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OPTICAL SOLITON PERTURBATION WITH NON-KERR LAW NONLINEARITIES

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Abstract—This paper studies solitons and its perturbations that is governed by the generalized nonlinear Schrödinger's equation with non-Kerr law nonlinearity. The quasi-stationarity is applied to the non-Kerr law case and an approximate solution is obtained. A few special cases of the non-Kerr law nonlinearity are considered, as examples, with the nonlinear damping type perturbation.

1. INTRODUCTION

The nonlinear Schrödinger's equation (NLSE) plays a vital role in the various areas of physical, biological and engineering sciences. It appears in many applied areas like Fluid Dynamics, Nonlinear Optics, Plasma Physics, Protein Chemistry just to name a few [1]. In this paper, we are going to study an important generalization of the NLSE known as the generalized Nonlinear Schrödinger's Equation (GNLSE) that is given by:

$$iq_t + \frac{1}{2}q_{xx} + F\left(|q|^2\right)q = 0. \tag{1}$$

where F is a real-valued algebraic function and we need to have the smoothness of the complex function $F\left(|q|^2\right)q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R^2, we say that the function $F\left(|q|^2\right)q$ is k times continuously differentiable so that we can write

$$F\left(|q|^2\right)q \in \cup_{m,n=1}^{\infty} C^k((-n,n)\times(-m,m);R^2)$$

Equation (1) is a nonlinear partial differential equation (PDE) of parabolic type that is not integrable, in general. The non-integrability is not necessarily related to the nonlinear term in it. Higher-order dispersion or birefringence, for example, can also make the system non-integrable, while it remains Hamiltonian. The special case, $F(s) = s$, also known as the Kerr law of nonlinearity, is integrable by the method of Inverse Scattering Transform (IST) [2]. The IST is the nonlinear analog of Fourier transform that is used for solving the linear partial differential equations. Schematically, the IST and the technique of Fourier transform are similar [3]. This special case falls in the category of S-integrable equation [4]. In this case, (1) is known as the cubic NLSE. The solutions are known as solitons. It arises in Fluid Dynamics, Nonlinear Optics and α-helix proteins in Protein Chemistry.

The general case $F(s) \neq s$ takes us away from the IST picture as it is not of Painleve type [2]. In a rigorous sense, the pulses of the non-integrable systems are not solitons. However, the term "solitons" has been used broadly for the solutions of the nonintegrable system as well, and this has become common. So in this paper we shall refer to the pulses as 'solitons'. Unlike the cubic NLSE which has an infinite number of conserved quantities, the GNLSE given by (1) has only a few. Although, stationary pulses exist, and some solutions can be written in the analytic form, their behaviour is different from that of the solutions of the cubic NLSE [5].

As we have mentioned, (1), unlike the Kerr law case, does not have infinitely many conserved quantities. In fact, it has as few as three [5]. They are the energy (E), linear momentum (M) and the Hamiltonian (H) that are respectively given by:

$$E = \int_{-\infty}^{\infty} |q|^2 \, dx \tag{2}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} (q^* q_x - q q_x^*) \, dx \tag{3}$$

$$H = \int_{-\infty}^{\infty} \left[\frac{1}{2}|q_x|^2 - f(I)\right] \, dx \tag{4}$$

where we have

$$f(I) = \int_0^I F(\xi) \, d\xi \tag{5}$$

and the intensity I is given by $I = |q|^2$. The first conserved quantity (E), given by (2), has various definitions depending on the context in which the equation arises. It is commonly known as the [6] *wave energy* and is also known as the *mass, wave action* or *plasmon number* while in optics it is called the *wave power* and mathematically speaking it is known as the L_2 norm.

One can see that (1) can be written in a canonical form [4]

$$iq_t = \frac{\delta H}{\delta q^*} \tag{6}$$

$$iq_t^* = -\frac{\delta H}{\delta q} \tag{7}$$

This defines a Hamiltonian dynamical system on an infinite dimensional phase space of two complex functions U and V which decrease to zero at infinity. It can be analyzed using the theory of Hamiltonian systems. This means that a behaviour of the solution is defined, to a large extent, by the singular points of the system, namely the stationary solutions of (1) and depends on the nature of these points as determined by the stability of its stationary solutions [7].

We shall assume that the soliton solution of (1), although not integrable, is given in the form [8]:

$$q(x,t) = A(t)g\left[B(t)\left\{\theta - \bar{\theta}(t)\right\}\right]e^{i\phi} \tag{8}$$

where

$$\frac{\partial\theta}{\partial x} = 1, \quad \frac{\partial\theta}{\partial t} = 0, \quad \frac{d\bar{\theta}}{dt} = v \tag{9}$$

and

$$\frac{\partial\phi}{\partial t} = \frac{B^2}{2}\frac{I_{0,0,2,0}}{I_{0,2,0,0}} - \frac{\kappa^2}{2} + \frac{1}{I_{0,2,0,0}}\int_{-\infty}^{\infty} g^2(s)F\left(A^2g^2(s)\right)ds \tag{10}$$

with

$$\frac{\partial\phi}{\partial x} = -\kappa \tag{11}$$

Here, we have defined the integral

$$I_{\alpha,\beta,\gamma,\nu} = \int_{-\infty}^{\infty} \tau^{\alpha} g^{\beta}(\tau)\left(\frac{dg}{d\tau}\right)^{\gamma}\left(\frac{d^2g}{d\tau^2}\right)^{\nu} d\tau \tag{12}$$

for non-negative integers α, β, γ and ν with $\tau = B(t)(\theta - \bar{\theta}(t))$. In (8), g represents the shape of the soliton described by the GNLSE and it depends on the type of nonlinearity in (1). The parameters $A(t)$ and $B(t)$, in (8), respectively represent the soliton amplitude and the width of the soliton respectively while ϕ represents the phase of the soliton and therefore κ is the frequency of the soliton while v is the velocity. The soliton width and the amplitude are related as $B(t) = \lambda(A(t))$ where the functional form λ depends on the type of nonlinearity in (1). Also, $\bar{\theta}(t)$ represents the mean position of the soliton. For such a general form of the soliton given by (8), we have the integrals of motion, from (2), (3) and (4), given by:

$$E = \int_{-\infty}^{\infty} |q|^2\ dx = \frac{A^2}{B}I_{0,2,0,0} \tag{13}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty}\left(qq_x^* - q^*q_x\right)\ dx = -\kappa\frac{A^2}{B}I_{0,2,0,0} \tag{14}$$

$$\begin{aligned} H &= \int_{-\infty}^{\infty}\left[\frac{1}{2}|q_x|^2 - f\left(|q|^2\right)\right]\ dx \\ &= \frac{A^2B}{2}I_{0,0,2,0} + \frac{\kappa^2A^2}{2B}I_{0,2,0,0} - \int_{-\infty}^{\infty}\int_0^I F(s)ds dx \end{aligned} \tag{15}$$

2. PARAMETER DYNAMICS

For the soliton given by (8), the parameters are now defined as follows [8]:

$$\kappa(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\left(qq_x^* - q^*q_x\right)\, dx}{\int_{-\infty}^{\infty}|q|^2\, dx} \tag{16}$$

$$\bar{\theta}(t) = \frac{\int_{-\infty}^{\infty}\theta\,|q|^2\, d\theta}{\int_{-\infty}^{\infty}|q|^2\, d\theta} \tag{17}$$

Also, the parameter dynamics of the unperturbed soliton is given by

$$\frac{d\kappa}{dt} = 0 \tag{18}$$

$$\frac{d\bar{\theta}}{dt} = -\kappa \tag{19}$$

We note that, the parameter dynamics for the amplitude and the width of the soliton can be obtained for the special cases of $F(s)$ once the functional form of F is known.

2.1. Perturbation Terms

We, now, consider the NLSE along with its perturbation terms that is given by

$$iq_t + \frac{1}{2}q_{xx} + F\left(|q|^2\right)q = i\epsilon R[q, q^*] \tag{20}$$

Here R is a spatio-differential operator and ϵ is a perturbation parameter with $0 < \epsilon \ll 1$. This perturbation parameter depends on the type of nonlinearity. For example, in the context of optics, ϵ is called the relative width of the spectrum that arises due to quasi-monochromaticity [9]. In presence of the perturbation terms, we have the adiabatic dynamics of the soliton parameters as:

$$\frac{dE}{dt} = \epsilon\int_{-\infty}^{\infty}\left(q^*R + qR^*\right)\, dx \tag{21}$$

$$\frac{d\kappa}{dt} = \frac{\epsilon}{I_{0,2,0,0}}\frac{B}{A^2}\left[i\int_{-\infty}^{\infty}(q_x^*R - q_xR^*)dx - \kappa\int_{-\infty}^{\infty}(q^*R + qR^*)dx\right] \tag{22}$$

$$\frac{d\bar{\theta}}{dt} = -\kappa + \frac{\epsilon}{I_{0,2,0,0}} \frac{B}{A^2} \int_{-\infty}^{\infty} \theta \left(q^* R + q R^*\right) \, d\theta \tag{23}$$

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{B^2}{2} \frac{I_{0,0,2,0}}{I_{0,2,0,0}} - \frac{\kappa^2}{2} + \frac{1}{I_{0,2,0,0}} \int_{-\infty}^{\infty} g^2(s) F\left(A^2 g^2(s)\right) \, ds \\ &\quad + \frac{i\epsilon}{I_{0,2,0,0}} \frac{B}{2A^2} \int_{-\infty}^{\infty} \left(q R^* - q^* R\right) \, dx \end{aligned} \tag{24}$$

We can, now, rewrite equations (20) to (23) in the following alternative form:

$$\frac{dE}{dt} = 2\epsilon \frac{A}{B} \int_{-\infty}^{\infty} g(\tau) \Re\left[R e^{i\phi}\right] \, d\tau \tag{25}$$

$$\begin{aligned} \frac{d\kappa}{dt} &= -\frac{2\epsilon}{I_{0,2,0,0}} \frac{\kappa}{A} \int_{-\infty}^{\infty} g(\tau) \Re\left[R e^{i\phi}\right] \, d\tau \\ &\quad - \frac{2\epsilon}{I_{0,2,0,0}} \frac{1}{A} \int_{-\infty}^{\infty} \left\{ \kappa g(\tau) \Re\left[R e^{i\phi}\right] - B \frac{dg}{d\tau} \Im\left[R e^{i\phi}\right] \right\} \, d\tau \end{aligned} \tag{26}$$

$$\frac{d\bar{\theta}}{dt} = -\kappa + \frac{2\epsilon}{I_{0,2,0,0}} \frac{1}{A} \int_{-\infty}^{\infty} x g(\tau) \Re\left[R e^{i\phi}\right] \, d\tau \tag{27}$$

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{B^2}{2} \frac{I_{0,0,2,0}}{I_{0,2,0,0}} - \frac{\kappa^2}{2} - \frac{1}{I_{0,2,0,0}} \int_{-\infty}^{\infty} g^2(s) F\left(A^2 g^2(s)\right) \, ds \\ &\quad + \frac{\epsilon}{I_{0,2,0,0}} \frac{1}{A} \int_{-\infty}^{\infty} g(\tau) \Im\left[R e^{i\phi}\right] d\tau \end{aligned} \tag{28}$$

where we have $\tau = B(t)(\theta - \bar{\theta})$. Equations (24)–(27) gives the adiabatic dynamics of the soliton parameters in presence of a perturbation terms.

