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SYMMETRY RELATIONS OF THE TRANSLATION COEFFICIENTS OF THE SPHERICAL SCALAR AND VECTOR MULTIPOLE FIELDS

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Abstract—We offer symmetry relations of the translation coefficients of the spherical scalar and vector multi-pole fields. These relations reduce the computational cost of evaluating and storing the translation coefficients and can be used to check the accuracy of their computed values. The symmetry relations investigated herein include not only those considered earlier for real wavenumbers by Peterson and Ström [9], but also the respective symmetries that arise when the translation vector is reflected about the xy-, yz-, and zx-planes. In addition, the symmetry relations presented in this paper are valid for complex wavenumbers and are given in a form suitable for exploitation in numerical applications.

1. INTRODUCTION

The spherical scalar and vector multipole fields, respectively the solutions of the scalar and vector Helmholtz wave equations in spherical coordinates, are of fundamental importance in acoustics and electromagnetics. Their importance lies not only in solving boundary values problems involving a spherical geometry, such as the scattering from a sphere, but also in the efficient expansion of the plane wave and the radiated field from a localized source distribution [1]. Their translation formulas [2–9], which express a spherical multipole field in one coordinate system in terms of the spherical multipole fields of another coordinate system that is related to the former by translation, have been a powerful analytic tool in many areas of electromagnetics. Early applications of the formulas include the plane-wave scattering from two metal spheres [10], which was later generalized to the scattering from many dielectric spheres in arbitrary configurations [11]; probe correction for spherical near-field scanning [12]; modeling of wave propagation through random discrete media [13]; extension of the T-matrix technique for many scatterers [9] and its efficient numerical implementation using FFT [16, 17]. As noted in [15], the cost of computing translation coefficients was found to be high and thus their recurrence relations [7, 14, 15] were derived in an attempt to contain this high cost. This paper examines the symmetry relations of the translation coefficients since they further reduce the computational cost of evaluating and storing the translation coefficients for the applications mentioned above.

As reviewed in the next section (see (6), (13) and (14)), the translation coefficients of the scalar and vector multipole fields are functions of six variables: the wavenumber k, the translation vector $\boldsymbol{R}$, and the mode indices of the original and translated multipole fields (l, m) and (l', m') [2–9]. Thus, we will use $\Gamma_{l',m'}^{l,m}(k, \boldsymbol{R})$ to represent them. It is important to note that the majority of the aforementioned applications requires the computation and storage of $\Gamma_{l',m'}^{l,m}(k, \boldsymbol{R})$ values for many mode combinations of (l, m) and (l', m') and at many $\boldsymbol{R}$ points. For example, the aforementioned algorithm of [16] and [17] requires the evaluation of $\Gamma_{l',m'}^{l,m}(k, \boldsymbol{R}_i - \boldsymbol{R}_j)$ for all combinations of (l, m) and (l', m') that satisfy $l, l' \leq L_{max}$ and $-l, -l' \leq m, m' \leq l, l'$ for some L_{max}, and at all $\boldsymbol{R}_i - \boldsymbol{R}_j$ spatial points, where $\boldsymbol{R}_i$ and $\boldsymbol{R}_j$ correspond to two well-separated nodes on a three-dimensional uniform grid containing the volume distribution of scatterers. Thus, even though $\Gamma_{l',m'}^{l,m}(k, \boldsymbol{R}_i - \boldsymbol{R}_j)$ is Toeplitz with respect to the spatial node indices, i and j, for a given mode combination, a scattering problem with a grid size $N_x = N_y = N_z = 128$ and with $L_{max} = 3$ would still require the evaluation of Γ approximately $4^4 \times (2 \times 128)^3 \cong 4.3 \times 10^9$ times for each type of Γ. Here, the factor 4^4 corresponds to the total number of mode combinations and $(2 \times 128)^3$ to the number of spatial node combinations. We note that the recurrence relations of [7, 14, 15], which relate Γs of different mode combinations at a given value of $\boldsymbol{R}$, reduce only the first factor.

Peterson and Ström [9] investigated the symmetry properties of the scalar and vector multipole fields with real k by taking advantage of the properties of the three-dimensional Euclidean group, E(3). According to [9], the translation coefficients exhibit symmetry properties under:

- spatial inversion of the translation vector: $\boldsymbol{R} \rightarrow -\boldsymbol{R}$
- interchange of the mode indices: $\{(l, m), (l', m')\} \rightarrow \{(l', m'), (l, m)\}$
- sign changes of the azimuthal indices: $\{(l, m), (l', m')\} \rightarrow \{(l, -m), (l', -m')\}$

Even though Peterson and Ström's derivation is elegant and concise, some of their results are expressed in a form that is ill suited for numerical applications. For example, some symmetry relations are expressed in terms of spherical Hankel functions with $-kR$ as their argument, while the aforementioned applications, including the scattering algorithm of [16] and [17], require the evaluation of the translation coefficients with $+kR$ as the argument of spherical Hankel functions. Furthermore, for the vector multipole fields, they choose to derive the symmetry relations of the translation coefficient of the general spherical tensor field that is defined in Appendix, while the numerical applications mentioned earlier all require only the translation coefficients of the two transverse fields.

In this paper, we extend the analysis of [9] in several important ways. First, in addition to the symmetry relations mentioned above, we consider the symmetry relations that arise when $\boldsymbol{R}$ is reflected about the xy-, yz-, and xz-planes. These reflection symmetry relations are particularly useful to the FFT-based, iterative solution technique [16, 17] that requires the computation and storage of translation coefficient values at a set of $\boldsymbol{R}$ points that are related to each other through reflection. Indeed, [19] shows that the reflection symmetry relations alone reduce the storage requirement of the FFT T-matrix method [16, 17] by a factor of 8. Second, unlike Peterson and Ström's results, all the symmetry relations presented herein are valid for complex k. For real k, we compare our symmetry relations with the corresponding results of [9]; except for one case, our results agree with the corresponding ones for the three types of symmetry operations considered in [9]. Third, we express the symmetry relations in a form suitable for numerical applications. For example, we combine the aforementioned expressions involving spherical Hankel functions with negative arguments with another type of symmetry relations to produce symmetry relations that are useful for numerical applications. [19] shows that this set of symmetry relations provide an additional reduction of the storage requirement of the FFT T-Matrix method. Moreover, as the aforementioned applications require translation coefficients of the transverse vector spherical multipole fields, we elect to derive their explicit symmetry relations diretly.

In Section 2 we define the scalar and vector spherical multipole fields and provide the explicit expressions for their respective translation coefficients for the purpose of establishing the notations and conventions used in the paper. In Section 3, we derive the symmetry relations of the translation coefficient of the scalar multipole field for the aforementioned four symmetry operations by taking advantage of the respective complex conjugation properties of the spherical harmonics and spherical Bessel and Hankel functions and the symmetry and recurrence relations of the Clebsch-Gordan coefficients. In Section 4, the symmetry relations of the translation coefficient of the scalar spherical multipole field are used to derive the corresponding relations of the two transverse vector spherical multipole fields. In Section 5, we present a summary of the symmetry relations derived. The appendix discusses the spherical tensor field and its relation to the spherical longitudinal and transverse multipole field [9, 20–22].

2. SCALAR AND VECTOR SPHERICAL MULTIPOLE FIELDS AND THEIR TRANSLATION FORMULAS

In this section we briefly review the respective translation formulas of the scalar and transverse vector spherical multipole fields. Consistent definitions of these fields are necessary to ensure that their symmetry relations be physically meaningful, as inconsistent definitions of these fields sometimes led to incorrect expressions for the translation coefficients in the literature. We will follow the notations used in [7] and [9].

