Progress In Electromagnetics Research, PIER 48, 249–278, 2004

NON-RELATIVESTIC SCATTERING BY TIME-VARYING BODIES AND MEDIA

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Abstract—We are interested in first order v/c velocity effects in scattering problems involving motion of media and scatterers. Previously constant velocities have been considered for scattering by cylindrical and spherical configurations. Presently time-varying motion — specifically harmonic oscillation — is investigated. A first-order quasi-Lorentz transformation is introduced heuristically, in order to establish relations to existing exact Special-Relativistic results.

We then consider simple problems of plane interfaces, normal incidence, and uniform motion, in order to introduce the model: Starting with an interface moving with respect to the medium in which the excitation wave is introduced, then considering the problem of an interface at rest and a moving medium contained in a half space. The latter corresponds to a Fizeau experiment configuration. Afterwards these configurations are considered for harmonic motion. This provides the method for dealing with the corresponding problems of scattering by a circular cylinder, involving harmonic motion.

The present formalism provides a systematic approach for solving scattering problems in the presence of time-varying media and boundaries.

1. INTRODUCTION

Most remote-sensing systems in use are based on scattering of external excitation waves. In many cases the geometry of the system prescribes back scattering, e.g., when the data collection is performed by flying machines (airplanes, balloons, missiles), or satellites, by ground or airborne radar systems, or when we are dealing with acoustic ultrasound systems. One of the main difficulties for performing such measurements, especially forward scattering remote sensing, is the fact that the scattered wave is often irretrievably masked by the excitation wave, especially in situations where we need narrow spectrum signals. In these cases traditional wide spectrum methods like time gating, used in radar and ultrasound, might not be feasible. It is therefore suggestive that we look for possibilities to distinguish between the excitation and reflected waves using spectral methods.

In order to achieve this goal, we need some modulating mechanism that will create new frequencies, e.g., mechanisms based on mechanical motion of the scatterer, which will then modulate the excitation wave and produce new spectral components. The mechanical oscillations can be artificially induced, or inherently present in the scatterer, due to its structural functionality, like the rotation of the helicopter blades, or other rotation or vibration modes of structures.

Once the mechanical motion creates new spectral components, these can be filtered out from the scattered wave, and unhampered by the presence of the excitation wave frequency, the multi-spectral scattered waves can be exploited to study the scatterer's signature, which might reveal further properties regarding its material properties, shape, orientation, etc.

The theoretical interest in scattering problems involving motion is motivated by two reasons: The extension of various canonical problems facilitates a better understanding of the general class of velocity-dependent scattering. And, since motion is almost always involved with new spectral components, the associated scattering problems allow for multi-spectral models for remote-sensing scatterers.

The modeling of such scattering problems is sometimes controversial. In acoustics there was a debate whether new spectral components are due to motional effects of the scatterer, the so called Doppler-effect kind, or result from the compression of the medium occurring when the scatterer moves, see relevant discussions and citing of earlier references in [1–3]. In electromagnetic wave scattering other complications arise, resulting from the lack of a definitive first principles model for the boundary conditions, transformation of fields, and transformation of coordinates for arbitrary non-uniform motion.

Before proceeding to the problems at hand, two comments are in order: Note that some symbols appear in different contexts, because of their similar role. On one hand this contributes to a clear presentation, but some care is called for. Note also that only first order in v/c terms are retained. This implies in a perturbation-iteration sense that in expressions multiplied by the velocity, the parameters are not unknowns, but are supposedly already known from the corresponding problem in which the velocity vanishes.

2. FIRST ORDER LORENTZ TRANSFORMATION

Einstein's Special-Relativity [4–6] involves inertial (un-accelerated) frames of reference moving at constant velocities. Coordinates are related by the Lorentz transformation, which to the first order in v/c, where c is the speed of light in free space, becomes

$$\begin{aligned} \boldsymbol{r}' &= \boldsymbol{r} - \boldsymbol{v}t \\ t' &= t - \boldsymbol{v}\cdot\boldsymbol{r}/c^2 \\ \hat{\boldsymbol{v}} &= \boldsymbol{v}/v, \quad \beta = v/c, \quad v = |\boldsymbol{v}| \end{aligned} \tag{1}$$

In (1) $\boldsymbol{r}$ is the position vector, and $\boldsymbol{v}$ is the velocity of the primed frame, as observed from the unprimed one. It is sometimes claimed that for low velocities the second equation (1) becomes $t' = t$, reducing (1) to the Galilean transformation. This is an incorrect statement if the two equations (1) are to be consistently considered to the first order in v/c. For convenience, the quadruplet of coordinates $\boldsymbol{r}$, t is compacted using the Minkowski four-vector notation, as $\boldsymbol{R} = (\boldsymbol{r}, ict)$. For brevity, (1) can be represented as $\boldsymbol{R}' = \boldsymbol{R}'[\boldsymbol{R}]$.

It has been observed previously, see for example [6, 7], that the role of spatiotemporally-dependent velocity needs special considerations, and in fact leads to a breakdown of the perfect axiomatic structure of relativistic electrodynamics. This is a well-known problem in physics: In his book [8] (see p. 162 ff.), Bohm discusses the hypothesis of the *locally co-moving un-accelerated frame*, whereby at each point in space and instance in time we attach to the accelerated system an inertial system, and use it until the discrepancy in the *instantaneous* velocity requires to define a new co-moving inertial frame. If we accept this idea, it follows that in (1) a local and *instantaneous* varying velocity $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{R})$ must be employed. But if one does not wish to treat the instantaneous velocity as a constant, as in the following problems involving oscillating boundaries and media, this heuristic proviso cannot be implemented and (1) must be modified.

As (1) stands, it is invertible, satisfying to the first order in v/c

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}' + \boldsymbol{v}t' \\ t &= t' + \boldsymbol{v} \cdot \boldsymbol{r}'/c^2 \end{aligned} \tag{2}$$

which can be represented as $\boldsymbol{R} = \boldsymbol{R}[\boldsymbol{R}']$. Clearly for $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{R})$ the invertibility is lost. We can look at it in another way: According to (1), $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{v}t$ is an equation of motion for a fixed initial position $\boldsymbol{r} = \boldsymbol{r}'$ at time $t = 0$. It follows that $d\boldsymbol{r}/dt = \boldsymbol{v}$, in accordance with the definition of velocity. On the other hand, if we use $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{R})$, then we get $d\boldsymbol{r}/dt = \boldsymbol{v} + t(d\boldsymbol{v}/dt)$, defying invertibility. Anyway, in general the position of a point along a trajectory should be given as an integral of the incremental distances covered by this point in time, which is not satisfied by simplistically using $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{R})$ in (1), (2). This point has been mentioned before [9]. Consequently, a new modified first order *quasi*-Lorentz transformation suggests itself

$$\begin{aligned} \boldsymbol{r}' &= \boldsymbol{r} - \int_{\boldsymbol{R}_0}^{\boldsymbol{R}} \boldsymbol{v}(\overline{\boldsymbol{R}}) d\bar{t} \\ t' &= t' - c^{-2} \int_{\boldsymbol{R}_0}^{\boldsymbol{R}} \boldsymbol{v}(\overline{\boldsymbol{R}}) \cdot d\overline{\boldsymbol{r}} \end{aligned} \tag{3}$$

In (3) $\boldsymbol{R}_0$ is a fixed reference position in Minkowski's space, and the bar indicates the integration variable, which is subsequently suppressed, assuming that the integration variable can be identified from the context. In (3) we have line integrals in the Minkowski space. The differentials of (3) yield

$$\begin{aligned} d\boldsymbol{r}' &= d\boldsymbol{r} - \boldsymbol{v}(\boldsymbol{R}) dt \\ dt' &= dt - \boldsymbol{v}(\boldsymbol{R}) \cdot d\boldsymbol{r}/c^2 \end{aligned} \tag{4}$$

but since $\partial_{\boldsymbol{r}} \times \boldsymbol{r} = 0$ is identically zero, the first line (4), hence also the first line (3), both require that the velocity field be laminar, i.e., $\partial_{\boldsymbol{r}} \times \boldsymbol{v} = 0$. Furthermore, deriving the second line (4) from the second line (3) also requires that $\boldsymbol{v}(\boldsymbol{R}) \cdot d\boldsymbol{r}/c^2$, be an entire differential, in order for the integral to be independent of the limits, which in turn entails $\partial_{\boldsymbol{r}} \times \boldsymbol{v} = 0$ once more.

Because of the restriction $\partial_{\boldsymbol{r}} \times \boldsymbol{v} = 0$, the present transformation (4) does not apply, for example, to systems rotating with a uniform angular velocity. See discussion on rotating media and references to the literature in Van Bladel [7]. Some results for scattering by rotating systems is given elsewhere [10].

For $d\boldsymbol{r}' = 0$ we now obtain $d\boldsymbol{r}/dt = \boldsymbol{v}(\boldsymbol{R})$ from the first line (4), retaining the original definition of the velocity. By substituting into the second line (4) we find the time dilatation $dt' = dt(1-(v/c)^2) \simeq dt$, which is a second order phenomenon in v/c, and therefore negligible in our present model.

Consequently, (4), and therefore (3), satisfies the relativistic transformation for velocities to the first order in v/c, becoming

$$\boldsymbol{u}' = (\boldsymbol{u} - \boldsymbol{v})/(1 - \boldsymbol{u} \cdot \boldsymbol{v}/c^2) \tag{5}$$

To the first order in v/c (4) can be recast as

$$\begin{aligned} d\boldsymbol{r} &= d\boldsymbol{r}' + \boldsymbol{v}(\boldsymbol{R}')dt' \\ dt &= dt' + \boldsymbol{v}(\boldsymbol{R}') \cdot d\boldsymbol{r}'/c^2, \quad \boldsymbol{v}(\boldsymbol{R}') = \boldsymbol{v}(\boldsymbol{R}[\boldsymbol{R}']) \end{aligned} \tag{6}$$

where in (6) $\boldsymbol{v}(\boldsymbol{R}')$ is understood as the initial velocity field, expressed in terms of the primed coordinates. In this sense (4) and (6) constitute a pair of invertible transformations, in the same sense as we had for the constant velocity in (1), (2).

We will be interested in the phase of plane waves. which can be represented as a line integral in Minkowski space, e.g., see [6]

$$\begin{aligned} \theta(\boldsymbol{R}) &= \int_{\theta(\boldsymbol{R}_0)}^{\theta(\boldsymbol{R})} d\theta = \int_{\boldsymbol{R}_0}^{\boldsymbol{R}} \partial_{\boldsymbol{R}}\theta(\boldsymbol{R}) \cdot d\boldsymbol{R} = \int_{\boldsymbol{R}_0}^{\boldsymbol{R}} \boldsymbol{K}(\boldsymbol{R}) \cdot d\boldsymbol{R} \\ \partial_{\boldsymbol{R}} &= \left(\partial_{\boldsymbol{r}}, -\frac{i}{c}\partial_t\right), \qquad \boldsymbol{K} = (\boldsymbol{k}, i\omega/c^2) \end{aligned} \tag{7}$$

where the last term in (7) defines the spectral-space coordinate quadruplet $\boldsymbol{K}$, also a Minkowski four-vector. Written in this manner as an entire differential, the line integral is independent of the integration path, depending on the limits only. This is tantamount to having $\partial_{\boldsymbol{R}} \times \boldsymbol{K}(\boldsymbol{R}) = 0$, but such a four-dimensional rotor operation is meaningless, and must be considered as symbolizing six equations $\frac{\partial K_i}{\partial R_j} - \frac{\partial K_j}{\partial R_i}$, $i, j = 1, 2, 3, 4$. This amounts to writing in three dimensions and time

$$\begin{aligned} \partial_{\boldsymbol{r}} \times \boldsymbol{k}(\boldsymbol{R}) &= 0 \\ \partial_t \boldsymbol{k}(\boldsymbol{R}) + \partial_{\boldsymbol{r}}\omega(\boldsymbol{R}) &= 0 \end{aligned} \tag{8}$$

where (8) is the so-called Sommerfeld-Runge law of refraction [11], whereby the first line (8) is recognized as the well-known Snell law (see also [6]).