2.2. Observation

Here, in this paper, as an example, we shall consider a particular form of the perturbation term R that is given by

$$R = \delta |q|^{2m} q \tag{29}$$

The parameter m is a non-negative integer depending on the degree of nonlinearity. This kind of perturbation is commonly known as the nonlinear damping (gain) depending on the parameter m as well as the coefficient δ. It arises, for example, in the context of fiber optics as a stauration term to suppress the unbounded growth [10, 11]. So, we are going to study the equation

$$i q_t + \frac{1}{2} q_{xx} + F\left(|q|^2\right) q = i \epsilon \delta |q|^{2m} q \tag{30}$$

For this perturbation, we have the adiabatic parameter dynamics as:

$$\frac{dE}{dt} = 2\epsilon\delta\frac{A^{2m+2}}{B}I_{0,2m+2,0,0} \tag{31}$$

$$\frac{d\kappa}{dt} = 0 \tag{32}$$

$$\frac{d\bar{\theta}}{dt} = -\kappa \tag{33}$$

$$\frac{\partial\phi}{\partial t} = \frac{B^2}{2}\frac{I_{0,0,2,0}}{I_{0,2,0,0}} - \frac{\kappa^2}{2} + \frac{1}{I_{0,2,0,0}}\int_{-\infty}^{\infty} g^2(s)F\left(A^2g^2(s)\right)\, ds \tag{34}$$

3. QUASI-STATIONARY SOLUTION

3.1. Introduction

The idea of quasi-stationarity for solving the nonlinear evolution equations was first introduced in 1981 [12] by Kodama and Ablowitz. Later, this idea was extended to study the NLSE with Hamiltonian and non-Hamiltonian type perturbation [10]. This paper is a generalization of the study of quasi-stationarity to the case of NLSE with non-Kerr law nonlinearity.

The basic idea of a quasi-stationary (QS) method can be explained as follows. In a general setting, we study the solution of a perturbed nonlinear dispersive wave equation that is of the type

$$K\left(q, q_t, q_x, \cdots\right) = \epsilon F\left(q, q_x, \cdots\right) \tag{35}$$

Here, K and F are nonlinear functions of q, q_x, $\cdots$ while $0 < \epsilon \ll 1$. The unperturbed equation (for $\epsilon = 0$)

$$K\left(q^{(0)}, q_t^{(0)}, q_x^{(0)}, \cdots\right) = 0 \tag{36}$$

has a solution $q^{(0)}$ that is taken as a solitary wave or a soliton solution. We write this solution in terms of certain natural fast and slow variables as

$$q^{(0)} = \hat{q}^{(0)}\left(\theta_1, \theta_2, \cdots \theta_m, T, X : P_1, P_2, \cdots, P_N\right) \tag{37}$$

where, θ_i for $1 \leq i \leq m$ are the, so called, *fast* variables while $T = \epsilon t$ and $X = \epsilon x$ are the *slow* variables and P_l for $1 \leq l \leq N$ are the parameters that depend on the slow variables. In many problems, we need only one fast variable, namely $\theta = x - P_1 t$ in the unperturbed problem. We generalize θ to satisfy $\partial\theta/\partial x = 1$ and $\partial\theta/\partial t = -P_1$

and use $P_1 = P_1(X,T)$ to remove the secular terms. With this, we can call such a solution (44) as a quasi-stationary solution and write $q = \hat{q}(\theta, X, T, \epsilon)$. It is necessary that we develop equations for the parameters $P_1, \ldots, P_N$ by using the appropriate conditions such as the secularity conditions. There must be N such conditions. Some of these conditions are formed from Green's identity, as follows.

We assume an expression for $\hat{q}$ of the form

$$\hat{q} = \hat{q}^{(0)} + \epsilon \hat{q}^{(1)} + \cdots$$

(after introducing the appropriate variables θ_i, X, T etc.). Then (36) is the leading order problem, and (if we assume that K has only first order in time derivatives)

$$L\left(\partial_{\theta_i}, \hat{q}^{(0)}\right)\hat{q}^{(1)} = F\left(\hat{q}^{(0}\right) - \frac{\partial K}{\partial q_t} \cdot q_T = F \tag{38}$$

is the first order equation. Here $L\left(\partial_{\theta_i}, \hat{q}^{(0)}\right), u = 0$ is the linearized equation of $K(q, q_t, q_x, \cdots) = 0$ after (x,t) is transformed to the appropriate coordinate θ_i. Calling v_i the M solutions of the homogenous adjoint problem satisfying the necessary boundary conditions (e.g. $v_i \to 0$, as $|\theta_i| \to \infty$)

$$L^A v_i = 0 \tag{39}$$

for $1 \leq i \leq M$ and $M \leq N$ with L^A being the adjoint operator to L, we form

$$L\hat{q}^{(1)} \cdot v_i - \left(L^A v_i\right) \cdot \hat{q}^{(1)} = \hat{F} v_i \tag{40}$$

The left side of (38) is always a divergence (Green's Theorem). It may be integrated to give the secularity conditions also known as the Fredholm's Alternative (FA). These secularity conditions allow us to compute a solution $\hat{q}^{(1)}$ to (37) that satisfies suitable boundary conditions (e.g. $\hat{q}^{(1)}$ is bounded as $|\theta| \to \infty$). However, as it is standard in perturbation problems there is still freedom in the solution. This is due to the fact that some terms in the solution $\hat{q}^{(1)}$ can be absorbed in the leading order solution $\hat{q}^{(0)}$ by shifting other parameters. The solution $\hat{q}^{(1)}$ can be made unique by imposing additional conditions which reflect specific initial conditions or other normalizations. Continuation to higher order $\hat{q}^{(N)}$ is straightforward.

3.2. Application

In this paper, we shall obtain a quasi-stationary solution to (29) using the method that was discussed in the previous sub-section. The main part of this work is to implement a perturbation scheme to solve (29) as follows:

$$q = \hat{q}(\theta, T, X; \epsilon) e^{\frac{i}{\epsilon}\rho(T,X;\epsilon)} \tag{41}$$

where

$$\frac{\partial \theta}{\partial x} = 1, \ \frac{\partial \theta}{\partial t} = 0$$

and

$$T = \epsilon t \ X = \epsilon x$$

Here, as mentioned, θ is a fast variable while X and T are the slow variables in space and time respectively. We note that, here in (39) we are allowing a slow variation in both the spatial and time variable. When we turn on the perturbation terms of the NLSE, we have the soliton parameters A and B are slowly varying functions, namely $A = A(X, T)$ and $B = B(X, T)$.

We now substitute (39) into (29) and expand

$$\begin{aligned} \hat{q} &= \hat{q}^{(0)} + \epsilon \hat{q}^{(1)} + \epsilon^2 \hat{q}^{(2)} + \cdots \\ \rho &= \rho^{(0)} + \epsilon \rho^{(1)} + \epsilon^2 \rho^{(2)} + \cdots \\ v &= v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots \end{aligned}$$

to get at the leading order

$$-\left\{\rho_T^{(0)} + \frac{1}{2}\left(\rho_X^{(0)}\right)^2\right\}\hat{q}^{(0)} + \frac{1}{2}\frac{\partial^2 \hat{q}^{(0)}}{\partial \theta^2} + \hat{q}^{(0)} F\left[\left(\hat{q}^{(0)}\right)^2\right] = 0 \tag{42}$$

and

$$\left(\rho_X^{(0)} - v^{(0)}\right)\frac{\partial \hat{q}^{(0)}}{\partial \theta} = 0. \tag{43}$$

Now, (41) implies

$$\rho_X^{(0)} = v^{(0)} \tag{44}$$

We now set:

$$h(B^2) = \rho_T^{(0)} + \frac{1}{2}\left(\rho_X^{(0)}\right)^2 = \rho_T^{(0)} + \frac{1}{2}\left(v^{(0)}\right)^2 \tag{45}$$

where the function h depends on the nonlinearity F. Thus, (40) changes to:

$$-h(B^2)\hat{q}^{(0)} + \frac{1}{2}\frac{\partial^2 \hat{q}^{(0)}}{\partial \theta^2} + \hat{q}^{(0)} F\left[\left(\hat{q}^{(0)}\right)^2\right] = 0 \tag{46}$$

whose solution is (on comparing with (8)):

$$\hat{q}^{(0)} = Ag[B(\theta - \bar{\theta})] \tag{47}$$

where

$$\frac{d\bar{\theta}}{dt} = v \tag{48}$$

At $O(\epsilon)$, we decompose $\hat{q}^{(1)} = \hat{\phi}^{(1)} + i\hat{\psi}^{(1)}$ into its real and imaginary parts. Now, the equations for $\hat{\phi}^{(1)}$ and $\hat{\psi}^{(1)}$, by virtue of (43), are respectively:

$$\begin{aligned} -h(B^2)\hat{\phi}^{(1)} + \frac{1}{2}\frac{\partial^2 \hat{\phi}^{(1)}}{\partial \theta^2} + 2\left(\hat{q}^{(0)}\right)^2 \hat{\phi}^{(1)} F'\left[\left(\hat{q}^{(0)}\right)^2\right] \\ +\hat{\phi}^{(1)} F\left[\left(\hat{q}^{(0)}\right)^2\right] = \left\{\rho_T^{(1)} + v^{(0)}\rho_X^{(1)}\right\}\hat{q}^{(0)} - \frac{\partial^2 \hat{q}^{(0)}}{\partial \theta \partial X} \end{aligned} \tag{49}$$

and

$$\begin{aligned} -h(B^2)\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2 \hat{\psi}^{(1)}}{\partial \theta^2} + F\left[\left(\hat{q}^{(0)}\right)^2\right]\hat{\psi}^{(1)} = \\ -\frac{\partial \hat{q}^{(0)}}{\partial T} - v^{(0)}\frac{\partial \hat{q}^{(0)}}{\partial X} - \left\{\rho_X^{(1)} - v^{(1)}\right\}\frac{\partial \hat{q}^{(0)}}{\partial \theta} - \rho_{XX}^{(0)}\hat{q}^{(0)} + \delta\left(\hat{q}^{(0)}\right)^{2m+1} \end{aligned} \tag{50}$$