The scalar spherical multipole field $\phi_{l,m}(k, \boldsymbol{r})$ of order (l, m) is defined as

$$\phi_{l,m}(k, \boldsymbol{r}) \equiv f_l(kr) Y_{l,m}(\theta, \phi), \tag{1}$$

where (r, θ, ϕ) represent the spherical coordinates of the position vector $\boldsymbol{r}$, and k is the fixed wavenumber which may be complex. $f_l(kr)$ is either the spherical Bessel function $j_l(kr)$ or spherical Hankel function $h_l^{(\pm)}(kr)$ of the first (second) kind. $Y_{l,m}(\theta, \phi)$ represents the spherical harmonics of order (l, m) [20–22],

$$Y_{lm}(\theta, \phi) \equiv (-1)^m \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{l,m}(\cos\theta) e^{im\phi},$$

where $P_{l,m}(\cos\theta)$ is the associated Legendre function,

$$P_{l,m}(\cos\theta) \equiv \frac{1}{2^l l!} (1 - \cos^2\theta)^{m/2} \frac{d^{l+m}}{d(\cos\theta)^{l+m}} (\cos^2\theta - 1)^l, \ -l \leq m \leq l.$$

With the above definition $Y_{l,m}(\theta, \phi)$ satisfies the complex-conjugation property [20–22],

$$Y^*_{l,m}(\theta, \phi) = (-1)^m Y_{l,-m}(\theta, \phi) \tag{2}$$

and the spatial-inversion property [20–22],

$$Y_{l,m}(\pi - \theta, \pi + \phi) = (-1)^l Y_{l,m}(\theta, \phi), \tag{3}$$

where $\pi - \theta$ and $\pi + \phi$. are, respectively, the polar and azimuth angles of the spatially-inverted position vector, $-\boldsymbol{r}$. Under reflection of $\boldsymbol{r}$ about the yz-, zx-, and xy-planes, its angular coordinates change from (θ, ϕ) to $(\theta, \pi - \phi,)$, $(\theta, -\phi)$, and $(\pi - \theta, \phi)$, respectively. Thus, $Y_{l,m}(\theta, \phi)$ satisfies the following reflection symmetry relations,

$$\begin{aligned} Y_{l,m}(\theta, \pi - \phi) &= Y_{l,-m}(\theta, \phi), \\ Y_{l,m}(\theta, -\phi) &= (-1)^m Y_{l,-m}(\theta, \phi), \\ \text{and } Y_{l,m}(\pi - \theta, \phi) &= (-1)^{l+m} Y_{l,m}(\theta, \phi). \end{aligned} \tag{4}$$

We note that the above reflection relations are consistent with the spatial-inversion symmetry relation, (3), as a spatial reflection is equivalent to successive reflections about the yz-, zx- and zy-plane:

$$\begin{aligned} Y_{l,m}(\pi-\theta,\pi+\phi) &= Y_{l,-m}(\pi-\theta,-\phi) \\ &= (-1)^m Y_{l,m}(\pi-\theta,\phi) = (-1)^l Y_{l,m}(\theta,\phi). \end{aligned}$$

Under translation of the coordinate system, $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}'$, the scalar spherical multipole field $\phi_{l,m}(k,\boldsymbol{r}) = f_l(kr)Y_{l,m}(\theta,\phi)$ of the original coordinate system can be expressed in terms of the scalar spherical multipole fields $\tilde{\phi}_{l',m'}(k,\boldsymbol{r}') = g_{l'}(kr')Y_{l',m'}(\theta',\phi')$ of the new coordinate system [2–9]:

$$\phi_{l,m}(k,\boldsymbol{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \alpha_{l',m'}^{l,m}(k,\boldsymbol{R})\tilde{\phi}_{l',m'}(k,\boldsymbol{r}'). \tag{5}$$

Here, (r',θ',ϕ') represent the spherical coordinates of the new position vector $\boldsymbol{r}'$. The tilde in $\tilde{\phi}_{l',m'}(k,\boldsymbol{r}')$ signifies that the radial function $g_{l'}$ of the new scalar multipole field in (5) may differ from the radial function $f_{l'}$, of the original multipole field depending on the ratio r'/R:

$$g_{l'}(kr') = \begin{cases} f_{l'}(kr') & \text{if } r' > R \\ j_{l'}(kr') & \text{otherwise.} \end{cases}$$

The translation coefficients $\alpha_{l',m'}^{l,m}(k,\boldsymbol{R})$ in (5) are given by [2–9]

$$\begin{aligned} \alpha_{l',m'}^{l,m}(k,\boldsymbol{R}) = &\sum_{l''=|l-l'|,2}^{l+l'} 4\pi i^{(l'-l+l'')}(-1)^{m'}[(2l+1)(2l'+1)/4\pi(2l''+1)]^{1/2} \\ &\cdot\mathcal{C}(l,l',l'';0,0,0)\mathcal{C}(l,l',l'';-m,m',-m+m') \\ &\cdot p_{l''}(kR)Y_{l'',m-m'}(\theta_R,\phi_R), \end{aligned} \tag{6}$$

where (R,θ_R,ϕ_R) represent the spherical coordinates of the translation vector $\boldsymbol{R}$. The Clebsch-Gordan coefficients $\mathcal{C}(l,l',l'';0,0,0)$ and $\mathcal{C}(l,l',l'';-m,m',-m+m')$ represent the strength of the coupling among the multipole modes (l,m), (l',m') and $(l'',-m+m')$. The summation in (6) is over the dummy variable l'' from $|l-l'|$ to $l+l'$ in steps of two, as $\mathcal{C}(l,l',l'';0,0,0)$ is non-zero only when $l+l'+l''$ is even [20–22]. The radial function $p_{l''}(kR)$ is determined according to

$$p_{l''}(kR) = \begin{cases} f_{l''}(kR) & \text{if } r' > R \\ j_{l''}(kR) & \text{otherwise.} \end{cases} \tag{7}$$

Where necessary, we will use $\alpha_{l',m'}^{l,m}(0,k,\boldsymbol{R})$ to denote $\alpha_{l',m'}^{l,m}(k,\boldsymbol{R})$ with $p_{l''}(kR) = j_{l''}(kR)$, and $\alpha_{l',m'}^{l,m}(\pm,k,\boldsymbol{R})$ with $p_{l''}(kR) = h_{l''}^{(\pm)}(kR)$.

The longitudinal spherical multipole field $\boldsymbol{L}_{l,m}(k,\boldsymbol{r})$ and transverse vector multipole fields $\boldsymbol{M}_{l,m}(k,\boldsymbol{r})$ and $\boldsymbol{N}_{l,m}(k,\boldsymbol{r})$ are defined in terms of $\phi_{l,m}(k,\boldsymbol{r})$ as [7, 20–22]

$$\boldsymbol{L}_{l,m}(k,\boldsymbol{r}) \equiv \frac{1}{k}\nabla\phi_{l,m}(k,\boldsymbol{r}), \tag{8}$$

$$\boldsymbol{M}_{l,m}(k,\boldsymbol{r}) \equiv \frac{1}{\sqrt{l(l+1)}}\frac{1}{i}\boldsymbol{r}\times\nabla\phi_{l,m}(k,\boldsymbol{r}), \tag{9}$$

$$\text{and } \boldsymbol{N}_{l,m}(k,\boldsymbol{r}) \equiv \frac{1}{k}\nabla\times\boldsymbol{M}_{l,m}(k,\boldsymbol{r}). \tag{10}$$

