad i, j = 1, 2, 3, 4 \tag{42}$$

$\partial_{\mathbf{X}} \times \Phi$ must be understood as merely a symbol, because a direct analog of the three-dimensional rotor operation does not exist. It is easily shown that in (42) only six independent equations exist, therefore $\partial_{\mathbf{X}} \times \Phi$ is referred to as a six-vector.

Carrying out the operations indicated in (42) yields the following six-vectors

$$\partial_{\mathbf{X}} \times \mathbf{\Phi}_e = (\mathbf{B}_e, -\frac{i}{c}\mathbf{E}_e), \qquad \partial_{\mathbf{X}} \times \mathbf{\Phi}_m = (\mathbf{H}_m, -ic\mathbf{D}_m) \tag{43}$$

and therefore the various fields in (36) can be expressed as the real or imaginary components of $\partial_{\mathbf{X}} \times \mathbf{\Phi}_e$, $\partial_{\mathbf{X}} \times \mathbf{\Phi}_m$, e.g., $\mathbf{B}_e = \Re(\partial_{\mathbf{X}} \times \mathbf{\Phi}_e)$ etc. Now these forms, together with the equations of continuity, can be chosen as a basis for the principle of relativity, i.e., as the sets that are form-invariant in all inertial systems. This approach is akin to the electromagnetic tensor representations, appearing in most mathematically-oriented statements of the Special Relativity theory. Obviously these approaches are only some of the many possibilities of playing the game.

## 8. CHARGE CONSERVATION, CONVECTION CURRENT AND THE LORENTZ TRANSFORMATION

Recently another possibility of replacing the Lorentz transformation has been considered [13]. The author adopts Einstein's principle of relativity, i.e., the form-invariance of (1), (5), and adds to it the conservation of charge of an isolated body. In this section we examine an approach based on (11).

Let us start with Einstein's principle of relativity, i.e., the stipulation of form-invariance in (1), (5). As long as the Lorentz transformation (6) is not included, we do not have (8) to work with, hence (10), (11) cannot be derived, let alone the comparison of observations in various inertial frames of reference, as suggested by (12), is not available.

The formal similarity of (11) and the Lorentz transformation (6) has already been noticed above as very suggestive for creating an alternative relativistic model. Let us now introduce the new concept of convection current density, and assume that such current densities can exist in all inertial frames

$$\mathbf{j}_c = \mathbf{u}_c \rho_c, \qquad \mathbf{j}'_c = \mathbf{u}'_c \rho'_c \qquad (44)$$

In (44) a statement has been made that a convection current density is created by moving charge densities. In other words, $\mathbf{u}_c$, $\mathbf{u}'_c$ are considered as proper velocities as understood in mechanics, i.e., like $\mathbf{u}$ in (13).

Note that Maxwell Equations (1) do not prescribe (44). In the Maxwell Equations the number of unknowns exceeds the number of equations, hence additional relations are necessary in order to solve (1). Such new equations relating quantities appearing in the Maxwell Equations, are termed constitutive relations. Therefore (44) constitutes such a constitutive relation, which has the status of a postulate.

Furthermore, let us now consider (11) as a postulate. For the convection current densities and the associated charges densities this prescribes

$$\mathbf{j}'_c = \tilde{\mathbf{U}} \cdot (\mathbf{j}_c - \mathbf{v}\rho_c), \qquad \rho'_c = \gamma(\rho_c - \mathbf{v} \cdot \mathbf{j}_c/c^2) \qquad (45)$$

By substituting (44) into (45), once again (13) is derived with $\mathbf{u}_c$ replacing $\mathbf{u}$. The assumption that $\mathbf{u}_c = d\mathbf{x}/dt$ is a proper velocity facilitates working our way backwards to derive (7) and therefore (6). Thus we have demonstrated that the principle of relativity, i.e., the form-invariance of (1), (5), together with (44), (45) constitutes a consistent Special Relativity model.

Having at our disposal the Lorentz transformation (6), or its differential form in (7), the convection current densities (44), and the postulate (45), we can now prove the conservation of charge. The following development is somewhat similar to the definition of the self (proper) time and the idea of time dilation in Special Relativity. Consider (45) with $\mathbf{j}'_c = 0$, and substitute the ensuing $\mathbf{j}_c = \mathbf{v}\rho_c$ into the second expression (45). This yields

$$\rho'_c = \rho_c/\gamma \qquad (46)$$

Note that (46) depends on our starting assumption $\mathbf{j}'_c = 0$. Had we started with $\mathbf{j}_c = 0$, we would end up with $\rho_c = \rho'_c/\gamma$. There is no conflict here, and both results show that in the proper frame, where the current density vanishes, the charge density is smaller than that observed in any other reference frame, i.e., $\rho'_c < \rho_c$.