By FA, applied to (47), we get:

$$B\frac{\partial A}{\partial X}I_{0,0,2,0} + A\frac{\partial B}{\partial X}I_{0,0,2,0} + A\frac{\partial B}{\partial X}I_{1,0,0,1} = 0 \tag{51}$$

and

$$\rho_T^{(1)} + v^{(0)}\rho_X^{(1)} = 0 \tag{52}$$

whereas, if applied to (48), yields:

$$\begin{aligned} B\frac{\partial A}{\partial T}I_{0,0,2,0} + A\frac{\partial B}{\partial T}I_{1,1,1,0} + v^{(0)}B\frac{\partial A}{\partial X}I_{0,0,2,0} + v^{(0)}A\frac{\partial B}{\partial X}I_{1,1,1,0} = \\ \delta A^{2m+1}BI_{0,2m+2,0,0} - \rho_{XX}^{(0)}ABI_{0,0,2,0} \end{aligned} \tag{53}$$

and

$$\rho_X^{(1)} = v^{(1)} \tag{54}$$

Since $A(t)$ and $B(t)$ are related, depending on the functional form of $F(s)$, (49) leads to the conclusion

$$\frac{\partial A}{\partial X} = \frac{\partial B}{\partial X} = 0 \tag{55}$$

so that A and B are functions of T only.

Also, in an ideal soliton-based communication system in fiber optics, input pulses launched into the fiber should be unchirped in order to avoid shedding part of the pulse energy as a dispersive tail during the process of soliton formation [13]. So, in (51) we use (53) and we set $\rho^{(0)}_{XX} = 0$ to eliminate frequency chirp to give

$$B\frac{dA}{dT}I_{0,0,2,0} + A\frac{dB}{dT}I_{1,1,1,0} = \delta A^{2m+1}BI_{0,2m+2,0,0} \tag{56}$$

Now (47), by virtue of (50) and (53), reduces to:

$$-h(B^2)\hat{\phi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\phi}^{(1)}}{\partial\theta^2} + 2\left(\hat{q}^{(0)}\right)^2\hat{\phi}^{(1)}F'\left[\left(\hat{q}^{(0)}\right)^2\right] + \hat{\phi}^{(1)}F\left[\left(\hat{q}^{(0)}\right)^2\right] = 0 \tag{57}$$

while (48), by virtue of (52) and (53), gives:

$$-h(B^2)\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\psi}^{(1)}}{\partial\theta^2} + F\left[\left(\hat{q}^{(0)}\right)^2\right]\hat{\psi}^{(1)} = -\frac{\partial\hat{q}^{(0)}}{\partial T} + \delta\left(\hat{q}^{(0)}\right)^{2m+1} \tag{58}$$

The solutions to (55) and (56) are respectively:

$$\hat{\phi}^{(1)} = 0 \tag{59}$$

and

$$\begin{aligned}
\hat{\psi}^{(1)} = {} & \frac{2A}{B}\frac{\partial\bar{\theta}}{\partial T}g(\tau)\int^{\tau}\frac{1}{g^2(s_2)}\left(\int^{s_2}g(s_1)\,g'(s_1)\,ds_1\right)ds_2 \\
& -\frac{2}{B^2}\frac{dA}{dT}g(\tau)\int^{\tau}\frac{1}{g^2(s_2)}\left(\int^{s_2}g^2(s_1)\,ds_1\right)ds_2 \\
& -\frac{2}{B^3}\frac{dB}{dT}g(\tau)\int^{\tau}\frac{1}{g^2(s_2)}\left(\int^{s_2}s_1 g(s_1)\,g'(s_1)\,ds_1\right)ds_2 \\
& +2\delta\frac{A^{2m+1}}{B^2}g(\tau)\int^{\tau}\frac{1}{g^2(s_2)}\left(\int^{s_2}g^{2m+2}(s_1)\,ds_1\right)ds_2
\end{aligned} \tag{60}$$

The $O(\epsilon)$ solution of (29) finally is:

$$q \approx Pe^{i\psi} \tag{61}$$

where

$$P = \hat{q}^{(0)} \tag{62}$$

and

$$\psi = \epsilon Q(\theta) + \frac{1}{\epsilon}\rho(X, T) \tag{63}$$

with

$$Q(\theta) = \hat{\psi}^{(1)}/\hat{q}^{(0)} \tag{64}$$

Equation (59) is finally the quasi-stationary solution to (29).

4. OBSERVATION

In this section, we shall implement the quasi-stationary solution, that was derived in the previous section, to three particular cases of the function F given in (1), apart from the Kerr law itself. The three cases are the power law, parabolic law and the dual-power law. These are studied in details in the following two subsections.

4.1. Kerr Law

In this case, we have $F(s) = s$ and so we get $f(s) = s^2/2$. So, the GNLSE modifies to

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q = 0 \tag{65}$$

This case, as we have mentioned, is integrable by the IST [2]. The form of the soliton is given by

$$q(x, t) = \frac{A}{\cosh\left[B(x - \bar{x}(t))\right]} e^{i(-\kappa x + \omega t + \sigma_0)} \tag{66}$$

where $A = B$ and

$$\kappa = -v \tag{67}$$

and

$$\omega = \frac{A^2 - \kappa^2}{2} \tag{68}$$

Here, κ is called the soliton frequency while ω is the wave number of the soliton and σ_0 is the center of phase of the soliton. The Kerr law of nonlinearity arises in the context of fiber optics where the propagation of solitons or pulses is governed by the NLSE given by (60). It is also studied in the context of Fluid Dynamics. Here, the corresponding parameter dynamics are:

$$\frac{dA}{dt} = 0 \tag{69}$$

$$\frac{dB}{dt} = 0 \tag{70}$$

$$\frac{d\kappa}{dt} = 0 \tag{71}$$

$$\frac{d\bar{x}}{dt} = -\kappa \tag{72}$$

In the Kerr law case, however, we have infinitely many integrals of motion. The first three integrals of motion that matches with those of the GNLSE are respectively

$$E = \int_{-\infty}^{\infty} |q|^2 \, dx = 2A \tag{73}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} (qq_x^* - q^* q_x) \, dx = -2\kappa A \tag{74}$$

$$H = \frac{1}{2}\int_{-\infty}^{\infty} (|q_x|^2 - |q|^4) \, dx = 2\left(\kappa^2 A - \frac{1}{3}A^3\right) \tag{75}$$

4.1.1. Perturbation Term

Here, we are going to study the perturbed equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q = i\epsilon\delta \, |q|^{2m} \, q \tag{76}$$

For the Kerr law case, we shall use the soliton given by (61) in equations (30)–(33) and in (62)–(63) to give

$$\frac{dE}{dt} = 2\epsilon\delta A^{2m+1} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+1\right)}{\Gamma\left(m+\frac{3}{2}\right)} \tag{77}$$

$$\frac{dA}{dt} = \epsilon\delta A^{2m+1} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+1\right)}{\Gamma\left(m+\frac{3}{2}\right)} \tag{78}$$

$$\frac{dB}{dt} = \epsilon\delta B^{2m+1} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+1\right)}{\Gamma\left(m+\frac{3}{2}\right)} \tag{79}$$

$$\frac{d\kappa}{dt} = 0 \tag{80}$$

$$\frac{d\bar{x}}{dt} = -\kappa \tag{81}$$

Here, $\Gamma(x)$ is the usual gamma function.