The translation formula for the longitudinal multipole field $\boldsymbol{L}_{l,m}(k,\boldsymbol{r})$ is the same as that of $\phi_{l,m}(k,\boldsymbol{r})$ since the gradient operator in (8) is translationally invariant; *i.e.*, $\nabla = \nabla'$. On the other hand, the translation formulas of the transverse fields $\boldsymbol{M}_{l,m}(k,\boldsymbol{r})$ and $\boldsymbol{N}_{l,m}(k,\boldsymbol{r})$ are more complicated since the operators $\nabla\times\boldsymbol{r}$ in (9) and $\nabla\times\nabla\times\boldsymbol{r}$ in (10) are not translationally invariant. It has be shown that $\boldsymbol{M}_{l,m}(k,\boldsymbol{r})$ and $\boldsymbol{N}_{l,m}(k,\boldsymbol{r})$ translate according to [2–8]

$$\begin{aligned}\boldsymbol{M}_{l,m}(k,\boldsymbol{r}) &= \sum_{L=1}^{\infty}\sum_{M=-L}^{L}\Big[\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})\tilde{\boldsymbol{M}}_{L,M}(k,\boldsymbol{r}') \\ &\qquad + \mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})\tilde{\boldsymbol{N}}_{L,M}(k,\boldsymbol{r}')\Big],\end{aligned} \tag{11}$$

$$\begin{aligned}\text{and } \boldsymbol{N}_{l,m}(k,\boldsymbol{r}) &= \sum_{L=1}^{\infty}\sum_{M=-L}^{L}\Big[\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})\tilde{\boldsymbol{N}}_{L,M}(k,\boldsymbol{r}') \\ &\qquad + \mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})\tilde{\boldsymbol{M}}_{L,M}(k,\boldsymbol{r}')\Big],\end{aligned} \tag{12}$$

where the tilde in $\tilde{\boldsymbol{M}}_{L,M}(k,\boldsymbol{r}')$ and $\tilde{\boldsymbol{N}}_{L,M}(k,\boldsymbol{r}')$ has the same meaning as in the scalar case. The translation coefficients $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ may be written in terms of $\alpha_{l',m'}^{l,m}(k,\boldsymbol{R})$ of (6) [2–8],

$$\begin{aligned}\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R}) \equiv\, & ik\left[\frac{L}{2L+1}\right]^{\frac{1}{2}}\beta_{l,m,L+1}^{L,M}(k,\boldsymbol{R}) \\ & -ik\left[\frac{L+1}{2L+1}\right]^{\frac{1}{2}}\beta_{l,m,L-1}^{L,M}(k,\boldsymbol{R}) + \left[\frac{L(L+1)}{l(l+1)}\right]^{\frac{1}{2}}\alpha_{L,M}^{l,m}(k,\boldsymbol{R}),\end{aligned}$$

and (13)

$$\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R}) \equiv k\beta_{l,m,L}^{L,M}(k,\boldsymbol{R}), \tag{14}$$

where $\beta^{L,M}_{l,m,l'}(k,\boldsymbol{R})$ is a linear combination of $\alpha^{l,m}_{L,M}(k,\boldsymbol{R})$ [7],

$$\begin{aligned}\beta^{L,M}_{l,m,l'}(k,\boldsymbol{R}) \;=\; & \sqrt{\frac{4\pi}{3}}\frac{R}{i\sqrt{l(l+1)}}\sum_{\mu=-1}^{1}(-1)^{\mu}Y_{1,-\mu}(\theta_R,\phi_R)\\ & \cdot\mathcal{C}(l',1,L;M-\mu,\mu,M)\alpha^{l,m}_{l',M-\mu}(k,\boldsymbol{R}).\end{aligned}\qquad(15)$$

We note that $\mathcal{A}^{l,m}_{L,M}(k,\boldsymbol{R})$ may also be written as [3, 8]

$$\begin{aligned}\mathcal{A}^{l,m}_{L,M}(k,\boldsymbol{R}) \;=\; & 2\pi(i)^{L-l}(-1)^{M}[l(l+1)L(L+1)]^{-1/2}\\ & \cdot\sum_{l''=|l-l'|,2}^{l+l'}(i)^{l''}[(2l+1)(2L+1)/4\pi(2l''+1)]^{1/2}\\ & \cdot[l(l+1)+L(L+1)-l''(l''+1)]\mathcal{C}(l,l',l'';0,0,0)\\ & \cdot\mathcal{C}(l,l',l'';-m,m',-m+m')p_{l''}(kR)Y_{l'',m-m'}(\theta_R,\phi_R),\end{aligned}\qquad(16)$$

which is more convenient for some applications. Since $\mathcal{A}^{l,m}_{L,M}(k,\boldsymbol{R})$ and $\mathcal{B}^{l,m}_{L,M}(k,\boldsymbol{R})$ can be expressed in terms of $\alpha^{l,m}_{l',m'}(k,\boldsymbol{R})$, the symmetry properties of the former are intimately related to those of the latter. This observation may be used to derive the symmetry relations of $\mathcal{A}^{l,m}_{L,M}(k,\boldsymbol{R})$ and $\mathcal{B}^{l,m}_{L,M}(k,\boldsymbol{R})$, once the corresponding relations of $\alpha^{l,m}_{l',m'}(k,\boldsymbol{R})$ are obtained. Some well known results [9, 20–23] of the spherical tensor field are presented in Appendix to facilitate comparison between the symmetry relations of [9] and the present work.

3. SYMMETRY RELATIONS OF THE TRANSLATION COEFFICIENT OF THE SCALAR SPHERICAL MULTIPOLE FIELD

Having defined the spherical multipole fields and their translation coefficients, we now proceed to derive the respective symmetry relations of $\alpha^{l,m}_{l',m'}(k,\boldsymbol{R})$ for the four symmetry operations mentioned earlier. These relations may be derived in several different ways. As (6) strongly suggests, however, we choose to derive them by taking advantage of the well-known symmetry properties of the Clebsch-Gordan coefficients and the scalar harmonics $Y_{l,m}(\theta_R,\phi_R)$. The wavenumber k is assumed to be complex.

3.1. Spatial Inversion of the Translation Vector

Under the spatial inversion of the translation vector, $\boldsymbol{R} \to -\boldsymbol{R}$, its spherical coordinates change from (R, θ_R, ϕ_R) to $(R, \pi - \theta_R, \pi + \phi_R)$. As a result, $Y_{l'',m-m'}(\theta_R, \phi_R)$ in (6) changes to $Y_{l'',m-m'}(\pi-\theta_R, \pi+\phi_R)$, which according to (3) equals $(-1)^{l''} Y_{l'',m-m'}(\theta_R, \phi_R)$. Thus, the net effect of the spatial inversion is the introduction of the phase factor, $(-1)^{l''}$. As noted earlier, $\mathcal{C}(l, l', l''; 0, 0, 0)$ is non-zero only when $l+l'+l''$ is even. This implies that $(-1)^{l''} = (-1)^{l+l'}$ and, therefore,

$$\alpha_{l',m'}^{l,m}(k, -\boldsymbol{R}) = (-1)^{l+l'} \alpha_{l',m'}^{l,m}(k, \boldsymbol{R}). \tag{17}$$

We note that (17) is identical to the result obtained for real k by Peterson and Ström [9]. However, as shown here, it is valid whether k is real or complex and whether $p_{l''}(kR) = j_{l''}(kR)$ or $p_{l''}(kR) = h_{l''}^{(\pm)}(kR)$ in (6).