To the first order in v/c, the relativistic transformation for the components of $\boldsymbol{K}$ becomes

$$\begin{aligned} \boldsymbol{k}' &= \boldsymbol{k} - \boldsymbol{v}(\boldsymbol{R})\omega/c^2 \\ \omega' &= \omega - \boldsymbol{v}(\boldsymbol{R}) \cdot \boldsymbol{k} \end{aligned} \tag{9}$$

where the first line (9) is related to the Fresnel drag effect, and the second line is the so-called relativistic Doppler effect. Similarly to (1), we can represent (9) as $\boldsymbol{K'} = \boldsymbol{K}'[\boldsymbol{K}]$. Once again, to the first order in v/c, the inverse of (9), similarly to (2), is written as $\boldsymbol{K} = \boldsymbol{K}[\boldsymbol{K}']$

$$\begin{aligned} \boldsymbol{k} &= \boldsymbol{k}' + \boldsymbol{v}(\boldsymbol{R}')\omega'/c^2 \\ \omega &= \omega' + \boldsymbol{v}(\boldsymbol{R}') \cdot \boldsymbol{k}', \quad \boldsymbol{R}' = \boldsymbol{R}'[\boldsymbol{R}] \end{aligned} \tag{10}$$

Note that in (9) we allow a field $\boldsymbol{v}(\boldsymbol{R})$, hence all the variables become spatiotemporally-dependent and in (9) all terms are dependent on space-time coordinates

$$\begin{aligned} \boldsymbol{k}'(\boldsymbol{R}') &= \boldsymbol{k}(\boldsymbol{R}) - \boldsymbol{v}(\boldsymbol{R})\omega(\boldsymbol{R})/c^2 \\ \omega'(\boldsymbol{R}') &= \omega(\boldsymbol{R}) - \boldsymbol{v}(\boldsymbol{R}) \cdot \boldsymbol{k}(\boldsymbol{R}) \end{aligned} \tag{11}$$

In the present context, according to (4), it is easily shown that to the first order in v/c

$$\begin{aligned} \theta(\boldsymbol{R}) &= \int_{\boldsymbol{R}_0}^{\boldsymbol{R}} \boldsymbol{K}(\boldsymbol{R}) \cdot d\boldsymbol{R} = \theta'(\boldsymbol{R}') = \int_{\boldsymbol{R}'_0}^{\boldsymbol{R}'} \boldsymbol{K}'(\boldsymbol{R}') \cdot d\boldsymbol{R}' \\ \boldsymbol{R}' &= \boldsymbol{R}'[\boldsymbol{R}], \quad \boldsymbol{K}(\boldsymbol{R}) \cdot d\boldsymbol{R} = \boldsymbol{k} \cdot d\boldsymbol{r} - \omega dt \end{aligned} \tag{12}$$

often referred to as "the invariance of the phase" or "covariance of the phase". For constant $\boldsymbol{K}$ (12) becomes after integration $\theta(\boldsymbol{R}) = \boldsymbol{K} \cdot \boldsymbol{R} = \theta'(\boldsymbol{R}') = \boldsymbol{K}' \cdot \boldsymbol{X}'$. with $\boldsymbol{K} \cdot \boldsymbol{R} = \boldsymbol{k} \cdot \boldsymbol{r} - \omega t$.

We are now ready to analyze scattering problems. For this we need boundary conditions. From relativistic electrodynamics considerations it is known that the velocity affects the field amplitudes too. To the first order in v/c we thus have

$$\begin{aligned} \boldsymbol{E}'(\boldsymbol{R}') &= \boldsymbol{E}(\boldsymbol{R}) + \boldsymbol{v}(\boldsymbol{R}) \times \boldsymbol{B}(\boldsymbol{R}) \\ \boldsymbol{B}'(\boldsymbol{R}') &= \boldsymbol{B}(\boldsymbol{R}) - \boldsymbol{v}(\boldsymbol{R}) \times \boldsymbol{E}(\boldsymbol{R})/c^2 \\ \boldsymbol{D}'(\boldsymbol{R}') &= \boldsymbol{D}(\boldsymbol{R}) + \boldsymbol{v}(\boldsymbol{R}) \times \boldsymbol{H}(\boldsymbol{R})/c^2 \\ \boldsymbol{H}'(\boldsymbol{R}') &= \boldsymbol{H}(\boldsymbol{R}) - \boldsymbol{v}(\boldsymbol{R}) \times \boldsymbol{D}(\boldsymbol{R}) \end{aligned} \tag{13}$$

where in (13) we also include the quasi-relativistic assumption that the formulas hold for spatiotemporally varying velocity. In the same sense as (9), (10), also (13) is invertible. It has been shown [12–14], that first order in v/c relativistic boundary conditions as in (13), and non-relativistic considerations based on the Lorentz force formulas lead to the same boundary conditions.

In the next sections, in order to introduce concepts and notation, we start with simple problems involving uniform motion and plane interface boundaries.

3. UNIFORMLY MOVING PLANE INTERFACE

The non-relativistic or quasi-relativistic first order in v/c model for scattering in velocity-dependent systems has been introduced and applied recently [12–14]. Instead of duplicating the essentials of the model, simple problems involving plane interface scattering will be analyzed.

The problem involves a plane interface and normal incidence. Two problems are considered: The first problem involves a half-space characterized by medium $\{\mathit{2}\}$, moving through an ambient material medium designated as $\{\mathit{1}\}$, in which the excitation and reflected waves propagate. In the next section the associated problem is analyzed, involving an interface at rest with respect to medium $\{\mathit{1}\}$, while medium $\{\mathit{2}\}$ moves with respect to the boundary.

In the first example we consider a medium $\{\mathit{1}\}$ with given parameters $\varepsilon^{(1)}$, $\mu^{(1)}$, in which the excitation wave is given by

$$\begin{aligned}\boldsymbol{E}_{ex} &= \hat{\boldsymbol{x}}E_{ex}e^{i\theta_{ex}},\ \boldsymbol{H}_{ex} = \hat{\boldsymbol{y}}H_{ex}e^{i\theta_{ex}},\ E_{ex}/H_{ex} = (\mu^{(1)}/\varepsilon^{(1)})^{1/2} = \zeta^{(1)}\\ \theta_{ex} &= k_{ex}z - \omega_{ex}t,\ k_{ex}/\omega_{ex} = (\mu^{(1)}\varepsilon^{(1)})^{1/2} = 1/v_{ph}^{(1)}\end{aligned} \tag{14}$$

The boundary is a plane interface moving through medium $\{\mathit{1}\}$ according to

$$z_T = z - vt \tag{15}$$

In (15) z_T denotes some arbitrary reference position in a local coordinate system of the boundary. The interface is at location $z_T = Z$. The origin $z_T = 0$ moves according to $z = vt$.

The phase of the wave (14) at $z_T = 0$ is given by

$$\theta_{ex0} = \theta_{ex}|_{z_T=0} = -\omega_{exT}t, \quad \omega_{exT} = \omega_{ex}(1-\beta^{(1)}), \quad \beta^{(1)} = v/v_{ph}^{(1)} \tag{16}$$

Note that $e^{i\theta_{ex0}}$ is not a wave, satisfying the wave equation, it is rather a *signal*.

According to (4) and the following remarks, to the first order in v/c it is immaterial whether we use t or t_T, i.e., whether the measurements are performed in the initial frame of reference attached to medium $\{1\}$, using sensors at different positions as prescribed by (15), or if measurements are performed by an observer attached to the boundary, in terms of t_T. This will be used for similar expressions below where the expressions refer to the vicinity of the boundary. The new frequency ω_{exT} in (16) is recognized as the first order in v/c relativistic Doppler frequency effect, as given in the second line (9).

To compute the phase shift between $z_T = 0$ and other points, specifically $z_T = Z$, we need to consider the effect of the moving medium, i.e., we need to include the Fresnel drag effect [5, 7, 15] embodied by the first line (9). The effect is usually associated with the celebrated Fizeau experiment (e.g., see [7]). Accordingly the phase shifts measured in a moving medium are different from those in the same medium at rest, or in free space, and are subject to the "drag effect" according to (9) and represented in the term of $A^{(1)}$ below.

In our case the phase at the boundary becomes

$$\begin{aligned}\theta_{exT} &= k_{exT}Z - \omega_{exT}t,\ k_{exT} = k_{ex} - v\omega_{ex}/c^2 = k_{ex}(1-\beta^{(1)}A^{(1)}),\\ &= q_{exT}(1-\beta^{(1)}A^{(1)})/(1-\beta^{(1)}) \simeq q_{exT}(1-\beta^{(1)}(A^{(1)}-1)) \quad (17)\\ q_{exT} &= \omega_{exT}/v_{ph}^{(1)}, \quad A^{(1)} = (v_{ph}^{(1)}/c)^2\end{aligned}$$

In free space $v_{ph}^{(1)} = c$, therefore $A^{(1)} = 1$ and the Fresnel drag effect vanishes.

An effective phase velocity can be defined using (17), to the first order in v/c

$$v_{eff,ex} = \omega_{exT}/k_{exT} = v_{ph}^{(1)}(1-\beta^{(1)})/(1-\beta^{(1)}A^{(1)}) \simeq v_{ph}^{(1)}(1+\beta^{(1)}(A^{(1)}-1)) \quad (18)$$

and for free space (18) reduces to $v_{eff,ex} = c$ as expected. As long as we deal with locations $z_T = Z$ in the vicinity of $z_T = 0$, we can leave the time as t, otherwise we have to replace it by $t' = t_T$ according to (1).

Except for the rare cases of dense materials where $A^{(1)}$ is on the order of $\beta^{(1)}$, the velocity effect on the propagation vector cannot be ignored. Ignoring the effect and assuming $k_{exT} = k_{ex}$ is tantamount to substituting a Galilean transformation in the phase (14), something we already know to be invalid for electromagnetic fields, because it violates Special-Relativity.

To find the reflected and transmitted fields the the boundary-value problem must be solved. Boundary conditions based on the Lorentz force concept have been recently proposed [12–14]. Essentially they involve (13). Accordingly, the boundary conditions are given by

$$\begin{gathered}\hat{\boldsymbol{n}}^{(b)} \times \left(\boldsymbol{E}_{eff}^{(1)} - \boldsymbol{E}^{(2)}\right) = 0, \quad \hat{\boldsymbol{n}}^{(b)} \times \left(\boldsymbol{H}_{eff}^{(1)} - \boldsymbol{H}^{(2)}\right) = 0 \\ \boldsymbol{E}_{eff}^{(1)} = \boldsymbol{E}^{(1)} + \boldsymbol{v} \times \boldsymbol{B}^{(1)}, \quad \boldsymbol{H}_{eff}^{(1)} = \boldsymbol{H}^{(1)} - \boldsymbol{v} \times \boldsymbol{D}^{(1)}\end{gathered} \tag{19}$$

In (19) superscripts correspond to media $\{\mathit{1}\}$ and $\{\mathit{2}\}$, and $\boldsymbol{E}_{eff}^{(1)}$, $\boldsymbol{H}_{eff}^{(1)}$ are the effective fields due to motion of medium $\{\mathit{1}\}$ when observed from the boundary, which is at rest with respect to medium $\{\mathit{2}\}$. To the first order in v/c, the unit vector $\hat{\boldsymbol{n}}^{(b)}$, normal to the boundary, is the same whether measurements are performed in $\{\mathit{1}\}$ or $\{\mathit{2}\}$.

Incorporating (14), (17), (19), we have at the boundary

$$\begin{gathered}\boldsymbol{E}_{exT} = \hat{\boldsymbol{x}} E_{exT} e^{i\theta_{exT}}, \quad \boldsymbol{H}_{exT} = \hat{\boldsymbol{y}} H_{exT} e^{i\theta_{exT}}, \\ E_{exT}/E_{ex} = H_{exT}/H_{ex} = 1 - \beta^{(1)}\end{gathered} \tag{20}$$

The reflected (scattered) wave is chosen as

$$\begin{gathered}\boldsymbol{E}_{sc} = \hat{\boldsymbol{x}} E_{sc} e^{i\theta_{sc}}, \quad \boldsymbol{H}_{sc} = -\hat{\boldsymbol{y}} H_{sc} e^{i\theta_{sc}}, \quad E_{sc}/H_{sc} = \zeta^{(1)} \\ \theta_{sc} = -k_{sc} z - \omega_{sc} t, \quad k_{sc}/\omega_{sc} = 1/v_{ph}^{(1)}\end{gathered} \tag{21}$$

Similarly to (16)

$$\begin{gathered}\theta_{sc0} = \theta_{sc}|_{z_T=0} = -\omega_{scT} t, \quad \omega_{scT} = \omega_{sc}(1 + \beta^{(1)}) = \omega_{exT} \\ \omega_{sc}/\omega_{ex} = k_{sc}/k_{ex} = (1 - \beta^{(1)})/(1 + \beta^{(1)}) \simeq 1 - 2\beta^{(1)}\end{gathered} \tag{22}$$

The proviso $\omega_{scT} = \omega_{exT}$ in (22) is prescribed by the boundary conditions (19): at the boundary the two waves must have identical time-dependence in order for the boundary conditions to be satisfied.