4.1.2. Quasistationarity

For obtaining a quasi-stationary solution to (71) we use the same asnatz as given by (39). Then, by virtue of Section 3.2, we get

$$-\left\{\rho_T^{(0)} + \frac{1}{2}\left(\rho_X^{(0)}\right)^2\right\}\hat{q}^{(0)} + \frac{1}{2}\frac{\partial^2 \hat{q}^{(0)}}{\partial\theta^2} + \left(\hat{q}^{(0)}\right)^3 = 0 \tag{82}$$

and

$$\left(\rho_X^{(0)} - v^{(0)}\right)\frac{\partial \hat{q}^{(0)}}{\partial\theta} = 0 \tag{83}$$

Now, (79) implies

$$\rho_X^{(0)} = v^{(0)} \tag{84}$$

We now set:

$$\frac{B^2}{2} = \rho_T^{(0)} + \frac{1}{2}\left(\rho_X^{(0)}\right)^2 = \rho_T^{(0)} + \frac{1}{2}\left(v^{(0)}\right)^2 \tag{85}$$

so that (78) changes to:

$$-\frac{B^2}{2}\hat{q}^{(0)} + \frac{1}{2}\frac{\partial^2 \hat{q}^{(0)}}{\partial\theta^2} + \left(\hat{q}^{(0)}\right)^3 = 0 \tag{86}$$

whose solution is:

$$\hat{q}^{(0)} = \frac{A}{\cosh\left\{B(\theta - \bar{\theta})\right\}} \tag{87}$$

where $A = B$ and

$$\frac{d\bar{\theta}}{dt} = v \tag{88}$$

At $O(\epsilon)$ level, we get on decomposing $\hat{q}^{(1)} = \hat{\phi}^{(1)} + i\hat{\psi}^{(1)}$

$$-\frac{B^2}{2}\hat{\phi}^{(1)} + \frac{1}{2}\frac{\partial^2 \hat{\phi}^{(1)}}{\partial\theta^2} + 3\left(\hat{q}^{(0)}\right)^2\hat{\phi}^{(1)} = \left\{\rho_T^{(1)} + v^{(0)}\rho_X^{(1)}\right\}\hat{q}^{(0)} - \frac{\partial^2 \hat{q}^{(0)}}{\partial\theta\partial X} \tag{89}$$

and

$$-\frac{B^2}{2}\hat{\psi}^{(1)}+\frac{1}{2}\frac{\partial^2\hat{\psi}^{(1)}}{\partial\theta^2}+\left(\hat{q}^{(0)}\right)^2\hat{\psi}^{(1)}=$$
$$-\frac{\partial\hat{q}^{(0)}}{\partial T}-v^{(0)}\frac{\partial\hat{q}^{(0)}}{\partial X}-\left\{\rho_X^{(1)}-v^{(1)}\right\}\frac{\partial\hat{q}^{(0)}}{\partial\theta}-\rho_{XX}^{(0)}\hat{q}^{(0)}+\delta\left(\hat{q}^{(0)}\right)^{2m+1} \quad (90)$$

Here, as discussed in the previous section, we set $\rho_{XX}^{(0)}=0$ in (86) to eliminate frequency chirp to give:

$$-\frac{B^2}{2}\hat{\psi}^{(1)}+\frac{1}{2}\frac{\partial^2\hat{\psi}^{(1)}}{\partial\theta^2}+\left(\hat{q}^{(0)}\right)^2\hat{\psi}^{(1)}=$$
$$-\frac{\partial\hat{q}^{(0)}}{\partial T}-v^{(0)}\frac{\partial\hat{q}^{(0)}}{\partial X}-\left\{\rho_X^{(1)}-v^{(1)}\right\}\frac{\partial\hat{q}^{(0)}}{\partial\theta}+\delta\left(\hat{q}^{(0)}\right)^{2m+1} \quad (91)$$

The FA, when applied to (85) gives

$$\frac{\partial B}{\partial X}=0 \quad (92)$$

and

$$\rho_T^{(1)}+v^{(0)}\rho_X^{(1)}=0 \quad (93)$$

whereas if, applied to (87), gives

$$\frac{dB}{dT}=\delta B^{2m+1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(m+1)}{\Gamma\left(m+\frac{3}{2}\right)} \quad (94)$$

and

$$\rho_X^{(1)}=v^{(1)} \quad (95)$$

Equation (88) shows that B is a function of T only and so is A, since $A=B$. Thus, these $\mathrm{O}(\epsilon)$ equations reduce to:

$$-\frac{B^2}{2}\hat{\phi}^{(1)}+\frac{1}{2}\frac{\partial^2\hat{\phi}^{(1)}}{\partial\theta^2}+3\left(\hat{q}^{(0)}\right)^2\hat{\phi}^{(1)}=0 \quad (96)$$

and

$$-\frac{B^2}{2}\hat{\psi}^{(1)}+\frac{1}{2}\frac{\partial^2\hat{\psi}^{(1)}}{\partial\theta^2}+\left(\hat{q}^{(0)}\right)^2\hat{\psi}^{(1)}=-\frac{\partial\hat{q}^{(0)}}{\partial T}+\delta\left(\hat{q}^{(0)}\right)^{2m+1} \quad (97)$$

whose solutions are respectively

$$\hat{\phi}^{(1)} = 0 \tag{98}$$

and

$$\begin{aligned}\hat{\psi}^{(1)} &= \frac{\partial \bar{\theta}}{\partial T}\frac{\tau}{\cosh\tau} - \frac{1}{2B^2}\frac{dB}{dT}\left(\frac{\tau^2}{\cosh\tau} + \cosh\tau\right) \\ &\quad + 2\delta\frac{B^{2m-1}}{\cosh\tau}\int^{\tau}\cosh^2 s_2\left(\int^{s_2}\frac{1}{\cosh^{2m+2}s_1}\,ds_1\right)\,ds_2\end{aligned} \tag{99}$$

which leads to the QS solution (59) for the Kerr law of nonlinearity.

4.2. Power Law

Here, we have $F(s) = s^p$ so that $f(s) = s^{p+1}/(p+1)$. We note that if we set $p = 1$, we recover the case of Kerr law nonlinearity. The GNLSE reduces to

$$iq_t + \frac{1}{2}q_{xx} + |q|^{2p}q = 0 \tag{100}$$

We note that here in (96) we have to have $0 < p < 2$ to avoid wave collapse and, in particular, we need $p \neq 2$ to avoid the issue of self-focussing singularity [2]. We write the soliton solution of (96) as [14]:

$$q(x,t) = \frac{A}{\cosh^{\frac{1}{p}}\left[B(x - \bar{x}(t))\right]}e^{i(-\kappa x + \omega t + \sigma_0)} \tag{101}$$

where

$$B = A^p\left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}} \tag{102}$$

and

$$\kappa = -v \tag{103}$$

with

$$\omega = \frac{B^2}{2p^2} - \frac{\kappa^2}{2} \tag{104}$$

The power law of nonlinearity arises in nonlinear plasmas that solves the problem of small K-condensation in weak turbulence theory. It also arises in the context of nonlinear optics. Physically, various materials including semiconductors, exhibit power law nonlinearities [15]. The corresponding parameter dynamics for the solitons are given by

$$\frac{dA}{dt} = 0 \tag{105}$$

$$\frac{dB}{dt} = 0 \tag{106}$$

$$\frac{d\kappa}{dt} = 0 \tag{107}$$

$$\frac{d\bar{x}}{dt} = -\kappa \tag{108}$$

The three integrals of motion, in this case, are respectively given by [14]

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |q|^2 \, dx = A^{2-p} \left(\frac{1+p}{2p^2}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} \\ &= B^{\frac{2-p}{p}} \left(\frac{1+p}{2p^2}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} \end{aligned} \tag{109}$$

$$\begin{aligned} M &= i\int_{-\infty}^{\infty} (q^* q_x - q q_x^*) \, dx \\ &= 2\kappa A^{2-p} \left(\frac{1+p}{2p^2}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} \\ &= 2\kappa B^{\frac{2-p}{p}} \left(\frac{1+p}{2p^2}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} \end{aligned} \tag{110}$$

and

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \left[\frac{1}{2}|q_x|^2 - \frac{1}{p+1}|q|^{2p+2}\right] dx \\ &= \frac{B^{\frac{2}{p}}}{2p^2}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{p}} \left[\frac{(B^2+\kappa^2 p^2)}{B} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} - 2B\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+1}{p}\right)}{\Gamma\left(\frac{p+1}{p}+\frac{1}{2}\right)}\right] \\ &= \frac{A^2}{2p^2}\left[\left\{A^p\left(\frac{2p^2}{1+p}\right)^{\frac{1}{p}} + \frac{\kappa^2 p^2}{A^p}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2}}\right\} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}\right. \end{aligned}$$

$$-2A^p\left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+1}{p}\right)}{\Gamma\left(\frac{p+1}{p}+\frac{1}{2}\right)}\Bigg] \tag{111}$$

4.2.1. Perturbation Terms

Now, we are going to study the perturbed NLSE as

$$iq_t + \frac{1}{2}q_{xx} + |q|^{2p}q = i\epsilon\delta\,|q|^{2m}\,q \tag{112}$$

For the power law case, we use the soliton form (97) in (30)-(33) and (99) to obtain the adiabatic parameter dynamics as [8]:

$$\begin{aligned}
\frac{dE}{dt} &= 2\epsilon\delta\left(\frac{1+p}{2p^2}\right)^{\frac{m+1}{p}} B^{\frac{2m+2-p}{p}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m+1}{p}\right)}{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)} \\
&= 2\epsilon\delta A^{2m-p+2}\left(\frac{2p^2}{1+p}\right)^{\frac{2m^2-2p^2+4m-mp-p+2}{2p(m+1)}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m+1}{p}\right)}{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)} && (113)\\
\frac{dA}{dt} &= \frac{2\epsilon\delta}{2-p}A^{2m+1}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2p}}\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)}\frac{\Gamma\left(\frac{m+1}{p}\right)}{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)} && (114)\\
\frac{dB}{dt} &= \frac{2\epsilon\delta p}{2-p}B^{\frac{2m+p}{p}}\left(\frac{1+p}{2p^2}\right)^{\frac{m}{p}}\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)}\frac{\Gamma\left(\frac{m+1}{p}\right)}{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)} && (115)\\
\frac{d\bar{x}}{dt} &= -\kappa && (116)\\
\frac{d\kappa}{dt} &= 0 && (117)
\end{aligned}$$

As pointed out, we see in (110) and (111) that we need $p \neq 2$ to avoid the self-focussing singularity issue.