3.2. Interchange of Mode Indices

We next investigate the symmetry properties of $\alpha_{l',m'}^{l,m}(k, \boldsymbol{R})$ under the interchange of mode indices $\{(l, m),\ (l', m')\} \to \{(l', m'),\ (l, m)\}$. From (6) we have

$$\begin{aligned}\alpha_{l,m}^{l',m'}(k, \boldsymbol{R}) = \sum_{l''=|l-l'|,2}^{l+l'} & 4\pi i^{(l-l'+l'')}(-1)^m [(2l+1)(2l'+1)/4\pi(2l''+1)]^{\frac{1}{2}} \\ & \cdot \mathcal{C}(l', l, l''; 0, 0, 0)\mathcal{C}(l', l, l''; -m', m, m-m') \\ & \cdot p_{l''}(kR) Y_{l'',-m+m'}(\theta_R, \phi_R).\end{aligned} \tag{18}$$

Comparison of the above equation with (6) shows that three terms are altered: (i) the phase factor $i^{(l-l'+l'')}(-1)^m$, (ii) the product of two Clebsch-Gordan coefficients $\mathcal{C}(l', l, l''; 0, 0, 0) \cdot \mathcal{C}(l', l, l''; -m', m, m-m')$, and (iii) the spherical harmonics, $Y_{l'',-m+m'}(\theta_R, \phi_R)$ which according to (2) equals $(-1)^{(-m+m')}[Y_{l'',m-m'}(\theta_R, \phi_R)]^*$. Since $l + l' + l''$ is even, the phase factor is real and satisfies

$$i^{(l-l'+l'')}(-1)^m = (-1)^{(l+l')} \left[i^{(l'-l+l'')} \right]^* (-1)^m. \tag{19}$$

Also, the Clebsch-Gordan coefficient $\mathcal{C}(a, b, c; m_a, m_b, m_c)$ undergoes a phase change under simultaneous interchanges of a and b, and m_a and m_b [20–22],

$$\mathcal{C}(a, b, c; m_a, m_b, m_c) = (-1)^{a+b+c} \mathcal{C}(a, b, c; m_a, m_b, m_c).$$

Therefore, the product of the two Clebcsh-Gordan coefficients becomes $\mathcal{C}(l',l,l'';0,0,0)\mathcal{C}(l',l,l'';-m',m,m-m') = \mathcal{C}(l,l',l'';0,0,0)\mathcal{C}(l,l',l'';m,-m',m-m')$. Substitution of these results into (18) and the fact that the Clebsch-Gordan coefficient is always real yield,

$$\alpha_{l,m}^{l',m'}(k,\boldsymbol{R}) = (-1)^{l+l'}\left[\sum_{l''=|l-l'|,2}^{l+l'} 4\pi i^{(l'-l+l'')}(-1)^{m'} \cdot[(2l+1)(2l'+1)/4\pi(2l''+1)]^{\frac{1}{2}}\mathcal{C}(l,l',l'';0,0,0) \cdot\mathcal{C}(l,l',l'';m,-m',m-m')p_{l''}^*(kR)Y_{l'',m-m'}(\theta_R,\phi_R)\right]^*.$$

If $p_{l''}(kR) = j_{l''}(kR)$, comparison of the above equation with (6) using $j_{l''}^*(kR) = j_{l''}(k^*R)$ yields

$$\alpha_{l,m}^{l',m'}(0,k,\boldsymbol{R}) = (-1)^{l+l'}\left[\alpha_{l',m'}^{l,m}(0,k^*,\boldsymbol{R})\right]^*. \tag{20}$$

On the other hand, if $p_{l''}(kR) = h_{l''}^{(\pm)}(kR)$, then $\left[h_{l''}^{(\pm)}(kR)\right]^* = h_{l''}^{(\mp)}(k^*R)$. Thus,

$$\alpha_{l,m}^{l',m'}(\pm,k,\boldsymbol{R}) = (-1)^{l+l'}\left[\alpha_{l',m'}^{l,m}(\mp,k^*,\boldsymbol{R})\right]^*. \tag{21}$$

If k is real, (20) and (21), respectively, reduce to

$$\alpha_{l,m}^{l',m'}(0,k,\boldsymbol{R}) = (-1)^{l+l'}\left[\alpha_{l',m'}^{l,m}(0,k,\boldsymbol{R})\right]^*, \tag{22}$$

$$\text{and } \alpha_{l,m}^{l',m'}(\pm,k,\boldsymbol{R}) = (-1)^{l+l'}\left[\alpha_{l',m'}^{l,m}(\mp,k,\boldsymbol{R})\right]^*. \tag{23}$$

We note that (22) is identical to the corresponding result of [9], while (23) can be shown to agree with the corresponding result of [9], namely, $\alpha_{l,m}^{l',m'}(\pm,k,\boldsymbol{R}) = \left[\alpha_{l',m'}^{l,m}(\pm,-k,\boldsymbol{R})\right]^*$ using the following symmetry property of the spherical Hankel function when $l+l'+l''$ is even,

$$h_{l''}^{(\pm)}(-kr) = (-1)^{l''}h_{l''}^{(\mp)}(kr) = (-1)^{l+l'}h_{l''}^{(\mp)}(kr). \tag{24}$$

3.3. Simultaneous Changes of the Signs of the Azimuthal Indices

We next investigate the symmetry property of $\alpha_{l',m'}^{l,m}(k,\boldsymbol{R})$ under simultaneous changes of the signs of the azimuthal indices

$\{(l,m),\,(l',m')\} \rightarrow \{(l,-m),\,(l',-m')\}$. This introduces two changes to (6). First, the Clebsch-Gordan coefficient $\mathcal{C}(l,l',l'';-m,m',-m+m')$ changes to $\mathcal{C}(l,l',l'';m,-m',m-m')$. Since the Clebsch-Gordan coefficient undergoes a phase change under simultaneous changes of the signs of the three azimuthal indices [20–22],

$$\mathcal{C}(a,b,c;-m_a,-m_b,-m_c) = (-1)^{(a+b-c)}\mathcal{C}(a,b,c;m_a,m_b,m_c),$$

and since $l + l' + l''$ is even, we have $\mathcal{C}(l,l',l'';-m,m',-m+m') = \mathcal{C}(l,l',l'';m,-m',m-m')$. Second, the spherical harmonics $Y_{l'',m-m'}(\theta_R,\phi_R)$ changes to $Y_{l'',-m+m'}(\theta_R,\phi_R)$ which according to (2) is $(-1)^{m+m'}[Y_{l'',-m+m'}(\theta_R,\phi_R)]^*$. Using the reasoning similar to that used in the previous subsection, we obtain the following symmetry relations:

$$\alpha_{l',-m'}^{l,-m}(0,k,\boldsymbol{R}) = (-1)^{m+m'}\left[\alpha_{l',m'}^{l,m}(0,k^*,\boldsymbol{R})\right]^* \text{ if } p_{l''}(kR) = j_{l''}(kR), \quad \text{and} \tag{25}$$

$$\alpha_{l',-m'}^{l,-m}(\pm,k,\boldsymbol{R}) = (-1)^{m+m'}\left[\alpha_{l',m'}^{l,m}(\mp,k^*,\boldsymbol{R})\right]^* \text{ if } p_{l''}(kR) = h_{l''}^{(\pm)}(kR). \tag{26}$$

If k is real, then (25) and (26), respectively, reduce to

$$\alpha_{l',-m'}^{l,-m}(0,k,\boldsymbol{R}) = (-1)^{m+m'}\left[\alpha_{l',m'}^{l,m}(0,k,\boldsymbol{R})\right]^* \text{ if } p_{l''}(kR) = j_{l''}(kR), \quad \text{and} \tag{27}$$

$$\alpha_{l',-m'}^{l,-m}(\pm,k,\boldsymbol{R}) = (-1)^{m+m'}\left[\alpha_{l',m'}^{l,m}(\mp,k,\boldsymbol{R})\right]^* \text{ if } p_{l''}(kR) = h_{l''}^{(\pm)}(kR). \tag{28}$$

(27) is identical to Peterson and Ström's result [9], while (28) can be shown to agree with their result $\alpha_{l',-m'}^{l,-m}(\pm,k,\boldsymbol{R}) = (-1)^{m+m'}\alpha_{l,m}^{l',m'}(\pm,k,-\boldsymbol{R})$ using (17) and (23).

As they stand, the symmetry relations of (20)–(23) and (25)–(28) individually are not useful for numerical applications. For example, (23) and (28) both require evaluations of the spherical Hankel function of different kinds with complex-conjugated k. However, combining these two types of symmetry relations effects the following relation that is suitable for numerical applications:

$$\alpha_{l,-m}^{l',-m'}(k,\boldsymbol{R}) = (-1)^{(l+l'+m+m')}\alpha_{l',m'}^{l,m}(k,\boldsymbol{R}). \tag{29}$$

We note that this relation holds whether k is real or complex and whether $p_{l''}(kR) = j_{l''}(kR)$ or $p_{l''}(kR) = h_{l''}^{(\pm)}(kR)$ in (6).