Similarly to (17), (18), (20), we have

$$\begin{gathered}\boldsymbol{E}_{scT} = \hat{\boldsymbol{x}} E_{scT} e^{i\theta_{scT}}, \quad \boldsymbol{H}_{scT} = -\hat{\boldsymbol{y}} E_{scT} e^{i\theta_{scT}}/\zeta^{(1)}, \\ E_{scT}/E_{sc} = H_{scT}/H_{sc} = 1 + \beta^{(1)} \quad \theta_{scT} = -k_{scT} Z - \omega_{exT} t, \\ k_{scT} = k_{sc} + v\omega_{sc}/c^2 = k_{sc}(1 + \beta^{(1)} A^{(1)}) = \omega_{exT}/v_{eff,sc} \\ = q_{exT}(1 + \beta^{(1)}(A^{(1)} - 1)) \\ v_{eff,sc} = \omega_{exT}/k_{scT} \simeq v_{ph}^{(1)}(1 - \beta^{(1)}(A^{(1)} - 1))\end{gathered} \tag{23}$$

Similarly to (14), in medium $\{\mathit{2}\}$ we stipulate a plane wave as well

$$\begin{aligned}\boldsymbol{E}_{in} &= \hat{\boldsymbol{x}} E_{in} e^{i\theta_{in}}, \ \boldsymbol{H}_{in} = \hat{\boldsymbol{y}} H_{in} e^{i\theta_{in}}, \ E_{in}/H_{in} = (\mu^{(2)}/\varepsilon^{(2)})^{1/2} = \zeta^{(2)} \\ \theta_{in} &= \kappa z_T - \omega_{exT} t, \ \kappa/\omega_{exT} = (\mu^{(2)}\varepsilon^{(2)})^{1/2} = 1/v_{ph}^{(2)}\end{aligned} \tag{24}$$

where in (24) ω_{exT} is prescribed by the boundary conditions at $z_T = Z$, ensuring the same time-dependence of all fields at the boundary.

We have now all the ingredients needed for the solution of the problem: The amplitude and phase for the excitation, scattered and internal waves are given by (17) and (20), (23), and (24), respectively, providing equations for determining the scattering and transmission coefficients E_{sc}/E_{ex}, E_{in}/E_{ex}, respectively. As a simple example, consider a perfectly conducting interface, prescribing the vanishing of the total tangential electrical field at the boundary, i.e.,

$$\begin{aligned} &\boldsymbol{E}_{exT} + \boldsymbol{E}_{scT} = 0 \\ &E_{ex}(1-\beta^{(1)})e^{iK_{ex}(1-\beta^{(1)}A^{(1)})} + E_{sc}(1+\beta^{(1)})e^{-iK_{sc}(1+\beta^{(1)}A^{(1)})} = 0 \\ &E_{sc}/E_{ex} \simeq -e^{i(K_{ex}+K_{sc})}\left[e^{i\beta^{(1)}(K_{sc}-K_{ex})A^{(1)}}(1-2\beta^{(1)})\right] \\ &\simeq -e^{i(K_{ex}+K_{sc})}(1-2\beta^{(1)}), \quad K_{sc} = k_{sc}Z, \quad K_{ex} = k_{ex}Z \end{aligned} \tag{25}$$

Interestingly, since $K_{sc} - K_{ex}$ is already of first order in $\beta^{(1)}$, the drag effect vanishes for this case to the first order in v/c. Clearly for $\beta^{(1)} = 0$ the problem reduces to the trivial case of scattering by a perfectly conducting plane in the absence of motion. For the special case $Z = 0$ the expression becomes even simpler, but for any scattering problem except the plane interface this simplification is inapplicable.

For an arbitrary medium $\{2\}$ we have instead of (25)

$$\begin{aligned} &\boldsymbol{E}_{exT} + \boldsymbol{E}_{scT} = \boldsymbol{E}_{inT}, \quad \boldsymbol{H}_{exT} + \boldsymbol{H}_{scT} = \boldsymbol{H}_{inT}, \quad K = \kappa Z \\ &E_{ex}(1-\beta^{(1)})e^{iK_{ex}(1-\beta^{(1)}A^{(1)})} + E_{sc}(1+\beta^{(1)})e^{-iK_{sc}(1+\beta^{(1)}A^{(1)})} \\ &\qquad = E_{in}e^{iK} \\ &E_{ex}(1-\beta^{(1)})e^{iK_{ex}(1-\beta^{(1)}A^{(1)})} - E_{sc}(1+\beta^{(1)})e^{-iK_{sc}(1+\beta^{(1)}A^{(1)})} \\ &\qquad = E_{in}e^{iK}\zeta^{(1)}/\zeta^{(2)} \end{aligned} \tag{26}$$

and (26) is solved for E_{sc}/E_{ex}, E_{in}/E_{ex} in an obvious manner.

Note that ω_{exT} computed above is a first order approximation in v/c, hence the dispersion properties of medium $\{2\}$ are only taken into account within this approximation.

4. UNIFORMLY MOVING HALF SPACE

Associated with the above moving boundary scattering problem, we have the so-called Fizeau experiment problem. Here the boundary is at rest with respect to medium $\{1\}$, while medium $\{2\}$ moves (cf. (15)) according to

$$z = \zeta - vt, \quad z = z_T \tag{27}$$

i.e., observed from medium $\{1\}$, a fixed point ζ = const. in medium $\{2\}$ is seen to move in the $-\hat{\boldsymbol{z}}$ direction, and observed from $\{2\}$, the boundary appears to move in the $+\hat{\boldsymbol{z}}$ direction.

We start with the excitation wave (14), and stipulate the scattered wave as (21) with (23) taken for the simple case $\beta^{(1)} = 0$, i.e., $\omega_{ex} = \omega_{exT} = \omega_{sc} = \omega_{scT}$. Instead of (24) we now consider a plane wave in the medium $\{2\}$ at rest, according to (27)

$$\begin{aligned} \boldsymbol{E}_{in} &= \hat{\boldsymbol{x}} E_{in} e^{i\theta_{in}}, \ \boldsymbol{H}_{in} = \hat{\boldsymbol{y}} H_{in} e^{i\theta_{in}}, \ E_{in}/H_{in} = (\mu^{(2)}/\varepsilon^{(2)})^{1/2} = \zeta^{(2)} \\ \theta_{in} &= \kappa\zeta - \omega_{in} t, \quad \kappa/\omega_{in} = (\mu^{(2)}\varepsilon^{(2)})^{1/2} = 1/v_{ph}^{(2)} \end{aligned} \tag{28}$$

The analog of (16) yields the phase at some reference location, say $z = 0$

$$\theta_{0in} = \theta_{in}|_{z=0} = -\omega_{ex} t = -\omega_{in} t(1 - \beta^{(2)}), \quad \beta^{(2)} = v/v_{ph}^{(2)} \tag{29}$$

In (29) ω_{ex} ensures that at the boundary all the waves have the same time-dependence. The analog of (17), (18), taking into account the Fresnel drag effect, includes the relative phase shift from $z = 0$ to the location $z = Z$ of the interface, yielding

$$\begin{aligned} &\theta_{inT} = \kappa_T Z - \omega_{ex} t, \ \kappa_T = \kappa - v\omega_{in}/c^2 = \kappa(1 - \beta^{(2)} A^{(2)}) \\ &= \omega_{in}(1 - \beta^{(2)} A^{(2)})/v_{ph}^{(2)} = \omega_{ex}/v_{eff,in} \\ &v_{eff,in} = v_{ph}^{(2)} \left(1 + \beta^{(2)}(A^{(2)} - 1)\right), \ A^{(2)} = (v_{ph}^{(2)}/c)^2 \end{aligned} \tag{30}$$

Similarly to (19), the boundary conditions are now given by

$$\begin{aligned} &\hat{\boldsymbol{n}}^{(b)} \times \left(\boldsymbol{E}_{eff}^{(2)} - \boldsymbol{E}^{(1)}\right) = 0, \quad \hat{\boldsymbol{n}}^{(b)} \times \left(\boldsymbol{H}_{eff}^{(2)} - \boldsymbol{H}^{(1)}\right) = 0 \\ &\boldsymbol{E}_{eff}^{(2)} = \boldsymbol{E}^{(2)} + \boldsymbol{v} \times \boldsymbol{B}^{(2)}, \quad \boldsymbol{H}_{eff}^{(2)} = \boldsymbol{H}^{(2)} - \boldsymbol{v} \times \boldsymbol{D}^{(2)} \end{aligned} \tag{31}$$

where $\boldsymbol{E}_{eff}^{(2)}$, $\boldsymbol{H}_{eff}^{(2)}$ are the effective fields due to motion of medium $\{2\}$, as observed at the boundary at rest with respect to medium $\{1\}$. In (19) $\boldsymbol{v}$ is the velocity of the boundary observed from medium $\{1\}$ at rest, in (31) $\boldsymbol{v}$ is the velocity of the boundary observed from medium $\{2\}$ at rest, therefore the signs in (19) and (31) are identical.

Accordingly, the analog of (20) is

$$\begin{aligned} &\boldsymbol{E}_{inT} = \hat{\boldsymbol{x}} E_{inT} e^{i\theta_{inT}}, \quad \boldsymbol{H}_{inT} = \hat{\boldsymbol{y}} H_{inT} e^{i\theta_{inT}} \\ &E_{inT}/E_{in} = H_{inT}/H_{in} = 1 - \beta^{(2)}, \ E_{int}/H_{inT} = (\mu^{(2)}/\varepsilon^{(2)})^{1/2} = \zeta^{(2)} \end{aligned} \tag{32}$$

We are now ready to compute the scattering coefficients. The boundary conditions (31) and the definitions of the various fields prescribe at $z = Z$ yield the analog of (26)

$$\begin{aligned} \boldsymbol{E}_{exT} + \boldsymbol{E}_{scT} &= \boldsymbol{E}_{inT}, \quad \boldsymbol{H}_{exT} + \boldsymbol{H}_{scT} = \boldsymbol{H}_{inT} \\ E_{ex}e^{iK_{ex}} + E_{ex}e^{-iK_{ex}} &= E_{in}(1-\beta^{(2)})e^{iK(1-\beta^{(2)}A^{(2)})} \\ E_{ex}e^{iK_{ex}} - E_{ex}e^{-iK_{ex}} &= E_{in}(1-\beta^{(2)})e^{iK(1-\beta^{(2)}A^{(2)})}\zeta^{(1)}/\zeta^{(2)}, \ \mathrm{K} = \kappa Z \end{aligned} \tag{33}$$

providing two equations for the two unknowns E_{sc}/E_{ex}, E_{in}/E_{ex}. Thus the problem is considered solved.

5. OSCILLATING PLANE INTERFACE

To the first order in v/c the above results are consistent with the relativistic formalism, for example see [16]. However, the present first order non-relativistic formalism is simpler to apply, and facilitates the solution of more intricate problems, as discussed in [12–14]. In the present study, examples involving non-uniform motion are investigated. Some relevant literature citations and results can be found in [7].

We start with the incident wave (14), and replace (15) by the equation of motion

$$z_T = z - z_0 S_{\Omega t}, \quad S_{\Omega t} = \sin \Omega t \tag{34}$$

which is commensurate with (3). Using (4) for $d\boldsymbol{r}' = 0$, i.e., $z_T = \text{const.}$ in (34), the instantaneous velocity $\boldsymbol{v}(\boldsymbol{R})$ is derived

$$v(t) = dz/dt = v_0 C_{\Omega t}, \quad v_0 = z_0\Omega, \quad C_{\Omega t} = \cos \Omega t \tag{35}$$

From (34) the local origin $z_T = 0$ moves according to $z = z_0 S_{\Omega t}$. Substituting in (14) yields the phase at this point (cf. (16))

$$\begin{aligned} \theta_{ex0} &= \theta_{ex}|_{z_T=0} = k_{ex}z_0 S_{\Omega t} - \omega_{ex}t, \quad e^{i\theta_{ex0}} = \Sigma_n I_n e^{-i\omega_n t} \\ \omega_n &= \omega_{ex} - n\Omega, \quad I_n = J_n(k_{ex}z_0), \quad \Sigma_n = \Sigma_{n=-\infty}^{n=\infty} \end{aligned} \tag{36}$$

where in principle the integer n in (36) covers, the range $-\infty$ to ∞, see (e.g., [17], p. 372), however the Bessel functions J_n spectrum tapers off as the sideband number n increases, therefore an adequate approximation is provided by truncating the series at some point.