4.2.2. Quasistationarity

If we apply the quasi-stationarity asnatz to (108) we get the leading order equation as

$$-\left\{\rho_T^{(0)} + \frac{1}{2}(\rho_X^{(0)})^2\right\}\hat{q}^{(0)} + \frac{1}{2}\frac{\partial^2\hat{q}^{(0)}}{\partial\theta^2} + \left(\hat{q}^{(0)}\right)^{2p+1} = 0 \tag{118}$$

and

$$\left(\rho_X^{(0)} - v^{(0)}\right) \frac{\partial \hat{q}^{(0)}}{\partial \theta} = 0 \tag{119}$$

Now (116) implies

$$\rho_X^{(0)} = v^{(0)} \tag{120}$$

For the power law nonlinearity, we set

$$\frac{B^2}{2p^2} = \rho_T^{(0)} + \frac{1}{2}\left(\rho_X^{(0)}\right)^2 = \rho_T^{(0)} + \frac{1}{2}\left(v^{(0)}\right)^2 \tag{121}$$

so that (115) changes to:

$$-\frac{B^2}{2p^2}\hat{q}^{(0)} + \frac{1}{2}\frac{\partial^2 \hat{q}^{(0)}}{\partial \theta^2} + \left(\hat{q}^{(0)}\right)^{2p+1} = 0 \tag{122}$$

whose solution is:

$$\hat{q}^{(0)} = \frac{A}{\cosh^{\frac{1}{p}}\left\{B(\theta - \bar{\theta})\right\}} \tag{123}$$

where

$$B = A^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}} \tag{124}$$

and

$$\frac{d\bar{\theta}}{dt} = v \tag{125}$$

The $O(\epsilon)$ equations, in this case, reduce to

$$\begin{aligned} -\frac{B^2}{2p^2}\hat{\phi}^{(1)} + \frac{1}{2}\frac{\partial^2 \hat{\phi}^{(1)}}{\partial \theta^2} + (2p+1)\left(\hat{q}^{(0)}\right)^{2p}\hat{\phi}^{(1)} = \\ \left(\rho_T^{(1)} + v^{(0)}\rho_X^{(1)}\right)\hat{q}^{(0)} - \frac{\partial^2 \hat{q}^{(0)}}{\partial \theta \partial X} \end{aligned} \tag{126}$$

and

$$\begin{aligned} -\frac{B^2}{2p^2}\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2 \hat{\psi}^{(1)}}{\partial \theta^2} + \left(\hat{q}^{(0)}\right)^{2p}\hat{\psi}^{(1)} = \\ -\frac{\partial \hat{q}^{(0)}}{\partial T} - v^{(0)}\frac{\partial \hat{q}^{(0)}}{\partial X} - \rho_{XX}^{(0)}\hat{q}^{(0)} + \left(v^{(1)} - \rho_X^{(1)}\right)\frac{\partial \hat{q}^{(0)}}{\partial \theta} + \delta\left(\hat{q}^{(0)}\right)^{2m+1} \end{aligned} \tag{127}$$

Again, in (124) we set $\rho^{(0)}_{XX} = 0$ to eliminate the frequency chirp as explained, in the previous section.

$$-\frac{B^2}{2p^2}\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\psi}^{(1)}}{\partial\theta^2} + \left(\hat{q}^{(0)}\right)^{2p}\hat{\psi}^{(1)} = -\frac{\partial\hat{q}^{(0)}}{\partial T} - v^{(0)}\frac{\partial\hat{q}^{(0)}}{\partial X} + \left(v^{(1)} - \rho^{(1)}_X\right)\frac{\partial\hat{q}^{(0)}}{\partial\theta} + \delta\left(\hat{q}^{(0)}\right)^{2m+1} \tag{128}$$

The FA applied to (123), gives:

$$\frac{\partial B}{\partial X} = 0 \tag{129}$$

and

$$\rho^{(1)}_T + v^{(0)}\rho^{(1)}_X = 0 \tag{130}$$

whereas if, applied to (125) gives

$$\frac{dB}{dT} = \frac{2\delta p}{2-p}\frac{(1+p)^{\frac{m}{p}}}{(2p^2)^{\frac{m}{p}}}B^{\left(\frac{2m+p}{p}\right)}\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)}\frac{\Gamma\left(\frac{m+1}{p}\right)}{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)} \tag{131}$$

and

$$\rho^{(1)}_X = v^{(1)} \tag{132}$$

By virtue of (126) we get

$$\frac{\partial A}{\partial X} = 0 \tag{133}$$

Thus, we see from (126) and (130) that A and B are functions of T alone. From (121) and (128) we get

$$\frac{dA}{dT} = \frac{2\delta}{2-p}A^{2m+1}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2p}}\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)}\frac{\Gamma\left(\frac{m+1}{p}\right)}{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)} \tag{134}$$

The O(ϵ) equations are now

$$-\frac{B^2}{2p^2}\hat{\phi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\phi}^{(1)}}{\partial\theta^2} + (2p+1)\left(\hat{q}^{(0)}\right)^{2p}\hat{\phi}^{(1)} = 0 \tag{135}$$

and

$$-\frac{B^2}{2p^2}\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\psi}^{(1)}}{\partial\theta^2} + \left(\hat{q}^{(0)}\right)^{2p}\hat{\psi}^{(1)} = -\frac{\partial\hat{q}^{(0)}}{\partial T} + \delta\left(\hat{q}^{(0)}\right)^{2m+1} \tag{136}$$

whose solutions are respectively

$$\hat{\phi}^{(1)} = 0 \tag{137}$$

and

$$\begin{aligned}\hat{\psi}^{(1)} = &-\frac{2B^{\frac{1}{p}}}{p}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2p}}\\ &\left[\frac{\partial\bar{\theta}}{\partial T}\frac{\tau}{\cosh^{\frac{1}{p}}\tau}\int^{\tau}\cosh^{\frac{2}{p}} s_2\left(\int^{s_2}\frac{\tanh s_1}{\cosh^{\frac{2}{p}} s_1}ds_1\right)ds_2\right.\\ &-\frac{1}{B^2}\frac{dB}{dT}\int^{\tau}\cosh^{\frac{2}{p}} s_2\left(\int^{s_2}\frac{\tanh s_1}{\cosh^{\frac{2}{p}} s_1}ds_1\right)ds_2\\ &\left.+\frac{1}{B^3}\frac{dB}{dT}\int^{\tau}\cosh^{\frac{2}{p}} s_2\left(\int^{s_2}\frac{\tanh s_1}{\cosh^{\frac{2}{p}} s_1}ds_1\right)ds_2\right]\\ &+2\delta B^{\frac{2m+1-2p}{p}}\left(\frac{1+p}{2p^2}\right)^{\frac{2m+1}{2p}}\int^{\tau}\cosh^{\frac{2}{p}} s_2\left(\int^{s_2}\frac{\tanh s_1}{\cosh^{\frac{2m+2}{p}} s_1}ds_1\right)ds_2\end{aligned} \tag{138}$$

Equation (135) now leads to the QS solution (59) for the power law of nonlinearity.

4.3. Parabolic Law

In this case, we have $F(s) = s + \nu s^2$ where ν is a constant, so that we get $f(s) = s^2/2 + \nu s^3/3$. The form of the GNLSE here is

$$iq_t + \frac{1}{2}q_{xx} + \left(|q|^2 + \nu\,|q|^4\right)q = 0 \tag{139}$$

The solution of (136) is now written as

$$q(x,t) = \frac{A}{\left[1 + a\cosh\left\{B\left(x - \bar{x}(t)\right)\right\}\right]^{\frac{1}{2}}}e^{i(-\kappa x + \omega t + \sigma_0)} \tag{140}$$

where

$$a = \sqrt{1 + \frac{4}{3}\nu A^2} \tag{141}$$

and

$$B(t) = \sqrt{2}A(t) \tag{142}$$

while

$$\kappa = -v \tag{143}$$

$$\omega = \frac{A^2}{4} - \frac{\kappa^2}{2} \tag{144}$$

This law is for constant ν is also known as the cubic-quintic nonlinearity. The term with ν is large for the case of p-toluene sulfonate crystals. It arises in the nonlinear interaction between Langmuir waves and electrons and describes the nonlinear interaction between the high frequency Langmuir waves and the ion-acoustic waves by pondermotive forces. Furthermore, the parabolic law of nonlinearity arises in the context of Fiber Optics to consider nonlinearities higher than the third order to obtain some knowledge of the diameter of the self-trapping beam. The existence of a significant $\chi^{(5)}$ nonlinearity effects for transparent glass in intense femtosecond pulses at 620 nm has been experimentally demonstrated [16]. The corresponding parameter dynamics is given by

$$\frac{dA}{dt} = 0 \tag{145}$$

$$\frac{dB}{dt} = 0 \tag{146}$$

$$\frac{d\kappa}{dt} = 0 \tag{147}$$

$$\frac{d\bar{x}}{dt} = -\kappa \tag{148}$$

The three integrals of motion for the parabolic law are

$$E = \int_{-\infty}^{\infty} |q|^2 \, dx = \begin{cases} \sqrt{\frac{3}{2\nu}} \tan^{-1}\left[2A\sqrt{\frac{\nu}{3}}\right] & : \ 0 < \nu < \infty \\ \sqrt{-\frac{3}{2\nu}} \tanh^{-1}\left[2A\sqrt{-\frac{\nu}{3}}\right] & : \ -\frac{3}{4A^2} < \nu < 0 \end{cases} \tag{149}$$

$$\begin{aligned} M &= \frac{i}{2}\int_{-\infty}^{\infty} \left(q q_x^* - q^* q_x\right) \, dx \\ &= \begin{cases} -\frac{\kappa}{2}\sqrt{\frac{3}{2\nu}} \tan^{-1}\left[2A\sqrt{\frac{\nu}{3}}\right] & : \ 0 < \nu < \infty \\ -\frac{\kappa}{2}\sqrt{-\frac{3}{2\nu}} \tanh^{-1}\left[2A\sqrt{-\frac{\nu}{3}}\right] & : \ -\frac{3}{4A^2} < \nu < 0 \end{cases} \end{aligned} \tag{150}$$

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \left[\frac{1}{2}|q_x|^2 - \frac{1}{2}|q|^4 - \frac{\nu}{3}|q|^6\right] dx \\ &= \begin{cases} -\frac{3A}{8\sqrt{2\nu}} + \frac{3}{8\nu}\sqrt{\frac{3}{2\nu}} \tan^{-1}\left[\frac{-\sqrt{3}+\sqrt{3+4\nu A^2}}{2A\sqrt{\nu}}\right] & : 0 < \nu < \infty \\ -\frac{3A}{8\sqrt{2\nu}} - \frac{3}{8\nu}\sqrt{-\frac{3}{2\nu}} \tanh^{-1}\left[\frac{-\sqrt{3}+\sqrt{3-4\nu A^2}}{2A\sqrt{-\nu}}\right] & : -\frac{3}{4A^2} < \nu < 0 \end{cases} \end{aligned} \tag{151}$$