3.4. Reflection of the Translation Vector about the yz-, zx-, and xy-Planes

We now proceed to investigate the respective symmetry relations of $\alpha_{l',m'}^{l,m}(k,\boldsymbol{R})$ when the translation vector $\boldsymbol{R}$ undergoes reflections about the yz-, zx-, and xy-planes. Substitution of (4) into (6) and the fact that $l+l'+l''$ is an even integer readily yield the following three reflection symmetry relations of $\alpha_{l',m'}^{l,m}(k,\boldsymbol{R})$.

$$\alpha_{l',m'}^{l,m}(k,-x,y,z) = \alpha_{l',-m'}^{l,-m}(k,x,y,z), \tag{30}$$

$$\alpha_{l',m'}^{l,m}(k,x,-y,z) = (-1)^{(m+m')}\alpha_{l',-m'}^{l,-m}(k,x,y,z), \tag{31}$$

$$\text{and } \alpha_{l',m'}^{l,m}(k,x,y,-z) = (-1)^{(l+l'+m+m')}\alpha_{l',m'}^{l,m}(k,x,y,z), \tag{32}$$

where x, y and z are the cartesian coordinates of $\boldsymbol{R}$. We note that each of the above relations holds whether k is complex or real and whether $p_{l''}(kR)$ in (6) is $j_{l''}(kR)$ or $h_{l''}^{(\pm)}(kR)$. Needless to say, symmetry relations involving multiple reflections can be obtained from the above relations. Indeed, we recover the spatial-inversion symmetry relation (17) from the above relations,

$$\begin{aligned}\alpha_{l',m'}^{l,m}(k,-x,-y,-z) &= \alpha_{l',-m'}^{l,-m}(k,x,-y,-z)\\ &= (-1)^{(m+m')}\alpha_{l',m'}^{l,m}(k,x,y,-z)\\ &= (-1)^{(l+l')}\alpha_{l',m'}^{l,m}(k,x,y,z).\end{aligned}$$

We further note that (30)–(32) are consistent with the corresponding reflection properties of $\phi_{l,m}(k,\boldsymbol{R})$. Substitution of (4) into (1) yields the following reflection properties of $\phi_{l,m}(k,\boldsymbol{R})$:

$$\begin{aligned}\phi_{l,m}(k,-x,y,z) &= \phi_{l,-m}(k,x,y,z)\\ \phi_{l,m}(k,x,-y,z) &= (-1)^{m}\phi_{l,-m}(k,x,y,z),\\ \text{and } \phi_{l,m}(k,x,y,-z) &= (-1)^{(l+m)}\phi_{l,m}(k,x,y,z).\end{aligned}$$

The consistency then can be established by applying the above relations and (30)–(32) to (5).

4. SYMMETRY RELATIONS OF THE TRANSLATION COEFFICIENTS OF THE TRANSVERSE VECTOR SPHERICAL MULTIPOLE FIELDS

We extend the analysis of the previous section to derive the corresponding symmetry relations of $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ for the same set of symmetry operations considered in the previous section. (13) and (14) show that $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ may be expressed as a sum of the products of $\alpha_{L,M}^{l,m}(k,\boldsymbol{R})$ and a geometrical factor that depends on $\{l,m\},\{L,M\}$ and $\boldsymbol{R}$. Therefore, the symmetry relations of $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ are intimately related to the combined symmetry properties of the geometrical factor and $\alpha_{L,M}^{l,m}(k,\boldsymbol{R})$. The symmetry relations of the latter are obtained in the previous section, while the symmetry properties of the former can be obtained by taking advantage of the well-known symmetry and recurrence relations of the Clebsch-Gordan coefficient [20–22] and the symmetry properties of the spherical harmonics as done in the previous section. As in the scalar case, k is allowed to be complex.

4.1. Spatial Inversion of the Translation Vector

(13) and (14), respectively, express $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ in terms of $\beta_{l,m,l'}^{L,M}(k,\boldsymbol{R})$ (15). Therefore, we first look into its spatial-inversion property. Application of (3) and (17) to (15) yields,

$$\beta_{l,m,l'}^{L,M}(k,-\boldsymbol{R}) = (-1)^{l+l'+1}\beta_{l,m,l'}^{L,M}(k,\boldsymbol{R}),$$

which, in turn, by virtue of (14) and (13) yields the following spatial-inversion relations:

$$\mathcal{B}_{L,M}^{l,m}(k,-\boldsymbol{R}) = (-1)^{l+L+1}\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R}), \text{ and} \tag{33}$$

$$\mathcal{A}_{L,M}^{l,m}(k,-\boldsymbol{R}) = (-1)^{l+L}\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R}). \tag{34}$$

Some comments are in order. First, $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ undergo different phase changes under the spatial inversion of the translation vector. This is due to the fact that $\boldsymbol{M}_{l,m}(k,\boldsymbol{r})$ and $\boldsymbol{N}_{l,m}(k,\boldsymbol{r})$ themselves undergo different phase changes under spatial inversion. From (9) and (10), we deduce $\boldsymbol{M}_{l,m}(k,-\boldsymbol{r}) = (-1)^l\boldsymbol{M}_{l,m}(k,\boldsymbol{r})$ and $\boldsymbol{N}_{l,m}(k,-\boldsymbol{r}) = (-1)^{l+1}\boldsymbol{N}_{l,m}(k,\boldsymbol{r})$. Substitution of these into (11) and (12) explains the phase difference between (33) and (34). Second, both (33) and (34) are valid whether k is complex or real, and whether $p_{l''}(kR) = j_{l''}(kR)$ or $p_{l''}(kR) = h_{l''}^{(\pm)}(kR)$ in (6). Third, one can show using (A1) and (A2) that (33) and (34) are consistent with the corresponding symmetry property of the translation coefficient of the spherical tensor field $S_{L,J,M}^{l,j,m}(k,\boldsymbol{R}) = (-1)^{(l+L)}S_{L,J,M}^{l,j,m}(k,-\boldsymbol{R})$ reported in [9].

4.2. Interchange of Modal Indices

We next investigate the symmetry relations of $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ under the interchange of mode indices. Comparison of (16) with (6) shows that they are of the same functional form. Following the reasoning used in the derivation of (20)–(23), we obtain

$$\mathcal{A}_{l,m}^{L,M}(0,k,\boldsymbol{r}) = (-1)^{l+L}\left[\mathcal{A}_{L,M}^{l,m}(0,k^*,\boldsymbol{r})\right]^*, \text{ if } p_{l''}(kR)=j_{l''}(kR), \quad \text{and} \tag{35}$$

$$\mathcal{A}_{l,m}^{L,M}(\pm,k,\boldsymbol{r}) = (-1)^{l+L}\left[\mathcal{A}_{L,M}^{l,m}(\mp,k^*,\boldsymbol{r})\right]^*, \text{ if } p_{l''}(kR)=h_{l''}^{(\pm)}(kR). \tag{36}$$

If k is real, then the above equations, respectively, reduce to

$$\mathcal{A}_{l,m}^{L,M}(0,k,\boldsymbol{r}) = (-1)^{l+L}\left[\mathcal{A}_{L,M}^{l,m}(0,k,\boldsymbol{r})\right]^*, \text{ if } p_{l''}(kR)=j_{l''}(kR), \quad \text{and} \tag{37}$$

$$\mathcal{A}_{l,m}^{L,M}(\pm,k,\boldsymbol{r}) = (-1)^{l+L}\left[\mathcal{A}_{L,M}^{l,m}(\mp,k,\boldsymbol{r})\right]^*, \text{ if } p_{l''}(kR)=h_{l''}^{(\pm)}(kR). \tag{38}$$

We note that (37) can be reconciled with Peterson and Ström's result $S_{L,J,M}^{l,j,m}(0,k,\boldsymbol{R}) = \left[S_{l,j,m}^{L,J,M}(0,-k,-\boldsymbol{R})\right]^*$ using (A1) and the spatial-inversion relation obtained earlier and that (38) can be reconciled with $S_{L,J,M}^{l,j,m}(\pm,k,\boldsymbol{R}) = \left[S_{l,j,m}^{L,J,M}(\pm,k,-\boldsymbol{R})\right]^*$ using (A1) and (24).