Corresponding to (9), (17), (18) we compute for each frequency ω_n the phase shift from $z_T = 0$ to $z_T = Z$, essentially using the effective phase velocity $v_{eff,ex}$ as in (18). Hence the phase θ_n, corresponding to each frequency ω_n in (36) becomes

$$\begin{aligned} \theta_{in} &= k_{nT} Z - \omega_n t \\ k_{nT} &= \omega_n / v_{eff,ex} = k_n \left(1 - \beta_0^{(1)} (A^{(1)} - 1) C_{\Omega t}\right) \\ k_n &= \omega_n / v_{ph}^{(1)}, \quad \beta_0^{(1)} = v_0 / v_{ph}^{(1)} \end{aligned} \tag{37}$$

Upon including the amplitude effect as expressed in (20), we find the corresponding expression

$$\begin{aligned} \boldsymbol{E}_{exT} &= \hat{\boldsymbol{x}} E_{exT}, \quad \boldsymbol{H}_{exT} = \hat{\boldsymbol{y}} H_{exT} = \hat{\boldsymbol{y}} E_{exT} / \zeta^{(1)} \\ E_{exT} &= (1 - \beta_0^{(1)} C_{\Omega t}) E_{ex} \Sigma_n I_n (k_{ex} z_0) e^{i\theta_n} \end{aligned} \tag{38}$$

Inasmuch as the Fresnel drag effect in (37) is a first order velocity effect, E_{exT} in (38) is recast as

$$\begin{aligned} E_{exT} &= E_{ex} \Sigma_n I_n e^{iK_n \left(1 - \beta_0^{(1)} (A^{(1)} - 1) C_{\Omega t}\right) - i\omega_n t} (1 - \beta_0^{(1)} C_{\Omega t}) \\ &= E_{ex} \Sigma_n I_n e^{iK_n - i\omega_n t} \left(1 + \beta_0^{(1)} B_n (e^{i\Omega t} + e^{-i\Omega t})\right) \\ &= E_{ex} \Sigma_n I_n e^{iK_n} \left(e^{-i\omega_n t} + \beta_0^{(1)} B_n (e^{-i\omega_{n+1} t} + e^{-i\omega_{n-1} t})\right) \\ &= \Sigma_n E_{ex;n} e^{-i\omega_n t} \\ E_{ex;n} &= E_{ex} \left(I_n e^{iK_n} + \beta_0^{(1)} \left(B_{n-1} I_{n-1} e^{iK_{n-1}} + B_{n+1} I_{n+1} e^{iK_{n+1}}\right)\right) \\ B_n &= \left(iK_n (1 - A^{(1)}) - 1\right)/2, \quad K_n = k_n Z \end{aligned} \tag{39}$$

where in (39) indices have been judiciously raised and lowered in order to end up with a spectrum of sidebands ω_n.

The internal field which was monochromatic in (24) must now have the spectral structure prescribed by (36)–(39), hence

$$\begin{aligned} \boldsymbol{E}_{in} &= \hat{\boldsymbol{x}} E_{in}, \quad \boldsymbol{H}_{in} = \hat{\boldsymbol{y}} H_{in} = \hat{\boldsymbol{y}} E_{in} / \zeta^{(2)} \\ E_{in} &= \Sigma_n E_{in;n} e^{\kappa_n z_T - i\omega_n t}, \ \kappa_n / \omega_n = (\mu^{(2)} \varepsilon^{(2)})^{1/2} = 1 / v_{ph}^{(2)} \end{aligned} \tag{40}$$

where in (40) the coefficients $E_{in;n}$ are to be determined by the boundary conditions at $z_T = Z$.

Unlike the monochromatic wave (21), the reflected wave must now be stipulated with the spectrum prescribed by (36)–(40)

$$\begin{aligned} &\boldsymbol{E}_{sc} = \hat{\boldsymbol{x}} E_{sc}, \ \boldsymbol{H}_{sc} = -\hat{\boldsymbol{y}} H_{sc} = -\hat{\boldsymbol{y}} E_{sc} / \zeta^{(1)}, \ E_{sc} = \Sigma_\nu E_{sc;\nu} e^{-ik_{sc;\nu} z - i\omega_{sc;\nu} t} \\ &\omega_{sc;\nu} = \omega_{ex} - \nu\Omega, \quad k_{sc;\nu} / \omega_{sc;\nu} = (\mu^{(1)} \varepsilon^{(1)})^{1/2} = 1 / v_{ph}^{(1)} \end{aligned} \tag{41}$$

At $z_T = 0$ (41) becomes (cf. (22))

$$\begin{aligned} E_{sc}|_{z_T=0} &= \Sigma_\nu E_{sc;\nu} e^{-ik_{sc;\nu} z_0 S_{\Omega t} - i\omega_{sc;\nu} t} \\ &= \Sigma_{\nu\mu} E_{sc;\nu} e^{-i\omega_{ex} t} J_\mu(k_{sc;\nu} z_0) e^{i(\nu-\mu)\Omega t} \\ &= \Sigma_{\nu\mu} E_{sc;\nu} e^{-i\omega_{sc;\nu-\mu} t} J_\mu(k_{sc;\nu} z_0) \\ \omega_{sc;\nu-\mu} &= \omega_{ex} - (\nu-\mu)\Omega, \quad k_{sc;\nu} z_0 = k_{ex} z_0 - \nu\beta_0^{(1)} \end{aligned} \tag{42}$$

which is now more complicated by virtue of having in (42) a double summation.

In order to satisfy the boundary conditions at all times, all signals at the boundary must have identical frequencies. For harmonic motion it has been shown in the transition from (36) to (39) that the phase shift does not introduce new frequencies. Hence we require that the frequencies in (36), (42) be identical, prescribing a constraint $n = \nu - \mu$. This is tantamount to including in (42) a Kronecker delta function $\delta_{n;\nu-\mu}$. Therefore in (42) when an arbitrary fixed n is considered, the time exponential $e^{-i\omega_{sc;n} t}$ is independent of the summation variables ν, μ, and due to the constraint, the double sum on ν, μ collapses to a single sum, on either ν or μ. For all possible sidebands we have to sum on n. In other words, when we eliminate μ, the double sum becomes $\Sigma_{\nu;\nu-n}$, but for any fixed ν, if we take n from $-\infty$ to $+\infty$, $\nu - n$ will cover the same range. Thus (42) is rewritten as

$$E_{sc}|_{z_T=0} = \Sigma_n e^{-i\omega_n t} E'_{sc;n}, \; E'_{sc;n} = \Sigma_\nu E_{sc;\nu} J_{\nu-n}(k_{sc;\nu} z_0), \; \omega_n = \omega_{sc;\nu-\mu} \tag{43}$$

From (19) and similarly to (38), (39), the amplitude effect introduces an additional factor into (43), yielding

$$E_{scT}/E_{sc} = E_{scT;\nu}/E_{sc;\nu} = H_{scT}/H_{sc} = H_{scT;\nu}/H_{sc;\nu} = 1 + \beta_0^{(1)} C_{\Omega t} \tag{44}$$

Similarly to (39), and using results from (23), the phase shift from $z_T = 0$ to $z_T = Z$ is now included too, yielding

$$\begin{aligned} &\boldsymbol{E}_{scT} = \hat{\boldsymbol{x}} E_{scT}, \quad \boldsymbol{H}_{scT} = -\hat{\boldsymbol{y}} H_{scT} = -\hat{\boldsymbol{y}} E_{scT}/\zeta^{(1)} \\ &E_{scT} = \Sigma_n E'_{sc;n} e^{-iK_n(1+\beta_0^{(1)}(A^{(1)}-1)C_{\Omega t}) - i\omega_n t} (1 + \beta_0^{(1)} C_{\Omega t}) \\ &\quad = \Sigma_n E'_{sc;n} e^{-iK_n - i\omega_n t} \left(1 + \beta_0^{(1)} B'_n (e^{i\Omega t} + e^{-i\Omega t})\right) \\ &\quad = \Sigma_n E'_{sc;n} e^{-iK_n} \left(e^{-i\omega_n t} + \beta_0^{(1)} B'_n (e^{-i\omega_{n+1} t} + e^{-i\omega_{n-1} t})\right) \\ &\quad = \Sigma_n e^{-i\omega_n t} E_{scT;n} \\ &E_{scT;n} = E'_{sc;n} e^{-iK_n} + \beta_0^{(1)} \left(B'_{n-1} E'_{sc;n-1} e^{-iK_{n-1}} + B'_{n+1} E'_{sc;n+1} e^{-iK_{n+1}}\right) \end{aligned} \tag{45}$$

$$B'_n = \left(iK_n(1 - A^{(1)}) + 1\right)/2$$

Inspecting (39), (40), (45) reveals that at the boundary all the signals possess the same spectra, and by applying the boundary conditions $\boldsymbol{E}_{exT} + \boldsymbol{E}_{scT} = \boldsymbol{E}_{inT}$, $\boldsymbol{H}_{exT} + \boldsymbol{H}_{scT} = \boldsymbol{H}_{inT}$, the amplitude for each spectral component can be found. Thus the problem can be considered as solved.

6. OSCILLATING HALF SPACE MEDIUM

The corresponding Fizeau experiment analog of the uniform motion (27) can be extended to harmonic motion. Here the boundary is at rest with respect to medium $\{1\}$, while medium $\{2\}$ moves according to (cf. (27))

$$z = \zeta - \zeta_0 S_{\Omega t} \tag{46}$$

which we associate with the velocity (cf. (35))

$$v(t) = d\zeta/dt = v_0 C_{\Omega t}, \quad v_0 = \zeta_0 \Omega \tag{47}$$

The excitation wave is given by (14), and because the boundary is at rest with respect to medium $\{1\}$, the value of its signal at the boundary at $z = Z$ is simply

$$\boldsymbol{E}_{exT} = \hat{\boldsymbol{x}} E_{exT} e^{i\theta_{exT}}, \; \boldsymbol{H}_{exT} = \hat{\boldsymbol{y}} H_{exT} e^{i\theta_{exT}}, \; \theta_{exT} = k_{ex} Z - \omega_{ex} t \tag{48}$$

Without the motion of medium $\{2\}$ we would be dealing with a boundary at rest, hence no Doppler frequency shifts would have been created. However, by virtue of the harmonic motion, medium $\{2\}$ is modulating the boundary condition. Consequently the reflected and transmitted fields will display a harmonic spectrum.