4.3.1. Perturbation Terms

Here, we are going to study the perturbed GNLSE with parabolic law nonlinearity, namely

$$iq_t + \frac{1}{2}q_{xx} + \left(|q|^2 + \nu\, |q|^4\right) q = i\epsilon\delta|q|^{2m}q \tag{152}$$

The adiabatic parameter dynamics here are

$$\frac{dE}{dt} = 2\epsilon\delta\frac{A^{2m+2}}{B}\int_{-\infty}^{\infty}\frac{1}{(1+a\cosh\tau)^{m+1}}d\tau \tag{153}$$

$$\frac{dA}{dt} = \frac{\sqrt{2}}{3}\epsilon\delta\frac{A^{2m+2}}{B}\left(3+4\nu A^2\right)\int_{-\infty}^{\infty}\frac{1}{(1+a\cosh\tau)^{m+1}}d\tau \tag{154}$$

$$\frac{dB}{dt} = \frac{2}{3}\epsilon\delta\frac{A^{2m+2}}{B}\left(3+4\nu A^2\right)\int_{-\infty}^{\infty}\frac{1}{(1+a\cosh\tau)^{m+1}}d\tau \tag{155}$$

$$\frac{d\kappa}{dt} = 0 \tag{156}$$

$$\frac{d\bar{x}}{dt} = -\kappa \tag{157}$$

4.3.2. Quasistationarity

For obtaining a quasi-stationary solution to (148) we use the same asnatz as given by (39) and we arrive at

$$-\left\{\rho_T^{(0)} + \frac{1}{2}\left(\rho_X^{(0)}\right)^2\right\}\hat{q}^{(0)} + \frac{1}{2}\frac{\partial^2\hat{q}^{(0)}}{\partial\theta^2} + \left(\hat{q}^{(0)}\right)^3 + \nu\left(\hat{q}^{(0)}\right)^5 = 0 \tag{158}$$

and

$$\left(\rho_X^{(0)} - v^{(0)}\right)\frac{\partial\hat{q}^{(0)}}{\partial\theta} = 0 \tag{159}$$

Now, (158) implies

$$\rho_X^{(0)} = v^{(0)} \tag{160}$$

We set

$$\frac{B^2}{8} = \rho_T^{(0)} + \frac{1}{2}\left(\rho_X^{(0)}\right)^2 = \rho_T^{(0)} + \frac{1}{2}\left(v^{(0)}\right)^2 \tag{161}$$

so that (157) changes to

$$-\frac{B^2}{8}\hat{q}^{(0)} + \frac{1}{2}\frac{\partial^2\hat{q}^{(0)}}{\partial\theta^2} + \left(\hat{q}^{(0)}\right)^3 + \nu\left(\hat{q}^{(0)}\right)^5 = 0 \tag{162}$$

whose solution is

$$\hat{q}^{(0)} = Ag\left[B(\theta - \bar{\theta})\right] \tag{163}$$

where

$$g(\tau) = \frac{1}{(1 + a\cosh\tau)^{\frac{1}{2}}} \tag{164}$$

and

$$B = A\sqrt{2} \tag{165}$$

along with

$$\tau = B(t)(\theta - \bar{\theta}) \tag{166}$$

while

$$\frac{d\bar{\theta}}{dt} = v \tag{167}$$

At $O(\epsilon)$ level, we get

$$-\frac{B^2}{8}\hat{\phi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\phi}^{(1)}}{\partial\theta^2} + \hat{\phi}^{(1)}\left\{3\left(\hat{q}^{(0)}\right)^2 + 5\nu\left(\hat{q}^{(0)}\right)^4\right\} = \left\{\rho_T^{(1)} + v^{(0)}\rho_X^{(1)}\right\}\hat{q}^{(0)} - \frac{\partial^2\hat{q}^{(0)}}{\partial\theta\partial X} \tag{168}$$

and

$$-\frac{B^2}{8}\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\psi}^{(1)}}{\partial\theta^2} + \left\{\left(\hat{q}^{(0)}\right)^2 + \nu\left(\hat{q}^{(0)}\right)^4\right\}\hat{\psi}^{(1)} = -\frac{\partial\hat{q}^{(0)}}{\partial T} - v^{(0)}\frac{\partial\hat{q}^{(0)}}{\partial X} - \left\{\rho_X^{(1)} - v^{(1)}\right\}\frac{\partial\hat{q}^{(0)}}{\partial\theta} - \rho_{XX}^{(0)}\hat{q}^{(0)} + \delta\left(\hat{q}^{(0)}\right)^{2m+1} \tag{169}$$

Here, as discussed in the previous section, we set $\rho_{XX}^{(0)} = 0$ in (166) to eliminate frequency chirp, to give:

$$-\frac{B^2}{8}\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\psi}^{(1)}}{\partial\theta^2} + \left\{\left(\hat{q}^{(0)}\right)^2 + \nu\left(\hat{q}^{(0)}\right)^4\right\}\hat{\psi}^{(1)} = -\frac{\partial\hat{q}^{(0)}}{\partial T} - v^{(0)}\frac{\partial\hat{q}^{(0)}}{\partial X} - \left\{\rho_X^{(1)} - v^{(1)}\right\}\frac{\partial\hat{q}^{(0)}}{\partial\theta} + \delta\left(\hat{q}^{(0)}\right)^{2m+1} \tag{170}$$

The FA, when applied to (165) gives

$$\frac{\partial B}{\partial X} = 0 \tag{171}$$

and

$$\rho_T^{(1)} + v^{(0)}\rho_X^{(1)} = 0 \tag{172}$$

whereas if, applied to (167), gives

$$\frac{dB}{dT} = \frac{2}{3}\delta\frac{A^{2m+2}}{B}\left(3 + 4\nu A^2\right)\int_{-\infty}^{\infty}\frac{1}{(1 + a\cosh\tau)^{m+1}}d\tau \tag{173}$$

and

$$\rho_X^{(1)} = v^{(1)} \tag{174}$$

By virtue of () we therefore get

$$\frac{dA}{dT} = \frac{\sqrt{2}}{3}\delta\frac{A^{2m+2}}{B}\left(3 + 4\nu A^2\right)\int_{-\infty}^{\infty}\frac{1}{(1 + a\cosh\tau)^{m+1}}d\tau \tag{175}$$

Equation (168) shows that B is a function of T only and so is A by virtue of (138). Thus, these O(ϵ) equations reduce to:

$$-\frac{B^2}{8}\hat{\phi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\phi}^{(1)}}{\partial\theta^2} + \hat{\phi}^{(1)}\left\{3\left(\hat{q}^{(0)}\right)^2 + 5\nu\left(\hat{q}^{(0)}\right)^4\right\} = 0 \tag{176}$$

and

$$-\frac{B^2}{8}\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\psi}^{(1)}}{\partial\theta^2} + \left\{\left(\hat{q}^{(0)}\right)^2 + \nu\left(\hat{q}^{(0)}\right)^4\right\}\hat{\psi}^{(1)} = -\frac{\partial\hat{q}^{(0)}}{\partial T} + \delta\left(\hat{q}^{(0)}\right)^{2m+1} \tag{177}$$

whose solutions are respectively

$$\hat{\phi}^{(1)} = 0 \tag{178}$$

and

$$\begin{aligned}\hat{\psi}^{(1)} = & -\frac{1}{\sqrt{2}}\frac{\partial\bar{\theta}}{\partial T}\frac{1}{(1 + a\cosh\tau)^{\frac{1}{2}}} \\ & \int^{\tau}(1 + a\cosh s_2)\left(\int^{s_2}\frac{a\sinh s_1}{1 + a\cosh s_1}\,ds_1\right)\,ds_2 \\ & + \frac{1}{2A^2}\frac{dA}{dT}\frac{1}{(1 + a\cosh\tau)^{\frac{1}{2}}}\end{aligned}$$

$$\int^{\tau}(1+a\cosh s_2)\left(\int^{s_2}\frac{as_1\sinh s_1}{(1+a\cosh s_1)^2}\,ds_1\right)\,ds_2$$
$$-\frac{1}{2A^2}\frac{dA}{dT}\frac{1}{(1+a\cosh\tau)^{\frac{1}{2}}}$$
$$\int^{\tau}(1+a\cosh s_2)\left(\int^{s_2}\frac{1}{1+a\cosh s_1}\,ds_1\right)\,ds_2$$
$$+\delta A^{2m-1}\frac{1}{(1+a\cosh\tau)^{\frac{1}{2}}}$$
$$\int^{\tau}(1+a\cosh s_2)\left(\int^{s_2}\frac{as_1\sinh s_1}{(1+a\cosh s_1)^{m+1}}ds_1\right)ds_2 \quad (179)$$

Equation (175) leads to the QS solution (59) for the parabolic law of nonlinearity where a is given by (155).

4.4. Dual-Power Law

In this case we have $F(s) = s^p + \nu s^{2p}$ so that we have $f(s) = s^{p+1}/(p+1) + \nu s^{2p+1}/(2p+1)$. We note that on setting $p = 1$ for the case of dual-power law, one recovers the parabolic law. The GNLSE, thus, is

$$iq_t + \frac{1}{2}q_{xx} + \left(|q|^{2p} + \nu\,|q|^{4p}\right)q = 0 \quad (180)$$

The solution of (180) is given by

$$q(x,t) = \frac{A}{\left[1+b\cosh\left\{B\left(x-\bar{x}(t)\right)\right\}\right]^{\frac{1}{2p}}}e^{i(-\kappa x+\omega t+\sigma_0)} \quad (181)$$

where

$$B = A^p\left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}} \quad (182)$$

with

$$\kappa = -v \quad (183)$$

$$\omega = \frac{A^{2p}}{2p+2} - \frac{\kappa^2}{2} \quad (184)$$

and

$$b = \sqrt{1 + \frac{\nu B^2}{2p^2}\frac{(1+p)^2}{1+2p}} \tag{185}$$

We note that for the dual-power law case the solitons exist for

$$-\frac{2p^2}{B^2}\frac{1+2p}{(1+p)^2} < \nu < 0 \tag{186}$$

The corresponding parameter dynamics of the soliton is given by

$$\frac{dA}{dt} = 0 \tag{187}$$

$$\frac{dB}{dt} = 0 \tag{188}$$

$$\frac{d\kappa}{dt} = 0 \tag{189}$$

$$\frac{d\bar{x}}{dt} = -\kappa \tag{190}$$

The three integrals of motion for the dual-power law case are

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |q|^2 \, dx \\ &= \frac{2}{B}\left[\frac{(b-1)(2p+1)}{2\nu(1+p)}\right]^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} F\left(\frac{1}{2}, \frac{1}{p}; \frac{1}{2}+\frac{1}{p}; \frac{1-b}{1+b}\right) \end{aligned} \tag{191}$$

$$\begin{aligned} M &= \frac{i}{2}\int_{-\infty}^{\infty} \left(qq_x^* - q^* q_x\right) \, dx \\ &= -\frac{2\kappa}{B}\left[\frac{(b-1)(2p+1)}{2\nu(1+p)}\right]^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} F\left(\frac{1}{2}, \frac{1}{p}; \frac{1}{2}+\frac{1}{p}; \frac{1-b}{1+b}\right) \end{aligned} \tag{192}$$