Similarly to the argument leading to (46), the relativistic time dilation is obtained from (6) upon assuming $\mathbf{x}' = 0$ and substituting $\mathbf{x} = \mathbf{v}t$ into the second expression (6), yielding $t' = t/\gamma$, i.e., $t' < t$, where in the proper frame, in which the clock is at rest, the amount of measured time is smaller than that measured by other observers, a phenomenon dubbed as time dilation.

We now wish to show how our analysis leads to conservation of charge. Consider in $\Gamma'$ an elementary charge defined by the charge density times a volume element

$$dq' = \rho'_c dV' = \rho'_c dx' dy' dz' \tag{47}$$

The corresponding charge in $\Gamma$ is moving according to $x = vt$. hence

$$dq = \rho_c dV = \rho_c d(x - vt) dy dz \tag{48}$$

By exploiting (7) and (46) in (47), (48), and some manipulation, we obtain

$$dq = (\rho_c/\gamma)(\gamma(dx - vdt)) dy dz = dq' \tag{49}$$

showing that the charge is conserved.

Conversely, we start with the postulate of charge conservation, as in (49). To derive (47) we need to postulate (46) and

$$dy = dy'; \qquad dz = dz' \tag{50}$$

Consequently (49), (50) establish the first expression of (7), hence also the corresponding one in (6). Postulating (44) we can backtrack to derive from (46) the second expression (45). Obviously, we still miss something, and that is either the second expression (6) or the first expression (45). One of them must be postulated, or some intermediate expression such as

$$dt'/dt = d\rho'_c/d\rho_c \tag{51}$$

Anyhow, this seems to be a very contrived game, although a legitimate one nevertheless.

The length contraction phenomenon is shown from (7) when the two ends of a length element are concurrently observed in $\Gamma$, at some fixed time i.e., $dt = 0$. Accordingly (7) yields $d\mathbf{x}' = \tilde{\mathbf{U}} \cdot d\mathbf{x}$, thus $d\mathbf{x}' > d\mathbf{x}$. By substituting $dt = 0$ into the second expression (7) it is seen that in $\Gamma'$ the two ends of the length element are observed at a time difference $dt' = -\gamma \mathbf{v} \cdot d\mathbf{x}/c^2$.

One may wonder if (11) displays a phenomenon analogous to the relativistic length contraction. Obviously if we assume $\rho = 0$ in (11), this leads formally to $d\mathbf{j}' = \tilde{\mathbf{U}} \cdot d\mathbf{j}$, $d\rho' = -\gamma \mathbf{v} \cdot d\mathbf{j}/c^2$, but it is not clear what this means physically.

## 9. SUMMARY AND CONCLUDING REMARKS

Einstein's Special Relativity theory combines with Maxwell's theory of electrodynamics into one of the pinnacles of physical theory, indeed of human intellect. This will always be associated with the names of Maxwell and Einstein.

The present research does not refute or replace the edifice of Relativistic Electrodynamics. The question asked here is: is the statement of Relativistic Electrodynamics unique? By providing many examples, it is shown that indeed it is not singular.

Einstein's original theory is based on a set of postulates and leads to conclusions. What we demonstrate above is the feasibility of choosing some conclusions to act in the role of postulates, and derive Einstein's original postulates as conclusions. Although no novel physical theory is proposed, the flexibility gained by this strategy allows for better understanding, and provides for more sophisticated teaching methods and the handling of specific problems. It also shows the way to determine when the theory is under-determined, i.e., needs more postulates and definitions, and when a certain approach might be over-determined, in the sense that too many postulates have been assumed, some of them already constituting results.

In the topsy-turvy approach, the constancy of $c$ for all inertial observers is replaced by the Lorentz transformation, and the principle of relativity, by which the set of the Maxwell Equations is form-invariant in all inertial systems, is replaced by a set of Maxwell Equations in one arbitrary frame of reference, and in addition the transformation formulas for the fields and sources, allowing to determine those in other inertial systems.

The equivalence of the Lorentz transformation and the relativistic Doppler Effect formulas is demonstrated, facilitating a consistent model in the spectral domain (related to the spatiotemporal domain by the four-dimensional Fourier transform). An interesting result of this approach is the constancy of slowness of light in vacuum, which is the counterpart of the constancy of light speed in vacuum. This is not merely some semantic game on words, it is an interesting result which allows all the equivalent theories to be stated in the spectral domain.

We have also examined the alternative of replacing the Maxwell Equations by potentials and equations and continuity.

Finally, it has been shown that the inclusion of the convection current in the postulates allows for a replacement of the Lorentz transformation by the current-charge density transformation formulas.

One theme which remains valid for all the models is the need for two elements: the theory needs a kinematical element, dealing with

transformation of coordinates, and a dynamical element dealing with measurable fields.

## ACKNOWLEDGMENT

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