This prescribes a harmonic spectrum in the transmitted wave when at rest with respect to medium $\{2\}$, as we did for the scattered wave in (42), (43). We therefore assume a spectrum of waves in medium $\{2\}$ at rest

$$\begin{aligned} \boldsymbol{E}_{in} &= \hat{\boldsymbol{x}} E_{in}, \quad \boldsymbol{H}_{in} = \hat{\boldsymbol{y}} H_{in} = \hat{\boldsymbol{y}} E_{in}/\zeta^{(2)}, \quad E_{in} = \Sigma_\nu E_{in;\nu} e^{i\kappa_\nu \zeta - i\omega_\nu t} \\ \omega_\nu &= \omega_{ex} - \nu\Omega, \quad \kappa_\nu/\omega_\nu = (\mu^{(2)}\varepsilon^{(2)})^{1/2} = 1/v_{ph}^{(2)} \end{aligned} \tag{49}$$

At $z = 0$, the origin for the coordinate system of the boundary and medium $\{1\}$, substitution from (46) yields the signal (cf. (42))

$$\begin{aligned} E_{in}|_{z=0} &= \Sigma_\nu E_{in;\nu} e^{i\kappa_\nu \zeta_0 S_{\Omega t} - i\omega_\nu t} = \Sigma_{\nu\mu} E_{in;\nu} e^{-i\omega_\nu t} J_\mu e^{i\mu\Omega t} \\ &= \Sigma_{\nu\mu} E_{in;\nu} e^{-i\omega_\nu t} J_\mu \\ J_\mu &= J_\mu(\kappa_\nu \zeta_0), \; \kappa_\nu \zeta_0 = \kappa\zeta_0 - \nu\beta_0^{(2)}, \; \kappa = \omega_{ex}/v_{ph}^{(2)}, \; \beta_0^{(2)} = v_0/v_{ph}^{(2)} \\ \omega_n &= \omega_{ex} - n\Omega, \quad n = \nu + \mu \end{aligned} \tag{50}$$

Using the instantaneous velocity (47), we take into account the phase shift from $z = 0$ to $z = Z$. Exploiting the same arguments that led to (16)–(18), (36)–(39), and including the amplitude effect prescribed by (31) yields (cf. (39), (45))

$$\begin{aligned}
\boldsymbol{E}_{inT} &= \hat{\boldsymbol{x}} E_{inT}, \quad \boldsymbol{H}_{inT} = \hat{\boldsymbol{y}} H_{inT} = \hat{\boldsymbol{y}} E_{inT}/\zeta^{(2)} \\
E_{inT}/E_{in} &= H_{inT}/H_{in} = (1 - \beta_0^{(2)} C_{\Omega t}) \\
E_{inT} &= \Sigma_{\nu\mu} E_{in;\nu\mu} e^{i\mathrm{K}_n (1-\beta_0^{(2)}(A^{(2)}-1)C_{\Omega t}) - i\omega_n t} (1 - \beta_0^{(2)} C_{\Omega t}) \\
&\simeq \Sigma_{\nu\mu} E_{in;\nu\mu} e^{-i\omega_n t} \left(1 + \beta_0^{(2)} B_{\nu\mu} (e^{i\Omega t} + e^{-i\Omega t})\right) \\
&= \Sigma_{\nu\mu} E_{in;\nu\mu} \left(e^{-i\omega_n t} + \beta_0^{(2)} B_{\nu\mu} (e^{-i\omega_{n+1} t} + e^{-i\omega_{n-1} t})\right) \\
&= \Sigma_{\nu\mu} e^{-i\omega_n t} E'_{in;\nu\mu}, \quad E_{in;\nu\mu} = E_{in;\nu} e^{i\mathrm{K}_n} J_\mu \\
\mathrm{K}_n &= \kappa_n Z, \quad B_{\nu\mu} = \left(i\mathrm{K}_n (1 - A^{(2)}) - 1\right)/2 \\
E'_{in;\nu\mu} &= E_{in;\nu\mu} + \beta_0^{(2)} (E_{in;\nu;\mu-1} B_{\nu;\mu-1} + E_{in;\nu;\mu+1} B_{\nu;\mu+1})
\end{aligned} \tag{51}$$

The scattered field must now have the periodic spectral structure prescribed by (51), hence at $z = Z$ we assume

$$\begin{aligned}
\boldsymbol{E}_{sc} &= \hat{\boldsymbol{x}} E_{sc}, \ \boldsymbol{H}_{sc} = -\hat{\boldsymbol{y}} H_{sc} = -\hat{\boldsymbol{y}} E_{sc}/\zeta^{(1)}, \\
E_{sc} &= \Sigma_n E_{sc;n} e^{-k_{sc;n} Z - i\omega_n t}, \ k_{sc;n}/\omega_n = (\mu^{(1)} \varepsilon^{(1)})^{1/2} = 1/v_{ph}^{(1)}
\end{aligned} \tag{52}$$

where in (52) the coefficients $E_{sc;n}$ are to be determined by the boundary conditions. So far the expression $n = \nu + \mu$ in (50) was only a notation, but it is now realized that in order to have the same frequencies at the boundary, this must be stipulated as a constraint, namely a Kronecker delta function $\delta_{n;\nu+\mu}$ as in (42)–(45), hence (51) is rewritten as

$$E_{inT} = \Sigma_n e^{-i\omega_{in;n} t} E_{inT;n}, \quad E_{inT;n} = \Sigma_\nu E_{in;\nu;n-\nu} \tag{53}$$

Thus we have now sufficient data for solving for the scattering coefficients $E_{sc;n}$ in (52). In practice, an approximation will have to be found by appropriately truncating the series on v in (53). With this the problem is considered to be solved.

7. BOUNDARY-VALUE PROBLEM: OSCILLATING CYLINDER

The problem of scattering by a uniformly moving cylinder has been discussed before [12, 13, 18]. Relevant references are also given in

[7, 15]. The particular model for scattering by a uniformly moving cylinder in material media has been investigated in [13]. This is the analog of the uniformly moving plane interface discussed above. Our task here is to analyze the problem of the harmonically oscillating cylinder, paralleling it with the analysis given above for the oscillating plane interface.

It has been shown that in the case of a uniformly moving cylinder a continuous spectrum is created due to the time-dependence [13, 18]. In the present case, in addition we also expect a discrete spectrum effect due to the harmonic motion of the scatterer, as found for the oscillating plane.

The geometry is defined by a circular cylinder of radius $\Re$ whose axis coincides with the direction of polarization of the excitation wave (14), along the x-axis. The motion is along the z-axis, in accordance with (34).

We start with the excitation wave (14), and derive (36) for the signal at $r_T = 0$, the center of the cylinder and the origin of the boundary's local coordinate system $\boldsymbol{r}_T$. We could choose an arbitrary reference point instead, but the symmetry used here simplifies the calculations.

Let us retrace the analysis up to (39), but here we substitute the projection $Z = \Re C_{\varphi_T}$, where φ_T is the azimuthal angle measured off the z-axis towards the negative y-axis (in a right-handed orientation), thus locating points on the cylinder's rim $r_T = \Re$. Accordingly (39) becomes (note the different definition of B_n below)

$$\begin{aligned}
\boldsymbol{E}_{exT} &= \hat{\boldsymbol{x}} E_{exT}, \quad \boldsymbol{H}_{exT} = \hat{\boldsymbol{y}} H_{exT} = \hat{\boldsymbol{y}} E_{exT}/\zeta^{(1)} \\
E_{exT} &= \Sigma_n E_{ex;n} e^{-i\omega_n t}, \quad E_{ex;n} = \Sigma_m i^m E_{ex;nm} e^{im\varphi_T} \\
E_{ex;nm} &= E_{ex} \Big[I_n L_{nm} - \beta_0^{(1)} (I_{n-1} L_{n-1;m} + I_{n+1} L_{n+1;m})/2 \\
&\quad -i\beta_0^{(1)} (B_{n-1} I_{n-1} L_{n-1;m-1} + B_{n+1} I_{n+1} L_{n+1;m-1}) \\
&\quad + i\beta_0^{(1)} (B_{n-1} I_{n-1} L_{n-1;m+1} + B_{n+1} I_{n+1} L_{n+1;m+1}) \Big] \\
B_n &= ik_n \Re (1 - A^{(1)})/4, \quad I_n = J_n(k_{ex} z_0), \quad L_{nm} = J_m(k_n \Re)
\end{aligned} \tag{54}$$

In (54) we have achieved a representation of $\boldsymbol{E}_{exT}$ at the boundary as a superposition of terms characterized by orthogonal discrete frequencies ω_n, each term being an orthogonal series in terms of φ_T. For the corresponding $\boldsymbol{H}_{exT}$ we note first that

$$\hat{\boldsymbol{y}} = -\hat{\boldsymbol{\varphi}}_T C_{\varphi_T} - \hat{\boldsymbol{r}}_T S_{\varphi_T} \tag{55}$$

Using (54), (55), and judiciously raising and lowering m indices, and exploiting a well known relation for cylindrical functions (see e.g.,

[17] p. 360), or alternatively, making the observation that multiplying $J_m(k_n\Re)$ by C_{φ_T} is tantamount to differentiating the function with respect to the argument and multiplying by $-i$. The tangential field at the boundary is obtained

$$\begin{aligned}
\hat{\boldsymbol{r}}_T\times\boldsymbol{H}_{exT} &= -\hat{\boldsymbol{x}}C_{\varphi_T}E_{exT}/\zeta^{(1)},\ C_{\varphi_T}E_{exT}=C_{\varphi_T}\Sigma_n E_{ex;n}e^{-i\omega_n t},\\
C_{\varphi_T}E_{ex;n} &= \Sigma_m i^m E_{ex;nm}e^{im\varphi_T}C_{\varphi_T}=\Sigma_m i^{m+1}E'_{ex;nm}e^{im\varphi_T}\\
E'_{ex;nm} &= -E_{ex}\left[I_nL'_{nm}-\beta_0^{(1)}(I_{n-1}L'_{n-1;m}+I_{n+1}L'_{n+1;m})/2\right.\\
&\quad -i\beta_0^{(1)}(B_{n-1}I_{n-1}L'_{n-1;m-1}+B_{n+1}I_{n+1}L'_{n+1;m-1})\\
&\quad \left.+\,i\beta_0^{(1)}(B_{n-1}I_{n-1}L'_{n-1;m+1}+B_{n+1}I_{n+1}L'_{n+1;m+1})\right],\\
L'_{nm} &= \partial_{k_n\Re}J_m(k_n\Re)
\end{aligned} \tag{56}$$

Once again (56) constitutes an orthogonal series with respect to both ω_n and φ_T.

In the internal domain we have the simple solution of the wave equation in media at rest. At the rim $r_T=\Re$ we have

$$\begin{aligned}
\boldsymbol{E}_{inT} &= \hat{\boldsymbol{x}}E_{inT},\ E_{inT}=\Sigma_n E_{in;n}e^{-i\omega_n t}\\
E_{in;n} &= \Sigma_m i^m E_{in;nm}J_m(k_{in;n}\Re)e^{im\varphi_T}\\
k_{in;n}/\omega_n &= (\mu^{(2)}\varepsilon^{(2)})^{1/2}=1/v_{ph}^{(2)}
\end{aligned} \tag{57}$$

where the coefficients $E_{in;nm}$ are unknowns, to be found by solving the boundary-value problem.

The corresponding field $\boldsymbol{H}_{inT}$ can be found directly from Maxwell's equations

$$\boldsymbol{H}_{inT}=\left(\hat{\boldsymbol{r}}_T\Re^{-1}\partial_{\varphi_T}-\hat{\boldsymbol{\varphi}}_T\partial_{\Re}\right)\Sigma_n E_{in;n}e^{-i\omega_n t}/(i\omega_n\mu^{(2)}) \tag{58}$$

For evaluation of the boundary-value problem we need the component of (58) tangential to the surface, given by

$$\begin{aligned}
\hat{\boldsymbol{r}}_T\times\boldsymbol{H}_{inT} &= -\hat{\boldsymbol{x}}\Sigma_n\kappa_n\partial_{\kappa_n R}E_{in;n}/(i\omega_n\mu^{(2)})=\hat{\boldsymbol{x}}\Sigma_n i\partial_{\kappa_n R}E_{in;n}/\zeta^{(2)}\\
&= \hat{\boldsymbol{x}}\Sigma_{nm}i^{m+1}E_{in;nm}J'_m(\kappa_n\Re)e^{im\varphi_T}/\zeta^{(2)}
\end{aligned} \tag{59}$$

In order to construct the scattered wave, we start with a plane wave superposition at frequencies corresponding to the expected sidebands, propagating in an arbitrary direction α (see also [12–14] for the uniform motion case)

$$\begin{aligned}
\boldsymbol{E}_\alpha &= \hat{\boldsymbol{x}}E_\alpha,\ \boldsymbol{H}_\alpha=\hat{\boldsymbol{k}}_\alpha\times\hat{\boldsymbol{x}}H_\alpha=\hat{\boldsymbol{k}}_\alpha\times\hat{\boldsymbol{x}}E_\alpha/\zeta^{(1)},\ E_\alpha=\Sigma_\mu E_{\alpha\mu}e^{i\theta_{\alpha\mu}}\\
\theta_{\alpha\mu} &= \boldsymbol{k}_{\alpha\mu}\cdot\boldsymbol{r}-\omega_\mu t=k_\mu rC_{\varphi-\alpha}-\omega_\mu t=k_{\mu z}z+k_{\mu y}y-\omega_\mu t\\
\omega_\mu &= \omega_{ex}-\mu\Omega,\ k_\mu/\omega_\mu=(\mu^{(1)}\varepsilon^{(1)})^{1/2}=1/v_{ph}^{(1)}
\end{aligned} \tag{60}$$

The origin of the local coordinate system $\boldsymbol{r}_T = 0$ moves according to (34). We obtain for $E_{\alpha 0}$, the field E_α of (60), evaluated at $z = z_0 S_{\Omega t},\ y = 0$ (cf. (36))

$$E_{\alpha 0} = E_\alpha|_{\boldsymbol{r}_T=0} = \Sigma_\mu E_{\alpha\mu} e^{i\theta_{\alpha\mu 0}},\ \theta_{\alpha\mu 0} = \theta_{\alpha\mu}|_{\boldsymbol{r}_T=0} = k_\mu z_0 C_\alpha S_{\Omega t} - \omega_\mu t$$
$$e^{i\theta_{\alpha\mu 0}} = \Sigma_\sigma J_\sigma(k_\mu z_0 C_\alpha) e^{-i\omega_n t},\ \omega_n = \omega_{ex} - n\Omega,\ n = \mu + \sigma \tag{61}$$