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \left[\frac{1}{2}|q_x|^2 - \frac{|q|^{2p+2}}{p+1} - \nu\frac{|q|^{4p+2}}{2p+1}\right] \, dx \\ &= \frac{B}{4p^2}\left[\frac{(b-1)(2p+1)}{2\nu(1+p)}\right]^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} \\ &\quad \left\{\frac{2p}{p+1}F\left(-\frac{1}{2}, \frac{1}{p}; \frac{3}{2}+\frac{1}{p}; \frac{1-b}{1+b}\right) - F\left(\frac{1}{2}, \frac{1}{p}; \frac{1}{2}+\frac{1}{p}; \frac{1-b}{1+b}\right)\right\} \end{aligned} \tag{193}$$

Here, $F(\alpha,\beta;\gamma;z)$ is the Gauss' hypergeometric function defined as

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\frac{z^n}{n!} \tag{194}$$

4.4.1. Perturbation Terms

The perturbed NLSE that is going to be considered in this paper is:

$$iq_t+\frac{1}{2}q_{xx}+\left(|q|^{2p}+\nu\,|q|^{4p}\right)q=i\epsilon\delta\,|q|^{2m}\,q \tag{195}$$

In this case, we have the adiabatic variation of the parameters as

$$\frac{dA}{dt}=\frac{\epsilon}{pLA^{p-1}}\left(\frac{p+1}{2p^2}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}(q^*R+qR^*)\,dx \tag{196}$$

$$\frac{dB}{dt}=\frac{\epsilon}{L}\int_{-\infty}^{\infty}(q^*R+qR^*)\,dx \tag{197}$$

$$\frac{d\kappa}{dt}=\frac{\epsilon}{E}\left[i\int_{-\infty}^{\infty}(q_x^*R-q_xR^*)\,dx-\kappa\int_{-\infty}^{\infty}(q^*R+qR^*)\,dx\right] \tag{198}$$

where we have E is the energy as given by (18) while

$$\begin{aligned}L=&\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}\left[\frac{(b-1)(2p+1)}{2\nu(1+p)}\right]^{\frac{1}{p}}\\&\left\{\frac{2\nu^2}{bp^3}\frac{(p+1)^3}{(b-1)(2p+1)^2}F\left(\frac{1}{2},\frac{1}{p};\frac{1}{2}+\frac{1}{p};\frac{1-b}{1+b}\right)\right.\\&-\frac{2}{B^2}F\left(\frac{1}{2},\frac{1}{p};\frac{1}{2}+\frac{1}{p};\frac{1-b}{1+b}\right)\\&\left.-\frac{2\nu}{bp^2}\frac{(p+1)^2}{(b-1)^2(p+2)(2p+1)}F\left(\frac{1}{2},\frac{1}{p};\frac{1}{2}+\frac{1}{p};\frac{1-b}{1+b}\right)\right\}\end{aligned} \tag{199}$$

Now, substituting the perturbation term R from (24) and carrying out the integrations in (25), (26) and (27) one finds:

$$\frac{dA}{dt}=\frac{2\epsilon\delta}{p}\frac{A^{2m-p+3}}{BL}\left(\frac{p+1}{2p^2}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{1}{(1+b\cosh\tau)^{\frac{m+1}{p}}}d\tau \tag{200}$$

$$\frac{dB}{dt}=2\epsilon\delta\frac{A^{2m+2}}{BL}\int_{-\infty}^{\infty}\frac{1}{(1+b\cosh\tau)^{\frac{m+1}{p}}}d\tau \tag{201}$$

and

$$\frac{d\kappa}{dt} = 0 \tag{202}$$

From (22) one can obtain the velocity of the soliton as

$$v = \frac{d\bar{x}}{dt} = -\kappa \tag{203}$$

Thus, we have obtained the basic adiabatic parameter dynamics of the solitons of the NLSE using the soliton perturbation theory (SPT).

4.4.2. Quasistationarity

Here, at the leading order, we arrive at

$$-\left\{\rho_T^{(0)} + \frac{1}{2}\left(\rho_X^{(0)}\right)^2\right\}\hat{q}^{(0)} + \frac{1}{2}\frac{\partial^2 \hat{q}^{(0)}}{\partial\theta^2} + \left(\hat{q}^{(0)}\right)^{2p+1} + \nu\left(\hat{q}^{(0)}\right)^{4p+1} = 0 \tag{204}$$

and

$$\left(\rho_X^{(0)} - v^{(0)}\right)\frac{\partial \hat{q}^{(0)}}{\partial\theta} = 0 \tag{205}$$

Now, (40) implies

$$\rho_X^{(0)} = v^{(0)} \tag{206}$$

We set

$$\frac{B^2}{4p^2} = \rho_T^{(0)} + \frac{1}{2}\left(\rho_X^{(0)}\right)^2 = \rho_T^{(0)} + \frac{1}{2}\left(v^{(0)}\right)^2 \tag{207}$$

so that (39) changes to

$$-\frac{B^2}{4p^2}\hat{q}^{(0)} + \frac{1}{2}\frac{\partial^2 \hat{q}^{(0)}}{\partial\theta^2} + \left(\hat{q}^{(0)}\right)^{2p+1} + \nu\left(\hat{q}^{(0)}\right)^{4p+1} = 0 \tag{208}$$

whose solution is

$$\hat{q}^{(0)} = Ag\left[B(\theta - \bar{\theta})\right] \tag{209}$$

where

$$g(\tau) = \frac{1}{(1 + b\cosh\tau)^{\frac{1}{2p}}} \tag{210}$$

and

$$B = A^p \left(\frac{2p^2}{1+p} \right)^{\frac{1}{2}} \tag{211}$$

along with

$$\tau = B(\theta - \bar{\theta}) \tag{212}$$

while

$$\frac{d\bar{\theta}}{dt} = v \tag{213}$$

At $O(\epsilon)$ level, we decompose $\hat{q}^{(1)} = \hat{\phi}^{(1)} + i\hat{\psi}^{(1)}$ into its real and imaginary parts. Now, the equations for $\hat{\phi}^{(1)}$ and $\hat{\psi}^{(1)}$, by virtue of (43), are respectively:

$$-\frac{B^2}{4p^2}\hat{\phi}^{(1)} + \frac{1}{2}\frac{\partial^2 \hat{\phi}^{(1)}}{\partial \theta^2} + \hat{\phi}^{(1)} \left\{ (2p+1) \left(\hat{q}^{(0)} \right)^{2p} + \nu(4p+1) \left(\hat{q}^{(0)} \right)^{4p} \right\}$$
$$= \left\{ \rho_T^{(1)} + v^{(0)} \rho_X^{(1)} \right\} \hat{q}^{(0)} - \frac{\partial^2 \hat{q}^{(0)}}{\partial \theta \partial X} \tag{214}$$

and

$$-\frac{B^2}{4p^2}\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2 \hat{\psi}^{(1)}}{\partial \theta^2} + \left\{ \left(\hat{q}^{(0)} \right)^{2p} + \nu \left(\hat{q}^{(0)} \right)^{4p} \right\} \hat{\psi}^{(1)} =$$
$$-\frac{\partial \hat{q}^{(0)}}{\partial T} - v^{(0)} \frac{\partial \hat{q}^{(0)}}{\partial X} - \left\{ \rho_X^{(1)} - v^{(1)} \right\} \frac{\partial \hat{q}^{(0)}}{\partial \theta} - \rho_{XX}^{(0)} \hat{q}^{(0)} + \delta \left(\hat{q}^{(0)} \right)^{2m+1} \tag{215}$$

Once again, setting, $\rho_{XX}^{(0)} = 0$, to eliminate the frequency chirp, gives

$$-\frac{B^2}{4p^2}\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2 \hat{\psi}^{(1)}}{\partial \theta^2} + \left\{ \left(\hat{q}^{(0)} \right)^{2p} + \nu \left(\hat{q}^{(0)} \right)^{4p} \right\} \hat{\psi}^{(1)} =$$
$$-\frac{\partial \hat{q}^{(0)}}{\partial T} - v^{(0)} \frac{\partial \hat{q}^{(0)}}{\partial X} - \left\{ \rho_X^{(1)} - v^{(1)} \right\} \frac{\partial \hat{q}^{(0)}}{\partial \theta} + \delta \left(\hat{q}^{(0)} \right)^{2m+1} \tag{216}$$

The FA, when applied to (49) gives

$$\frac{\partial B}{\partial X} = 0 \tag{217}$$

and

$$\rho_T^{(1)} + v^{(0)} \rho_X^{(1)} = 0 \tag{218}$$

whereas if, applied to (51), gives

$$\frac{dB}{dT} = 2\delta \frac{A^{2m+2}}{BL} \int_{-\infty}^{\infty} \frac{1}{(1 + b\cosh\tau)^{\frac{m+1}{p}}} d\tau \tag{219}$$

and

$$\rho_X^{(1)} = v^{(1)} \tag{220}$$

Equation (52) shows that B is a function of T only and so is A by virtue of (46) so that we obtain

$$\frac{dA}{dt} = \frac{2\delta}{p} \frac{A^{2m-p+3}}{BL} \left(\frac{p+1}{2p^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{(1 + b\cosh\tau)^{\frac{m+1}{p}}} d\tau \tag{221}$$