The derivation of the corresponding symmetry relation for $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ is more complicated since (14) has not been reduced to a simple form comparable to (16). Using (14) and (6), $\mathcal{B}_{l,m}^{L,M}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ can, respectively, be written as

$$\begin{aligned}\mathcal{B}_{l,m}^{L,M}(k,\boldsymbol{R}) = (-1)^m kR \sum_{l'=|L-l|,2}^{l+L} \frac{\Lambda(L,l,l'')}{\sqrt{2l''+1}} \mathcal{C}(L,l,l'';0,0,0)\, p_{l''}(kR)\\ \cdot\left[\sum_{\mu=-1}^{1} \mathcal{C}(L,l,l'';-M,m-\mu,m-M-\mu)\right.\\ \left.\cdot\,\mathcal{C}(l,1,l;m-\mu,\mu,m)\, Y_{l'',M-m+\mu}(\theta_R,\phi_R)\, Y_{1,-\mu}(\theta_R,\phi_R)\right],\end{aligned} \tag{39}$$

and

$$\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R}) = (-1)^m kR \sum_{l'=|L-l|,2}^{l+L} \frac{\Lambda(l,L,l'')}{\sqrt{2l''+1}} \mathcal{C}(L,l,l'';0,0,0) p_{l''}(kR)$$
$$\cdot \left[\sum_{\mu=-1}^{1} \mathcal{C}(L,l,l'';M+\mu,-m,M-m+\mu) \right.$$
$$\left. \cdot \mathcal{C}(L,1,L;M{+}\mu,-\mu,M) Y_{l'',M-m+\mu}(\theta_R,\phi_R) Y_{1,-\mu}(\theta_R,\phi_R) \right]^*, \quad (40)$$

where $\Lambda(l,L,l'')$ and $\Lambda(L,l,l'')$ are defined, respectively, as $\Lambda(L,l,l'') = 4\pi i^{(-L+l+l''-1)}\sqrt{\frac{(2L+1)(2l+1)}{3L(L+1)}}$ and $\Lambda(l,L,l'') = 4\pi i^{(-l+L+l''-1)} \cdot \sqrt{\frac{(2L+1)(2l+1)}{3L(L+1)}}$. (19) indicates that $\Lambda(l,L,l'')$ and $\Lambda(L,l,l'')$ are both real and related to each other by

$$\sqrt{l(l+1)}\Lambda(l,L,l'') = (-1)^{l+L}\sqrt{L(L+1)}\Lambda(L,l,l''). \quad (41)$$

The quantities in the square brackets in (39) and (40) can be related to each other by using the recurrence relation of the Clebsch-Gordan [20–22],

$$\begin{aligned} &\sqrt{J(J+1)}\mathcal{C}(J,1,J;M+\mu,-\mu,M)\mathcal{C}(j_1,j_2,J;m_1,m_2,M) \\ =\ &\sqrt{j_1(j_1{+}1)}\mathcal{C}(j_1,1,j_1;m_1{+}\mu,{-}\mu,m_1)\mathcal{C}(j_1,j_2,J;m_1{+}\mu,m_2,M{+}\mu) \\ +\ &\sqrt{j_2(j_2{+}1)}\mathcal{C}(j_2,1,j_2;m_2{+}\mu,{-}\mu,m_2)\mathcal{C}(j_1,j_2,J;m_1,m_2{+}\mu,M{+}\mu) \end{aligned}$$

and the symmetry relation [20–22],

$$\begin{aligned} \mathcal{C}(a,b,c;m_a,m_b,m_c) \ = \ &(-1)^{(b+m_b)}\sqrt{(2c+1)/(2a+1)} \\ &\cdot\mathcal{C}(c,b,a;m_c,m_b,m_a). \end{aligned}$$

Using these relations, the product of the Clebsch-Gordan coefficients in the square bracket in (40) may be written as

$$\begin{aligned} &\mathcal{C}(L,1,L:M+\mu,-\mu,M)\mathcal{C}(L,l,l'';M+\mu,-m,M-m+\mu) \\ =\ &\frac{1}{\sqrt{L(L+1)}}\left[\sqrt{l''(l''+1)}\mathcal{C}(l'',1,l'';M-m+\mu,-\mu,M-m)\right. \\ &\cdot\mathcal{C}(L,l,l'';M,-m,M-m) \\ &\left.+\sqrt{l(l{+}1)}\mathcal{C}(l,1,l;m{-}\mu,\mu,m)(L,l,l'';-M,m{-}\mu,m{-}M{-}\mu)\right]. \quad (42) \end{aligned}$$

Since $\sum_{\mu=-1}^{1} \mathcal{C}(l'', 1, l''; M-m+\mu, -\mu, M-m) Y_{l'',M-m+\mu}(\theta_R, \phi_R) Y_{1,-\mu}$ $(\theta_R, \phi_R) = 0$, the quantity in the square bracket, with the aid of (41), may be written as

$$(-1)^{m+M} \sqrt{\frac{l(l+1)}{L(L+1)}} \left[\sum_{\mu=-1}^{1} \mathcal{C}(l, 1, l; m-\mu, \mu, m) \right.$$
$$\left. \mathcal{C}(L, l, l''; -M, m-\mu, m-M-\mu) Y_{l'',M-m+\mu}(\theta_R, \phi_R) Y_{1,-\mu}(\theta_R, \phi_R) \right]^* .$$

Comparing this equation with the corresponding term in (39) and using (41), we obtain the symmetry relations of $\mathcal{B}_{L,M}^{l,m}(k, \boldsymbol{R})$,

$$\mathcal{B}_{l,m}^{L,M}(0, k, \boldsymbol{r}) = (-1)^{l+L+1} \left[\mathcal{B}_{L,M}^{l,m}(0, k^*, \boldsymbol{r})\right]^*, \text{ if } p_{l''}(kR) = j_{l''}(kR),$$
$$\text{and} \tag{43}$$
$$\mathcal{B}_{l,m}^{L,M}(\pm, k, \boldsymbol{r}) = (-1)^{l+L+1} \left[\mathcal{B}_{L,M}^{l,m}(\mp, k^*, \boldsymbol{r})\right]^*, \text{ if } p_{l''}(kR) = h_{l''}^{(\pm)}(kR). \tag{44}$$

If the k is real, then the above equations, respectively, reduce to

$$\mathcal{B}_{l,m}^{L,M}(0, k, \boldsymbol{r}) = (-1)^{l+L+1} \left[\mathcal{B}_{L,M}^{l,m}(0, k, \boldsymbol{r})\right]^*, \text{ if } p_{l''}(kR) = j_{l''}(kR),$$
$$\text{and} \tag{45}$$
$$\mathcal{B}_{l,m}^{L,M}(\pm, k, \boldsymbol{r}) = (-1)^{l+L+1} \left[\mathcal{B}_{L,M}^{l,m}(\mp, k, \boldsymbol{r})\right]^*, \text{ if } p_{l''}(kR) = h_{l''}^{(\pm)}(kR), \tag{46}$$

both of which again can be reconciled with Peterson and Ström's result for $S_{L,J,M}^{l,j,m}(k, \boldsymbol{R})$ [9]. As in the spatial-inversion symmetry relations, $\mathcal{A}_{L,M}^{l,m}(k, \boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k, \boldsymbol{R})$ undergo different phase changes under interchange of model indices.