In (61) $n = \mu + \sigma$ is a constraint $\delta_{n;\mu+\sigma}$ prescribed by (54). Consequently (61) is rewritten as (cf. (42), (43))

$$E_{\alpha 0} = \Sigma_n E'_{\alpha n} e^{-i\omega_n t}, \quad E'_{\alpha n} = \Sigma_\mu E_{\alpha\mu} J_{n-\mu}(k_\mu z_0 C_\alpha) \tag{62}$$

With each frequency in (62) we associate a phase shift according to (9) (see also (37)) from the origin $\boldsymbol{r}_T = 0$ to the cylinder's rim at points $r_T = \Re$. Accordingly we replace in (61) $-\omega_n t$ by the appropriate $\theta_{\alpha n T}$. This is determined from the effective phase velocity prescribed as in (37), affecting the component of the propagation vector along the velocity.

$$\theta_{\alpha n T} = \boldsymbol{k}_{\alpha n T} \cdot \Re\hat{\boldsymbol{r}}_T - \omega_n t = k_{\alpha n T y} \Re S_{\varphi_T} + k_{\alpha n T z} \Re C_{\varphi_T} - \omega_n t$$
$$k_{\alpha n T y} = k_{\alpha n y},\ k_{\alpha n T z} = k_{\alpha n z}(1 - \beta_0^{(1)}(A^{(1)} - 1)C_{\Omega t}),\ k_{\alpha n z} = k_n C_\alpha$$
$$\theta_{\alpha n T} = \theta_{\alpha n R} - \beta_0^{(1)} K_n (A^{(1)} - 1) C_{\Omega t} C_\alpha C_{\varphi_T} \tag{63}$$
$$\theta_{\alpha n R} = \boldsymbol{k}_{\alpha n} \cdot \Re\hat{\boldsymbol{r}}_T - \omega_n t = K_n C_{\varphi_T - \alpha} - \omega_n t,\ K_n = k_n \Re,\ k_n = \omega_n / v_{ph}^{(1)}$$

We also multiply (61) by a factor $1 - \beta_0^{(1)} C_{\Omega t} C_\alpha$ prescribed by the boundary conditions (19) (cf. (38)), obtaining

$$\begin{aligned}
\boldsymbol{E}_{\alpha T} &= \hat{\boldsymbol{x}} E_{\alpha T},\ E_{\alpha T} = \Sigma_n E'_{\alpha n} e^{i\theta_{\alpha n T}} (1 - \beta_0^{(1)} C_{\Omega t} C_\alpha) = \Sigma_n E_{\alpha n T} e^{-i\omega_n t} \\
&\simeq \Sigma_n E'_{\alpha n} e^{i\theta_{\alpha n R}} (1 - \beta_0^{(1)} C_{\Omega t} C_\alpha)(1 - iK_n \beta_0^{(1)} (A^{(1)} - 1) C_{\Omega t} C_\alpha C_{\varphi_T}) \\
&\simeq \Sigma_n E'_{\alpha n} e^{i\theta_{\alpha n R}} (1 - \beta_0^{(1)} B_{\alpha n} (e^{i\Omega t} + e^{-i\Omega t})) \qquad (64) \\
B_{\alpha n} &= C_\alpha (iK_n (A^{(1)} - 1) C_{\varphi_T} + 1)/2, \quad E_{\alpha n T} = e^{iK_n C_{\varphi_T - \alpha}} E'_{\alpha n} \\
&\quad - \beta_0^{(1)} (e^{iK_{n-1} C_{\varphi_T - \alpha}} B_{\alpha;n-1} E'_{\alpha;n-1} + e^{iK_{n+1} C_{\varphi_T - \alpha}} B_{\alpha;n+1} E'_{\alpha;n+1})
\end{aligned}$$

The associated $\boldsymbol{H}_{\alpha T}$ field follows from (60). We employ (19), (35), (55) and compute the amplitude of the tangential field $\boldsymbol{H}_{\alpha T}$ at the boundary

$$\begin{aligned}
\hat{\boldsymbol{r}}_T \times \boldsymbol{H}_{\alpha T} &= \hat{\boldsymbol{r}}_T \times (\boldsymbol{H}_\alpha - \boldsymbol{v} \times \boldsymbol{D}_\alpha) = \hat{\boldsymbol{r}}_T \times \left(\hat{\boldsymbol{k}}_\alpha \times \hat{\boldsymbol{x}} H_\alpha - \boldsymbol{v} \times \hat{\boldsymbol{x}} \varepsilon^{(1)} E_\alpha\right) \\
&= E_\alpha \left(\hat{\boldsymbol{r}}_T \times \hat{\boldsymbol{k}}_\alpha \times \hat{\boldsymbol{x}} - \hat{\boldsymbol{r}}_T \times \hat{\boldsymbol{y}} \beta_0^{(1)} C_{\Omega t}\right) / \zeta^{(1)}
\end{aligned}$$

$$= E_\alpha \left(\hat{\boldsymbol{r}}_T \times \hat{\boldsymbol{k}}_\alpha \times \hat{\boldsymbol{x}} + \hat{\boldsymbol{r}}_T \times \hat{\boldsymbol{\varphi}}_T C_{\varphi_T} \beta_0^{(1)} C_{\Omega t} \right) / \zeta^{(1)}$$
$$= -\hat{\boldsymbol{x}} E_\alpha \left(C_{\varphi_T - \alpha} - \beta_0^{(1)} C_{\varphi_T} C_{\Omega t} \right) / \zeta^{(1)} \tag{65}$$

It follows that instead of the factor $1 - \beta_0^{(1)} C_{\Omega t} C_\alpha$ in (64) we now have from (65) $C_{\varphi_T - \alpha} - \beta_0^{(1)} C_{\varphi_T} C_{\Omega t}$, with (63) remaining unchanged, hence

$$\begin{aligned} \hat{\boldsymbol{r}}_T \times \boldsymbol{H}_{\alpha T} &= -\hat{\boldsymbol{x}} \Sigma_n e^{-i\omega_n t} E''_{\alpha n T} / \zeta^{(1)} \\ P_{\alpha n} &= C_{\varphi_T} \left(1 + i C_{\varphi_T - \alpha} C_\alpha K_n (A^{(1)} - 1) \right) / 2 \\ E''_{\alpha n T} &= E'_{\alpha n} e^{i K_n C_{\varphi_T - \alpha}} C_{\varphi_T - \alpha} \\ &\quad - \beta_0^{(1)} \Big(P_{\alpha; n-1} E'_{\alpha; n-1} e^{i K_{n-1} C_{\varphi_T - \alpha}} + P_{\alpha; n+1} E'_{\alpha; n+1} e^{i K_{n+1} C_{\varphi_T - \alpha}} \Big) \end{aligned} \tag{66}$$

Inspecting (64) it is seen that $K_n - K_{n\pm1} = \pm \Omega \Re / v_{ph}^{(1)}$, hence if we restrict the discussion to cases where the oscillation amplitude z_0 (34) is on the order of the cylinder radius $\Re$, then $\Omega \Re / v_{ph}^{(1)}$ is on the order of $\beta_0^{(1)}$, (37). Therefore, to the first order in $\beta_0^{(1)}$ (64) can be rewritten as

$$\begin{aligned} \boldsymbol{E}_{\alpha T} &= \hat{\boldsymbol{x}} E_{\alpha T} = \hat{\boldsymbol{x}} \Sigma_n E_{\alpha n T} e^{-i\omega_n t} \\ E_{\alpha n T} &= e^{i K_n C_{\varphi_T - \alpha}} \left(E'_{\alpha n} - \beta_0^{(1)} B_{\alpha n} (E'_{\alpha; n-1} + E'_{\alpha; n+1}) \right) \end{aligned} \tag{67}$$

In cases where the radius is much larger than the oscillation amplitude $\Re \gg z_0$, (67) is not valid and (64) must be retained.

Now express $E'_{\alpha n}$ as a Fourier series in terms of angle α

$$E'_{\alpha n} = \Sigma_m a_{nm} e^{im\alpha} \tag{68}$$

Therefore $E_{\alpha n T}$, (67), becomes

$$\begin{aligned} E_{\alpha n T} &= e^{i K_n C_{\varphi_T - \alpha}} \Sigma_m a'_{nm} e^{im\alpha} \\ a'_{nm} &= a_{nm} - \beta_0^{(1)} B_n \Sigma_{\nu = n\pm1; \mu = m\pm1} a_{\nu\mu} \\ B_n &= (i K_n (A^{(1)} - 1) C_{\varphi_T} + 1)/4 \\ \Sigma_{\nu = n\pm1; \mu = m\pm1} a_{\nu\mu} &= a_{n-1; m-1} + a_{n-1; m+1} + a_{n+1; m-1} + a_{n+1; m+1} \end{aligned} \tag{69}$$

where in (69) we have raised and lowered m indices, and compacted the notation of the sum.

Corresponding to (67)–(69), the scattered field signal is now constructed as a superposition (integral) of properly weighted plane waves propagating in various directions α, with the proviso that it leads to outgoing waves

$$\begin{aligned}\boldsymbol{E}_{scT} &= \hat{\boldsymbol{x}}\Sigma_n e^{-i\omega_n t}\frac{1}{\pi}\int E_{\alpha nT} d\alpha \\ &= \hat{\boldsymbol{x}}\Sigma_{nm} e^{-i\omega_n t}\frac{1}{\pi}\int e^{iK_n C_{\varphi_T-\alpha}} a'_{nm} e^{im\alpha} d\alpha \\ &= \hat{\boldsymbol{x}}\Sigma_{nm} e^{-i\omega_n t} i^m a'_{nm} H_m(K_n) e^{im\varphi_T}, \\ \int &= \int_{\alpha=\varphi_T-(\pi/2)+i\infty}^{\alpha=\varphi_T+(\pi/2)-i\infty}\end{aligned} \tag{70}$$

Following from the appropriate Sommerfeld integral (e.g., see [17]), in (70) H_m denotes the Hankel function of the first kind. Note that the dependence of $E_{\alpha nT}$ on φ_T does not affect the integration on α in (70). However, in the resulting series, $a'_{nm}(\varphi_T)$ means that the series needs to be modified once more in order to have an orthogonal series with respect to φ_T. Thus we obtain

$$\begin{aligned}\boldsymbol{E}_{scT} &= \hat{\boldsymbol{x}}\Sigma_{nm} e^{-i\omega_n t} i^m e^{im\varphi_T} F_{nm} \\ F_{nm} &= a_{nm}H_m - \beta_0^{(1)}\Sigma_{\nu=n\pm1;\mu=m\pm1} a_{\nu\mu}H_m/4 \\ &\quad +\beta_0^{(1)}V_n(\Sigma_{\nu=n\pm1;\mu=m+1\pm1} a_{\nu\mu}H_{m+1} - \Sigma_{\nu=n\pm1;\mu=m-1\pm1} a_{\nu\mu}H_{m-1}) \\ H_m &= H_m(K_n), \quad V_n = K_n(A^{(1)}-1)/8\end{aligned} \tag{71}$$

and (71) constitutes a double series, orthogonal with respect to the two indices n, m, as needed for solving the boundary-value problem.

We now waive the approximation (67) and return to (64). By inspection it becomes clear that the correct indices can be recovered by using the proper indices of $E'_{\alpha n}$. Accordingly (70) is now replaced by three integrals, with the same contour of integration, corresponding to the three exponentials in the last expression (64)

$$\begin{aligned}\boldsymbol{E}_{scT} &= \hat{\boldsymbol{x}}\Sigma_n e^{-i\omega_n t}\frac{1}{\pi}\int \Big[e^{iK_n C_{\varphi_T-\alpha}} E'_{\alpha n} \\ &\quad -\beta_0^{(1)} e^{iK_{n-1}C_{\varphi_T-\alpha}} B_{\alpha;n-1}E'_{\alpha;n-1} \\ &\quad -\beta_0^{(1)} e^{iK_{n+1}C_{\varphi_T-\alpha}} B_{\alpha;n+1}E'_{\alpha;n+1}\Big] d\alpha\end{aligned} \tag{72}$$

where for (72) $E'_{\alpha n}$, $B_{\alpha n}$ are defined in (62), (64), respectively.