Although (54) and (56) were obtained before by the SPT, we note that the relations (52), (53) and (55) cannot be recovered by the SPT and therefore this method has its limitations. Thus, these O(ϵ) equations reduce to:

$$-\frac{B^2}{4p^2}\hat{\phi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\phi}^{(1)}}{\partial\theta^2} + \hat{\phi}^{(1)}\left\{(2p+1)\left(\hat{q}^{(0)}\right)^{2p} + \nu(4p+1)\left(\hat{q}^{(0)}\right)^{4p}\right\} = 0 \tag{222}$$

and

$$\begin{aligned} -\frac{B^2}{4p^2}\hat{\psi}^{(1)} + \frac{1}{2}\frac{\partial^2\hat{\psi}^{(1)}}{\partial\theta^2} + \left\{\left(\hat{q}^{(0)}\right)^{2p} + \nu\left(\hat{q}^{(0)}\right)^{4p}\right\}\hat{\psi}^{(1)} \\ = -\frac{\partial\hat{q}^{(0)}}{\partial T} + \delta\left(\hat{q}^{(0)}\right)^{2m+1} \end{aligned} \tag{223}$$

whose solutions are respectively

$$\hat{\phi}^{(1)} = 0 \tag{224}$$

and

$$\begin{aligned} \hat{\psi}^{(1)} = & -\frac{A}{Bp}\frac{\partial\bar{\theta}}{\partial T}\frac{1}{(1+b\cosh\tau)^{\frac{1}{2p}}} \\ & \int^{\tau}(1+b\cosh s_2)^{\frac{1}{p}}\left(\int^{s_2}\frac{b\sinh s_1}{(1+b\cosh s_1)^{\frac{p+1}{p}}}ds_1\right)ds_2 \\ & +\frac{A}{B^3p}\frac{dB}{dT}\frac{1}{(1+b\cosh\tau)^{\frac{1}{2p}}} \end{aligned}$$

$$
\begin{aligned}
&\int^{\tau}(1+b\cosh s_2)^{\frac{1}{p}}\left(\int^{s_2}\frac{bs_1\sinh s_1}{(1+b\cosh s_1)^{\frac{p+1}{p}}}ds_1\right)ds_2\\
&-\frac{2A}{B^2}\frac{dA}{dT}\frac{1}{(1+b\cosh\tau)^{\frac{1}{2p}}}\\
&\int^{\tau}(1+b\cosh s_2)^{\frac{1}{p}}\left(\int^{s_2}\frac{1}{(1+b\cosh s_1)^{\frac{1}{p}}}ds_1\right)ds_2\\
&+2\delta\frac{A^{2m}}{B^2}\frac{1}{(1+b\cosh\tau)^{\frac{1}{2p}}}\\
&\int^{\tau}(1+b\cosh s_2)^{\frac{1}{p}}\left(\int^{s_2}\frac{1}{(1+b\cosh s_1)^{\frac{m+1}{p}}}ds_1\right)ds_2 \quad (225)
\end{aligned}
$$

which leads to the QS solution of the perturbed NLSE with dual-power law nonlinearity.

5. NUMERICAL SIMULATION

We have now carried out the direct numerical simulation of the equation (78). We have used a hyperbolic secant profile, here, for the soliton. The Fast Fourier Transform (FFT) of the profile in space variable is used. The different modes of the FFT are studied. The program proceeds in the time step by Picard iteration. The evolution of the nonlinear terms is carried out by the convolution integrals. The iteration ceases when the difference of values between successive iterations is at $O(h^2)$ where h is the time step.

In the following figures, we have obtained the numerical and the analytical variation of the amplitude of the perturbed soliton. They are plotted on the same set of axes for a direct comparison.

Figure 1 is the numerical and analytical variation of the soliton amplitude given by (73) or (74). Here, the special case $m = 0$ is plotted for $\delta = -0.5$.

Now, Figure 2 numerical and analytical variation of the soliton amplitude in (73) or (74) for $m = 1$ and $\delta = -0.5$.

Finally, in Figure 3, we have the numerical and analytical variation of the amplitude of the soliton for $m = 2$ with $\delta = -0.5$.

Thus, in Figures 1, 2 and 3 we see that the agreement between the theory and the numerics of the variation of the amplitude of the soliton, for the Kerr law case, is very good.

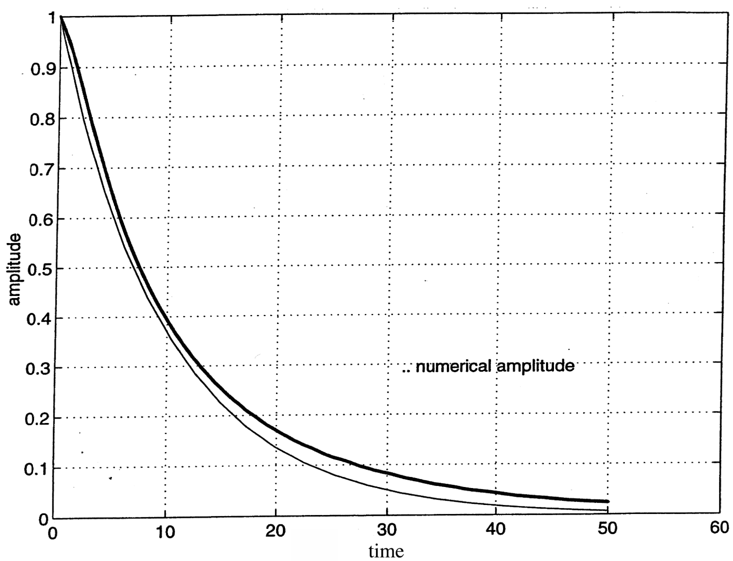

Figure 1. Amplitude variation for $m = 0,\ \delta = -0.5$.

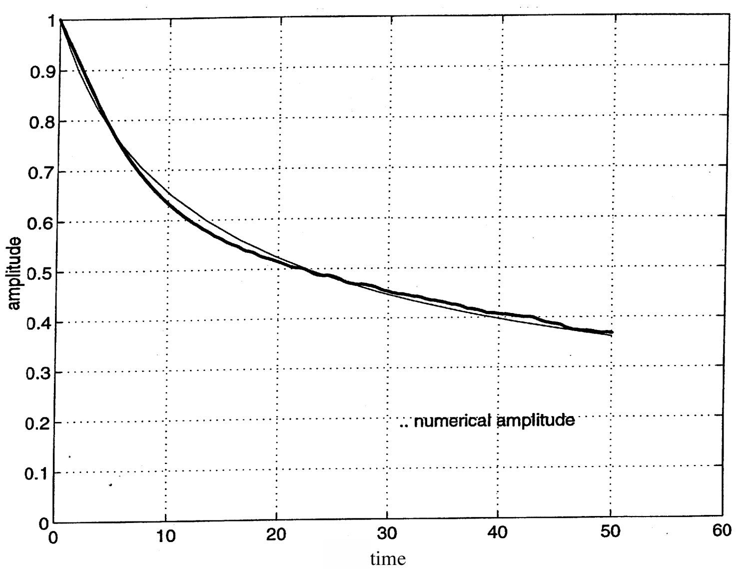

Figure 2. Amplitude variation for $m = 1,\ \delta = -0.5$.

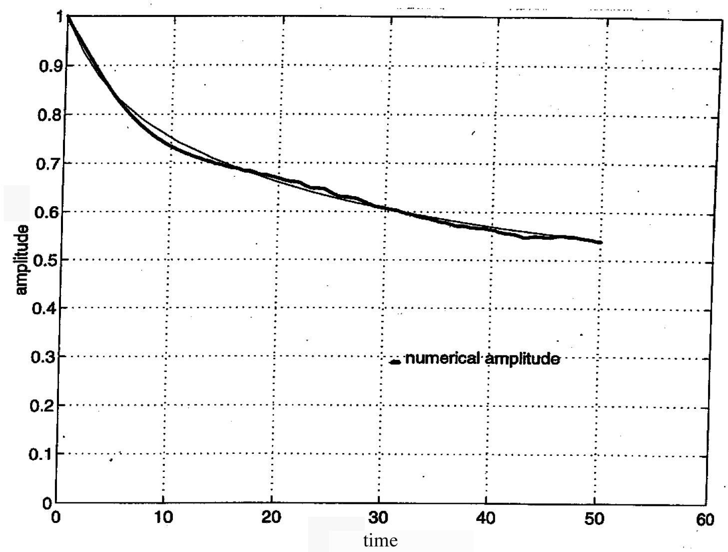

Figure 3. Amplitude variation for $m = 2,\ \delta = -0.5$.

6. CONCLUSIONS

In this paper, we have obtained the quasi-stationary solution to the generalized NLSE in presence of the nonlinear damping perturbation term. The special cases of the Kerr law, power law, parabolic law as well as the dual-power law of nonlinearity are considered here. The effect of these four laws are summarized in the following table.

Law	$F(s)$	$f(s)$	$g(\tau)$	ω
Kerr	s	$s^2/2$	$\frac{1}{\cosh\tau}$	$\frac{B^2-\kappa^2}{2}$
Power	s^p	$s^{p+1}/(p+1)$	$\frac{1}{\cosh^{\frac{1}{p}}\tau}$	$\frac{B^2}{2p^2}-\frac{\kappa^2}{2}$
Parabolic	$s+\nu s^2$	$s^2/2+\nu s^3/3$	$\frac{1}{\sqrt{1+a\cosh\tau}}$	$\frac{A^2}{4}-\frac{\kappa^2}{2}$
Dual-Power	$s^p+\nu s^{2p}$	$s^{p+1}/(p+1)+\nu s^{2p+1}/(2p+1)$	$\frac{1}{(1+b\cosh\tau)^{\frac{1}{2p}}}$	$\frac{A^{2p}}{2p+2}-\frac{\kappa^2}{2}$

This technique of QS with the WKB ansatz of the solution can be used for Hamiltonian as well as the non-Hamiltonian type perturbation. The application to the conservative type of perturbation will be reported elsewhere, in future. Moreover, the extension of such studies to the case of vector solitons will also be carried out in future.

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