4.3. Simultaneous Changes of the Signs of the Azimuthal Indices

We investigate the symmetry properties of $\mathcal{A}_{L,M}^{l,m}(k, \boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k, \boldsymbol{R})$ under simultaneous changes of the signs of the azimuthal indices m and M. Using (2), (25), and (26), we can show that

$$\beta_{l,-m,l'}^{L,-M}(0, k, \boldsymbol{R}) = (-1)^{l'+L+m+M} \left[\beta_{l,m,l'}^{L,M}(0, k^*, \boldsymbol{R})\right]^*,$$

$$\text{if } p_{l''}(kR) = j_{l''}(kR) \tag{47}$$

$$\text{and } \beta_{l,-m,l'}^{L,-M}(\pm, k, \boldsymbol{R}) = (-1)^{l'+L+m+M} \left[\beta_{l,m,l'}^{L,M}(\mp, k^*, \boldsymbol{R})\right]^*, \quad \text{if } p_{l''}(kR) = h_{l''}^{(\pm)}(kR). \tag{48}$$

The desired symmetry relations of $\mathcal{A}_{L,M}^{l,m}(k, \boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k, \boldsymbol{R})$ then can be readily obtained by substituting the above equations into (13) and (14).

$$\mathcal{A}_{L,-M}^{l,-m}(0, k, \boldsymbol{R}) = (-1)^{m+M} \left[\mathcal{A}_{L,M}^{l,m}(0, k^*, \boldsymbol{R})\right]^*, \text{ if } p_{l''}(kR) = j_{l''}(kR), \tag{49}$$

$$\mathcal{A}_{L,-M}^{l,-m}(\pm, k, \boldsymbol{R}) = (-1)^{m+M} \left[\mathcal{A}_{L,M}^{l,m}(\mp, k^*, \boldsymbol{R})\right]^*, \text{ if } p_{l''}(kR) = h_{l''}^{(\pm)}(kR), \tag{50}$$

$$\mathcal{B}_{L,-M}^{l,-m}(0, k, \boldsymbol{R}) = (-1)^{m+M} \left[\mathcal{B}_{L,M}^{l,m}(0, k^*, \boldsymbol{R})\right]^*, \text{ if } p_{l''}(kR) = j_{l''}(kR), \quad \text{and} \tag{51}$$

$$\mathcal{B}_{L,-M}^{l,-m}(\pm, k, \boldsymbol{R}) = (-1)^{m+M} \left[\mathcal{B}_{L,M}^{l,m}(\mp, k^*, \boldsymbol{R})\right]^*, \text{ if } p_{l''}(kR) = h_{l''}^{(\pm)}(kR). \tag{52}$$

We note that when k is real, the above symmetry relations fail to reduce to the corresponding relations of $S_{L,J,M}^{l,j,m}(k, \boldsymbol{R})$ reported in [9],

$$S_{L,j,-M}^{l,j,-m}(0, k, \boldsymbol{R}) = (-1)^{(J-L+M)-(j-l+m)} \left[S_{L,j,M}^{l,j,m}(0, k, -\boldsymbol{R})\right]^* \quad \text{if } p_{l''}(kR) = j_{l''}(kR),$$

$$\text{and } S_{L,j,-M}^{l,j,-m}(\pm, k, \boldsymbol{R}) = (-1)^{(J-L+M)-(j-l+m)} \left[S_{L,j,M}^{l,j,m}(\pm, k, -\boldsymbol{R})\right]^* \quad \text{if } p_{l''}(kR) = h_{l''}^{(\pm)}(kR),$$

both of which are believed to be in error.

As they stand, the symmetry relations of (35), (36), (43), (44), and (49)–(52) are not useful for numerical applications for the reasons stated in the previous section. However, as in the scalar case, combining two types of symmetry relations produces results that are useful in numerical applications,

$$\mathcal{A}_{L,-M}^{l,-m}(k, \boldsymbol{R}) = (-1)^{l+L+m+M} \mathcal{A}_{l,m}^{L,M}(k, \boldsymbol{R}),$$

$$\text{and } \mathcal{B}_{L,-M}^{l,-m}(k, \boldsymbol{R}) = (-1)^{l+L+m+M+1} \mathcal{B}_{l,m}^{L,M}(k, \boldsymbol{R}).$$

We note that both of the above relations are valid whether k is complex or real and whether $p_{l''}(kR) = j_{l''}(kR)$ or $p_{l''}(kR) = h_{l''}^{(\pm)}(kR)$ in (6).

4.4. Reflection of the Translation Vector about the xy-, zy-, and zy-Planes

We investigate the respective symmetry properties of $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ under reflection of $\boldsymbol{R}$ about the yz-, xz- and xy-planes. Using the respective reflection symmetry relations of $\alpha_{L,M}^{l,m}(k,\boldsymbol{R})$ and $Y_{l,m}(\theta,\phi)$ we obtain the following reflection symmetry relations of $\beta_{l,m,l'}^{L,M}(k,\boldsymbol{R})$.

$$\begin{aligned}
\beta_{l,m,l'}^{L,M}(k,-x,y,z) &= (-1)^{(l'+L+1)}\beta_{l,-m,l'}^{L,-M}(k,x,y,z),\\
\beta_{l,m,l'}^{L,M}(k,x,-y,z) &= (-1)^{(l'+L+m+M+1)}\beta_{l,-m,l'}^{L,-M}(k,x,y,z),\\
\text{and } \beta_{l,m,l'}^{L,M}(k,x,y,-z) &= (-1)^{(l+l'+m+M+1)}\beta_{l,m,l'}^{L,M}(k,x,y,z).
\end{aligned}$$

Substitution of these relations into (13) and (14), respectively, yields the following reflection symmetry relations $A_{L,M}^{l,m}(k,\boldsymbol{r})$ and $B_{L,M}^{l,m}(k,\boldsymbol{r})$,

$$\begin{aligned}
A_{L,M}^{l,m}(k,-x,y,z) &= A_{L,-M}^{l,-m}(k,x,y,z),\\
A_{L,M}^{l,m}(k,x,-y,z) &= (-1)^{(m+M)}A_{L,-M}^{l,-m}(k,x,y,z),\\
A_{L,M}^{l,m}(k,x,y,-z) &= (-1)^{(l+L+m+M)}A_{L,M}^{l,m}(k,x,y,z)
\end{aligned}$$

and

$$\begin{aligned}
B_{L,M}^{l,m}(k,-x,y,z) &= -B_{L,-M}^{l,-m}(k,x,y,z),\\
B_{L,M}^{l,m}(k,x,-y,z) &= (-1)^{(m+M+1)}B_{L,-M}^{l,-m}(k,x,y,z),\\
B_{L,M}^{l,m}(k,x,y,-z) &= (-1)^{(l+L+m+M+1)}B_{L,M}^{l,m}(k,x,y,z).
\end{aligned}$$

Several comments are in order. First, each of the above relations holds whether k is real or complex, and whether $p_{l''}(kR)$ is $j_{l''}(kR)$ or $h_{l''}^{(\pm)}(kR)$. Second, they are consistent with the spatial-inversion symmetry relations (33) and (34) since

$$\begin{aligned}
A_{L,M}^{l,m}(k,-x,-y,-z) &= A_{L,-M}^{l,-m}(k,x,-y,-z),\\
&= (-1)^{m+M}A_{L,M}^{l,m}(k,x,y,-z)\\
&= (-1)^{l+L}A_{L,M}^{l,m}(k,x,y,z)\\
\text{and } B_{L,M}^{l,m}(k,-x,-y,-z) &= -B_{L,-M}^{l,-m}(k,x,-y,-z),\\
&= (-1)^{m+M}B_{L,M}^{l,m}(k,x,y,-z)\\
&= (-1)^{l+L+1}B_{L,M}^{l,m}(k,x,y,z).
\end{aligned}$$