Manipulating the integrals in a similar manner and incorporaing (68), we obtain the analog of (70)

$$\boldsymbol{E}_{scT} = \hat{\boldsymbol{x}}\Sigma_{nm} e^{-i\omega_n t}\frac{1}{\pi}\int \Big[e^{iK_n C_{\varphi_T-\alpha}} a_{nm}$$

$$
\begin{aligned}
& -\beta_0^{(1)} e^{iK_{n-1}C_{\varphi_T-\alpha}} B_{\alpha;n-1} a_{n-1;m} \\
& -\beta_0^{(1)} e^{iK_{n+1}C_{\varphi_T-\alpha}} B_{\alpha;n+1} a_{n+1;m}\Big] e^{im\alpha} d\alpha \\
U_n &= (iK_n(A^{(1)}-1)C_{\varphi_T}+1)/4 \\
&= \hat{\boldsymbol{x}}\Sigma_{nm} e^{-i\omega_n t}\frac{1}{\pi}\int \Big[e^{iK_n C_{\varphi_T-\alpha}} a_{nm} \\
& -\beta_0^{(1)}\Sigma_{\nu=n\pm1;\mu=m\pm1} a_{\nu\mu} U_\nu e^{iK_\nu C_{\varphi_T-\alpha}}\Big] e^{im\alpha} d\alpha \qquad (73)
\end{aligned}
$$

Expressing (73) in terms of Hankel functions series yields, similar to (71)

$$
\begin{aligned}
\boldsymbol{E}_{scT} = \hat{\boldsymbol{x}}\Sigma_{nm} e^{-i\omega_n t} i^m e^{im\varphi_T}\big[a_{nm}M_{nm} \\
-\beta_0^{(1)}\Sigma_{\nu=n\pm1;\mu=m\pm1} a_{\nu\mu} U_\nu M_{\nu m}\big], \ M_{nm} = H_m(K_n) \quad (74)
\end{aligned}
$$

Like we did in (71), the series must be modified to become orthogonal with respect to exponentials involving φ_T. This yields

$$
\begin{aligned}
\boldsymbol{E}_{scT} &= \hat{\boldsymbol{x}}\Sigma_{nm} e^{-i\omega_n t} i^m e^{im\varphi_T} F_{nm} \\
F_{nm} &= a_{nm}M_{nm} + \beta_0^{(1)}\Sigma_{\nu=n\pm1;\mu=m+1\pm1} a_{\nu\mu} V_\nu M_{\nu;m+1} \\
& -\beta_0^{(1)}\Sigma_{\nu=n\pm1;\mu=m-1\pm1} a_{\nu\mu} V_\nu M_{\nu;m-1} \qquad (75) \\
& -\beta_0^{(1)}\Sigma_{\nu=n\pm1;\mu=m\pm1} a_{\nu\mu} M_{\nu m}/4
\end{aligned}
$$

and the consistency of (71), (75) can be easily verified.

We are now in the position of solving boundary-value problems. For example, for a perfectly conducting cylinder (54), (75) are combined to evaluate the boundary condition

$$
\boldsymbol{E}_{exT} + \boldsymbol{E}_{scT} = 0|_{\Re} \qquad (76)
$$

For arbitrary material (57) is included in the form

$$
\boldsymbol{E}_{exT} + \boldsymbol{E}_{scT} - \boldsymbol{E}_{inT} = 0|_{\Re} \qquad (77)
$$

but now we also need conditions on the component of the magnetic field tangential to the boundary. Returning to (66) and implementing the approximation used in (67) we derive

$$
\begin{aligned}
\hat{\boldsymbol{r}}_T \times \boldsymbol{H}_{\alpha T} = -\hat{\boldsymbol{x}}\Sigma_n e^{-i\omega_n t} E''_{\alpha n T}/\zeta^{(1)} \\
E''_{\alpha n T} = e^{iK_n C_{\varphi_T-\alpha}}\left(E'_{\alpha n} C_{\varphi_T-\alpha} - \beta_0^{(1)} P_{\alpha n}(E'_{\alpha;n-1} + E'_{\alpha;n+1})\right)
\end{aligned} \qquad (78)
$$

Incorporating (68) we obtain the analog of (69)

$$\begin{aligned} E''_{\alpha nT} &= \Sigma_m a''_{nm} e^{iK_n C_{\varphi_T-\alpha}} e^{im\alpha}, \ U'_n = C_{\varphi_T} K_n (A^{(1)} - 1)/4 \\ a''_{nm} &= a_{nm} C_{\varphi_T-\alpha} - \beta_0^{(1)} C_{\varphi_T} \Sigma_{\nu=n\pm 1;\mu=m} a_{\nu\mu}/2 \\ &\quad -\beta_0^{(1)} i C_{\varphi_T-\alpha} \Sigma_{\nu=n\pm 1;\mu=m\pm 1} a_{\nu\mu} U'_n \end{aligned} \tag{79}$$

Note that in (79) we can replace

$$C_{\varphi_T-\alpha} e^{iK_n C_{\varphi_T-\alpha}} = -i\partial_{K_n} e^{iK_n C_{\varphi_T-\alpha}} \tag{80}$$

therefore the analog of (70) follows as

$$\begin{aligned} \hat{\boldsymbol{r}}_T \times \boldsymbol{H}_{scT} &= -\hat{\boldsymbol{x}} \Sigma_n e^{-i\omega_n t} \frac{1}{\pi} \int E''_{\alpha nT} d\alpha/\zeta^{(1)} \\ &= -\hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_n t} \frac{1}{\pi} \int a''_{nm} e^{iK_n C_{\varphi_T-\alpha}} e^{im\alpha} d\alpha/\zeta^{(1)} \\ &= -\hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_n t} \frac{1}{\pi} \int \Big[-i a_{nm} \partial_{K_n} - \beta_0^{(1)} C_{\varphi_T} \Sigma_{\nu=n\pm 1;\mu=m} a_{\nu\mu}/2 \\ &\quad - \beta_0^{(1)} \Sigma_{\nu=n\pm 1;\mu=m\pm 1} a_{\nu\mu} U'_n \partial_{K_n} \Big] e^{iK_n C_{\varphi_T-\alpha}} e^{im\alpha} d\alpha/\zeta^{(1)} \end{aligned} \tag{81}$$

In (81), similarly to (70), the dependence on φ_T does not affect the integration on α. Like (70), we recast (81) in a series of cylindrical functions

$$\begin{aligned} \hat{\boldsymbol{r}}_T \times \boldsymbol{H}_{scT} = \hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_n t} i^m e^{im\varphi_T} \Big[& i a_{nm} H'_m \\ & + \beta_0^{(1)} C_{\varphi_T} \Sigma_{\nu=n\pm 1;\mu=m} a_{\nu\mu} H_m/2 \\ & + \beta_0^{(1)} \Sigma_{\nu=n\pm 1;\mu=m\pm 1} a_{\nu\mu} U'_n H'_m \Big] /\zeta^{(1)} \\ H_m = H_m(K_n), \quad & H'_m = \partial_{K_n} H_m(K_n) \end{aligned} \tag{82}$$

Once again, similarly to (71), we recast (82) to obtain coefficients independent of φ_T

$$\begin{aligned} \hat{\boldsymbol{r}}_T \times \boldsymbol{H}_{scT} &= \hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_n t} i^{m+1} e^{im\varphi_T} G_{nm}/\zeta^{(1)} \\ G_{nm} &= a_{nm} H'_m + \beta_0^{(1)} (\Sigma_{\nu=n\pm 1;\mu=m+1} a_{\nu\mu} H_{m+1} \\ &\quad - \Sigma_{\nu=n\pm 1;\mu=m-1} a_{\nu\mu} H_{m-1})/4 \\ &\quad + \beta_0^{(1)} V_n (\Sigma_{\nu=n\pm 1;\mu=m+1\pm 1} a_{\nu\mu} H'_{m+1} \\ &\quad - \Sigma_{\nu=n\pm 1;\mu=m-1\pm 1} a_{\nu\mu} H'_{m-1}) \end{aligned} \tag{83}$$

Thus (83) is now an orthogonal series with respect to φ_T, as well as the frequencies ω_n.

Returning to (66), which led to the approximate expression (78), we now wish to waive the approximation. Let us first inspect the analogous expressions for the electric field. Here we started with (64) and derived the corresponding approximate form (67). It is seen that $E'_{\alpha n}$, $E'_{\alpha;n-1}$, $E'_{\alpha;n+1}$ retain their indices, and this is the clue for recovering the indices for the remaining factors. This is consistent for all expressions, e.g., compare (71), (75), all we have to do is to modify the index of K. Similarly, the approximate (83) is replaced by the exact expression

$$\begin{aligned}
\hat{\boldsymbol{r}}_T\times\boldsymbol{H}_{scT} &= \hat{\boldsymbol{x}}\Sigma_{nm}e^{-i\omega_n t}i^{m+1}e^{im\varphi_T}G_{nm}/\zeta^{(1)}\\
G_{nm} &= a_{nm}M'_m+\beta_0^{(1)}(\Sigma_{\nu=n\pm1;\mu=m+1}a_{\nu\mu}M_{\nu\mu}\\
&\quad -\Sigma_{\nu=n\pm1;\mu=m-1}a_{\nu\mu}M_{\nu\mu})/4\\
&\quad +\beta_0^{(1)}(\Sigma_{\nu=n\pm1;\mu=m+1\pm1}a_{\nu\mu}V_\nu M'_{\nu;m+1}\\
&\quad -\Sigma_{\nu=n\pm1;\mu=m-1\pm1}a_{\nu\mu}V_\nu M'_{\nu;m-1})\\
M_{nm} &= H_m(K_n),\quad M'_{nm}=H'_m(K_n)
\end{aligned}\tag{84}$$

In addition to (77) we therefore have

$$\hat{\boldsymbol{r}}_T\times(\boldsymbol{H}_{exT}+\boldsymbol{H}_{scT}-\boldsymbol{H}_{inT})|_{\Re}=0\tag{85}$$

together providing sufficient equations for deriving from the supposedly known coefficients $E_{ex;nm}, E'_{ex;nm}$, (54), (56), respectively, the unknowns $E_{in;n}$, (57), (59), and a_{nm}, (71) or (75) with (83) or (84), respectively.

It is noted once more that the coefficients having $\beta_0^{(1)}$ as a factor are known — these are the coefficients for scattering by a cylinder in the absence of motion, at the proper frequency involved.

8. DERIVATION OF THE SCATTERED FIELD

We have studied the boundary-value problem for the oscillating cylinder. Now we have to demonstrate how the scattered field is computed. The elements of the present approach have been discussed before [13, 14], but here they are adapted to the specific problem at hand. The method is based on the representation of the scattered field as a plane wave superposition, with the phase invariance property (12) holding when the phase is expressed in terms of $\boldsymbol{r}_T$, t_T coordinates, and the boundary conditions (19) excluded.

We therefore return to (64) and retain only terms multiplied by $\beta_0^{(1)}$ containing K_n, because these are the terms originating from the phase exponential and not the boundary conditions term $(1 - \beta_0^{(1)} C_{\Omega t} C_\alpha)$. We also replace K_n by $\overline{K}_n = k_n r_T$ to take into account various distances. This leads to the scattered field $\boldsymbol{E}_{sc}$, expressed in terms of coordinates $(\boldsymbol{r}_T, t_T)$. Accordingly we replace (71) by

$$\begin{aligned} \boldsymbol{E}_{sc} &= \hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_n t_T} i^m e^{im\varphi_T} \overline{F}_{nm} \\ \overline{F}_{nm} &= a_{nm} \overline{M}_{nm} + \beta_0^{(1)} \overline{V}_n (\Sigma_{\nu=n\pm1;\mu=m+1\pm1} a_{\nu\mu} \overline{H}_{m+1} \\ &\quad - \Sigma_{\nu=n\pm1;\mu=m-1\pm1} a_{\nu\mu} \overline{H}_{m-1}) \\ \overline{V}_n &= \overline{K}_n (A^{(1)} - 1)/8, \ \overline{K}_n = k_n r_n, \ \overline{H}_m = H_m(\overline{K}_n) \end{aligned} \tag{86}$$

and similarly (75) is replaced by

$$\begin{aligned} \boldsymbol{E}_{sc} &= \hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_n t_T} i^m e^{im\varphi_T} \overline{F}_{nm}, \ \overline{M}_{nm} = H_m(\overline{K}_n) \\ \overline{F}_{nm} &= a_{nm} \overline{M}_{nm} + \beta_0^{(1)} \Sigma_{\nu=n\pm1;\mu=m+1\pm1} a_{\nu\mu} \overline{V}_\nu \overline{M}_{\nu;m+1} \\ &\quad - \beta_0^{(1)} \Sigma_{\nu=n\pm1;\mu=m-1\pm1} a_{\nu\mu} \overline{V}_\nu \overline{M}_{\nu;m-1} \end{aligned} \tag{87}$$

Finally, by substitution of the initial space and time coordinates (see (1), (34), (35))

$$z_T = z - z_0 S_{\Omega t}, \quad y_T = y, \quad t_T = t - v_0 z C_{\Omega t}/c^2 \tag{88}$$

the field can be expressed in terms of the initial coordinate system parameters $(\boldsymbol{r}, t)$. The corresponding magnetic field $\boldsymbol{H}_{sc}$ is then found by effecting Maxwell's equation in terms of $(\boldsymbol{r}, t)$.