Third, the sign difference between the symmetry relations of $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ for each reflection is consistent with the reflection properties of $\boldsymbol{M}_{l,m}(k,\boldsymbol{r})$ and $\boldsymbol{N}_{l,m}(k,\boldsymbol{r})$. Application of (4) to (9) and (10) yields the following reflection symmetry relations of $\boldsymbol{M}_{l,m}(k,\boldsymbol{r})$ and $\boldsymbol{N}_{l,m}(k,\boldsymbol{r})$.

$$\begin{aligned}
\boldsymbol{M}_{l,m}(k,-x,y,z) &= -\Big[-M_{l,-m}^{(x)}(k,x,y,z)\tilde{\boldsymbol{x}} \\
&\quad + M_{l,-m}^{(y)}(k,x,y,z)\hat{\boldsymbol{y}} + M_{l,-m}^{(z)}(k,x,y,z)\hat{\boldsymbol{z}}\Big], \\
\boldsymbol{M}_{l,m}(k,x,-y,z) &= (-1)^{(m+1)}\Big[M_{l,-m}^{(x)}(k,x,y,z)\tilde{\boldsymbol{x}} \\
&\quad - M_{l,-m}^{(y)}(k,x,y,z)\hat{\boldsymbol{y}} + M_{l,-m}^{(z)}(k,x,y,z)\hat{\boldsymbol{z}}\Big], \\
\boldsymbol{M}_{l,m}(k,x,y,-z) &= (-1)^{(l+m+1)}\Big[M_{l,m}^{(x)}(k,x,y,z)\tilde{\boldsymbol{x}} \\
&\quad + M_{l,m}^{(y)}(k,x,y,z)\hat{\boldsymbol{y}} - M_{l,m}^{(z)}(k,x,y,z)\hat{\boldsymbol{z}}\Big],
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{N}_{l,m}(k,-x,y,z) &= -N_{l,-m}^{(x)}(k,x,y,z)\tilde{\boldsymbol{x}} \\
&\quad + N_{l,-m}^{(y)}(k,x,y,z)\hat{\boldsymbol{y}} + N_{l,-m}^{(z)}(k,x,y,z)\hat{\boldsymbol{z}}, \\
\boldsymbol{N}_{l,m}(k,x,-y,z) &= (-1)^{(m)}\Big[N_{l,-m}^{(x)}(k,x,y,z)\tilde{\boldsymbol{x}} \\
&\quad - N_{l,-m}^{(y)}(k,x,y,z)\hat{\boldsymbol{y}} + N_{l,-m}^{(z)}(k,x,y,z)\hat{\boldsymbol{z}}\Big], \\
\boldsymbol{N}_{l,m}(k,x,y,-z) &= (-1)^{(l+m)}\Big[M_{l,m}^{(x)}(k,x,y,z)\tilde{\boldsymbol{x}} \\
&\quad + N_{l,m}^{(y)}(k,x,y,z)\hat{\boldsymbol{y}} - N_{l,m}^{(z)}(k,x,y,z)\hat{\boldsymbol{z}}\Big],
\end{aligned}$$

which explain the phase difference.

5. SUMMARY

We have examined the symmetry properties of the translation coefficients of the scalar and vector spherical multipole fields. The symmetry relations considered include not only those considered earlier for real wavenumbers in [9], but also the respective symmetry relations that arise when the translation vector is reflected about the xy-, yz-, and zx-planes. In addition, all the symmetry relations considered in this work are valid for complex wavenumbers. These symmetry relations may be used to reduce the computational cost of evaluating and storing the translation coefficients for the applications mentioned earlier because they are expressed in a form suitable for exploitation in numerical applications. These relations were successfully applied to significantly reduce the memory requirement of the FFT T-matrix technique [19].

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APPENDIX A.

It is possible to define a more general spherical vector multipole field than $\boldsymbol{L}_{l,m}(k,\boldsymbol{r})$, $\boldsymbol{M}_{l,m}(k,\boldsymbol{r})$ and $\boldsymbol{N}_{l,m}(k,\boldsymbol{r})$. The spherical tensor field $\boldsymbol{T}_{l,1}^{j,m}(k,\boldsymbol{r})$ of order (j,m) [9, 20–22] is defined by

$$\begin{aligned}\boldsymbol{T}_{l,1}^{j,m}(k,\boldsymbol{r}) &\equiv f_l(kr)\boldsymbol{Y}_{l,1}^{j,m}(\theta,\phi)\\ &\equiv f_l(kr)\sum_{\mu=-1}^{1}\mathcal{C}(l,1,j;m-\mu,\mu,m)Y_{l,m-\mu}(\theta,\phi)\hat{\boldsymbol{e}}_\mu,\end{aligned}$$

where the spherical basis vectors $\hat{\boldsymbol{e}}_\mu$ are related to the Cartesian unit vectors by $\hat{\boldsymbol{e}}_0 = \hat{z}$ and $\hat{\boldsymbol{e}}_\pm = \mp\frac{1}{\sqrt{2}}(\hat{x}\pm i\hat{y})$. Under translation, $\boldsymbol{r}=\boldsymbol{R}+\boldsymbol{r}'$, $\boldsymbol{T}_{l,1}^{j,m}(k,\boldsymbol{r})$ translates according to

$$\boldsymbol{T}_{l,1}^{j,m}(k,\boldsymbol{r}) = \sum_{L,M,J} S_{L,J,M}^{l,j,m}(k,\boldsymbol{R})\tilde{\boldsymbol{T}}_{L,1}^{J,M}(k,\boldsymbol{r}'),$$

where $S_{L,J,M}^{l,j,m}(k,\boldsymbol{R})$ is the translation coefficient for $\boldsymbol{T}_{l,1}^{j,m}(k,\boldsymbol{r})$ [9]. The more familiar $\boldsymbol{L}_{l,m}(k,\boldsymbol{r})$, $\boldsymbol{M}_{l,m}(k,\boldsymbol{r})$ and $\boldsymbol{N}_{l,m}(k,\boldsymbol{r})$ can be expressed as a linear combination of $\boldsymbol{T}_{l,1}^{j,m}(k,\boldsymbol{r})$ [9, 20–22],

$$\begin{aligned}\boldsymbol{L}_{l,m}(k,\boldsymbol{r}) &= \left[\frac{l+1}{2l+1}\right]^{1/2}\boldsymbol{T}_{l+1,1}^{j,m}(k,\boldsymbol{r}) + \left[\frac{l}{2l+1}\right]^{1/2}\boldsymbol{T}_{l-1,1}^{j,m}(k,\boldsymbol{r}),\\ \boldsymbol{M}_{l,m}(k,\boldsymbol{r}) &= \boldsymbol{T}_{l,1}^{l,m}(k,\boldsymbol{r}),\\ \boldsymbol{N}_{l,m}(k,\boldsymbol{r}) &= -\left[\frac{l}{2l+1}\right]^{1/2}\boldsymbol{T}_{l+1,1}^{j,m}(k,\boldsymbol{r}) + \left[\frac{l+1}{2l+1}\right]^{1/2}\boldsymbol{T}_{l-1,1}^{j,m}(k,\boldsymbol{r}).\end{aligned}$$

The latter two relations may be used to relate $S_{L,J,M}^{l,j,m}(k,\boldsymbol{R})$ to $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ [9],

$$\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R}) = S_{L,J,M}^{l,j,m}(k,\boldsymbol{R}), \tag{A1}$$

$$\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R}) = \frac{i}{(2l+1)^{1/2}}\left[\sqrt{l+1}S_{L,L,M}^{l-1,j,m}(k,\boldsymbol{R}) - \sqrt{l}S_{L,L,M}^{l+1,j,m}(k,\boldsymbol{R})\right], \tag{A2}$$

which can be used to compare the symmetry relations of $\mathcal{A}_{L,M}^{l,m}(k,\boldsymbol{R})$ and $\mathcal{B}_{L,M}^{l,m}(k,\boldsymbol{R})$ derived in this paper with the corresponding symmetry relations of $S_{L,J,M}^{l,j,m}(k,\boldsymbol{R})$ reported in [9].

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