Unfortunately, the formulas given by (86), (87), are adequate for small $k_n r_n$ only, i.e., in the instantaneous vicinity of the scatterer. Such expressions are adequate for the solution of the boundary-value problem, but fail for representing the scattered field at arbitrary distances, as the scatterer moves away from some observer's fixed position $\boldsymbol{r} =$ constant. To overcome this difficulty we have to return to (63), (64). Inasmuch as we are not trying to derive orthogonal series here, we keep the original phase, retain the velocity-dependent term in the exponent, and exclude the term $1 - \beta_0^{(1)} C_{\Omega t} C_{\varphi_T}$. This yields

$$\begin{aligned} \boldsymbol{E}_\alpha &= \hat{\boldsymbol{x}} \Sigma_n E'_{\alpha n} e^{i\overline{\theta}_{\alpha n T}} = \hat{\boldsymbol{x}} \Sigma_n \overline{E}_{\alpha n} e^{i\overline{\theta}_{\alpha n R}} \\ \overline{\theta}_{\alpha n T} &= \boldsymbol{k}_{\alpha n T} \cdot \boldsymbol{r}_T - \omega_n t_T = \overline{\theta}_{\alpha n R} - \beta_0^{(1)} \overline{K}_n (A^{(1)} - 1) C_{\Omega t_T} C_\alpha C_{\varphi_T} \\ \theta_{\alpha n R} &= \boldsymbol{k}_{\alpha n} \cdot \boldsymbol{r}_T - \omega_n t_T = \overline{K}_n C_{\varphi_T - \alpha} - \omega_n t_T \\ \overline{E}_{\alpha n} &= E'_{\alpha n} e^{-i\beta_0^{(1)} \overline{K}_n (A^{(1)} - 1) C_{\Omega t_T} C_\alpha C_{\varphi_T}} \end{aligned} \tag{89}$$

where in (89) the coefficients $E'_{\alpha n}$ retain their original definition (62), (68), and are by now supposedly known from the solution of the boundary-value problem. Therefore instead of (68) we define now

$$\overline{E}_{\alpha n} = \Sigma_m \overline{a}_{nm} e^{im\alpha} \tag{90}$$

obviously $\overline{E}_{\alpha n}$ in (90) is periodic in α, hence it can be considered as a Fourier series with respect to α, with new coefficients. Similarly to (70), the scattered wave is assembled as a superposition of plane waves, resulting in an outgoing wave

$$\boldsymbol{E}_{sc} = \hat{\boldsymbol{x}}\Sigma_n \frac{1}{\pi}\int \overline{E}_{\alpha n} e^{i\overline{\theta}_{\alpha n T}} d\alpha = \hat{\boldsymbol{x}}\Sigma_n e^{-i\omega_n t_T} \frac{1}{\pi}\int e^{i\overline{K}_n C_{\varphi_T - \alpha}} \overline{E}_{\alpha n}(\alpha) d\alpha \tag{91}$$

Note that in (91) the integration is on α, hence the dependence of $\overline{E}_{\alpha n}$ on φ_T, t_T does not affect the integration.

In [13, 14], a method of using Twersky's asymptotic or exact differential-operator series in inverse powers of the distance [19, 20], has been adapted. In the present case the asymptotic representation, for example, reads

$$\begin{aligned}
&\boldsymbol{E}_{sc} = \hat{\boldsymbol{x}}\Sigma_n e^{-i\omega_n t_T} D_\alpha \left\{\overline{E}_{\alpha n}(\alpha)\right\} \\
&D_\alpha \left\{\overline{E}_{\alpha n}(\alpha)\right\} = H\left(1 + \frac{1+4\partial_\alpha^2}{i8\overline{K}_n} - \frac{9+40\partial_\alpha^2+16\partial_\alpha^4}{128\overline{K}_n^2}\cdots\right)\overline{E}_{\alpha n}(\alpha)\Bigg|_{\alpha=\varphi_T} \\
&\qquad = H\sum_{\mu=0} \frac{(1+4\partial_\alpha^2)(9+4\partial_\alpha^2)\cdots([2\mu-1]^2+4\partial_\alpha^2)}{(i8\overline{K}_n)^\mu \mu!}\overline{E}_{\alpha n}(\alpha)\Bigg|_{\alpha=\varphi_T} \\
&H = H(\overline{K}_n) = (2/(i\pi\overline{K}_n))^{1/2} e^{i\overline{K}_n}
\end{aligned} \tag{92}$$

Once again, to express (92) in terms of the native coordinates $(\boldsymbol{r}, t)$, we have to substitute from (88). Once this is effected, the associated magnetic field $\boldsymbol{H}_{sc}$ is found by applying the relevant Maxwell equation. Thus finally the problem is considered as solved.

9. BOUNDARY-VALUE PROBLEM: OSCILLATING CYLINDRICAL MEDIUM

In the same way that the oscillating plane interface relates to the oscillating half space medium, also the oscillating cylinder discussed above can be related to a cylindrical boundary at rest with an oscillating medium within. These problems are dubbed as the Fizeau experiment configuration because the vessel containing the moving medium is at rest relative to the experimenter.

The incident wave is given by (14) and the tangential fields evaluated at the cylinder rim $r = \Re$, are given in terms cylindrical functions

$$\begin{aligned} \boldsymbol{E}_{ex} &= \hat{\boldsymbol{x}} E_{ex} e^{i\theta_{ex}} = \hat{\boldsymbol{x}} E_{ex} \Sigma_m i^m e^{-i\omega_{ex} t} J_m(K_{ex}) e^{im\varphi} \\ \hat{\boldsymbol{r}} \times \boldsymbol{H}_{ex} &= \hat{\boldsymbol{x}} E_{ex} \Sigma_m i^{m+1} J'_m(K_{ex}) e^{im\varphi} / \zeta^{(1)}, \ K_{ex} = k_{ex} \Re \end{aligned} \tag{93}$$

Inasmuch as the oscillating medium inside the cylinder is harmonically affecting the boundary conditions, we assume a scattered wave containing all the sidebands, similarly to (52). The tangential fields at the cylinder rim $r = \Re$, in terms of cylindrical functions corresponding to outgoing waves, are given by

$$\begin{aligned} \boldsymbol{E}_{sc} &= \hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_n t} i^m E_{sc;nm} H_m(K_{sc;n}) e^{im\varphi} \\ \hat{\boldsymbol{r}} \times \boldsymbol{H}_{sc} &= \hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_n t} i^{m+1} E_{sc;nm} \partial_{K_{sc;n}} H_m(K_{sc;n}) e^{im\varphi} / \zeta^{(1)} \\ \omega_n &= \omega_{ex} - n\Omega, \ k_{sc;n}/\omega_n = (\mu^{(1)} \varepsilon^{(1)})^{1/2} = 1/v_{ph}^{(1)}, \ K_{sc;n} = k_{sc;n} \Re \end{aligned} \tag{94}$$

In the interior medium at rest we define a reference system $\boldsymbol{\rho} = \boldsymbol{\rho}(\zeta, \eta) = \boldsymbol{\rho}(\rho, \psi)$, in Cartesian, and cylindrical coordinates, respectively. and consider plane waves propagating in an arbitrary direction α. Similarly to (60), we assume a spectrum of sidebands, yielding

$$\begin{aligned} \boldsymbol{E}_\alpha &= \hat{\boldsymbol{x}} E_\alpha, \ \boldsymbol{H}_\alpha = \hat{\boldsymbol{\kappa}}_\alpha \times \hat{\boldsymbol{x}} H_\alpha = \hat{\boldsymbol{\kappa}}_\alpha \times \hat{\boldsymbol{x}} E_\alpha / \zeta^{(2)}, \ E_\alpha = \Sigma_\mu E_{\alpha\mu} e^{i\theta_{\alpha\mu}} \\ \theta_{\alpha\mu} &= \boldsymbol{\kappa}_{\alpha\mu} \cdot \boldsymbol{\rho} - \omega_\mu t = \kappa_\mu \rho C_{\psi-\alpha} - \omega_\mu t = \kappa_{\mu\zeta} \zeta + \kappa_{\mu\eta} \eta - \omega_\mu t \\ \omega_\mu &= \omega_{ex} - \mu\Omega, \quad \kappa_\mu / \omega_\mu = (\mu^{(2)} \varepsilon^{(2)})^{1/2} = 1/v_{ph}^{(2)} \end{aligned} \tag{95}$$

Using (46) and $\eta = y$ we now find the phase $\theta_{\alpha\mu}$ (95) at $\boldsymbol{r} = 0$. To obtain the analog of (61) all we have to do is to interchange z_0 of (61) with ζ_0, thusly

$$\begin{aligned} E_{\alpha 0} &= \Sigma_\mu E_{\alpha\mu} e^{i\theta_{\alpha\mu 0}}, \quad \theta_{\alpha\mu 0} = \kappa_\mu \zeta_0 C_\alpha S_{\Omega t} - \omega_\mu t \\ e^{i\theta_{\alpha\mu 0}} &= \Sigma_\sigma J_\sigma(\kappa_\mu \zeta_0 C_\alpha) e^{-i\omega_n t}, \ \omega_n = \omega_{ex} - n\Omega, \ n = \mu + \sigma \end{aligned} \tag{96}$$

The frequencies at the boundary must be the same in the external and internal regimes, like the constraint we had in (42), (52), (61). Here (94), (96) prescribe the constraint $\delta_{n;\mu+\sigma}$, hence we get, similarly to (62)

$$E_{\alpha 0} = \Sigma_n E'_{\alpha n} e^{-i\omega_n t}, \quad E'_{\alpha n} = \Sigma_\mu E_{\alpha\mu} J_{n-\mu}(\kappa_\mu \zeta_0 C_\alpha) \tag{97}$$

and the analog of (63), (64) follows

$$\begin{aligned}
\boldsymbol{E}_{\alpha T} &= \hat{\boldsymbol{x}} E_{\alpha T} = \hat{\boldsymbol{x}} \Sigma_n E_{\alpha n T} e^{-i\omega_n t},\ E_{\alpha n T} = \Sigma_n E'_{\alpha n} e^{i\theta_{\alpha n T}} (1 - \beta_0^{(2)} C_{\Omega t} C_\alpha) \\
\theta_{\alpha n T} &= \boldsymbol{\kappa}_{\alpha n T} \cdot \Re \hat{\boldsymbol{r}} - \omega_n t = \theta_{\alpha n R} - \beta_0^{(2)} \mathrm{K}_n (A^{(2)} - 1) C_{\Omega t} C_\alpha C_\varphi \\
\mathrm{K}_n &= \kappa_n \Re,\ \theta_{\alpha n R} = \boldsymbol{\kappa}_{\alpha n} \cdot \Re \hat{\boldsymbol{r}} - \omega_n t = \mathrm{K}_n C_{\varphi - \alpha} - \omega_n t \qquad (98) \\
B_{\alpha n} &= C_\alpha (i \mathrm{K}_n (A^{(2)} - 1) C_\varphi + 1)/2,\ E_{\alpha n T} = E'_{\alpha n} e^{i \mathrm{K}_n C_{\varphi-\alpha}} - \beta_0^{(2)} \\
&\quad \cdot \left(B_{\alpha; n-1} E'_{\alpha; n-1} e^{i \mathrm{K}_{n-1} C_{\varphi-\alpha}} + B_{\alpha; n+1} E'_{\alpha; n+1} e^{i \mathrm{K}_{n+1} C_{\varphi-\alpha}} \right)
\end{aligned}$$

The analogs of (65)–(69) follow in an obvious manner, and will not be shown in detail. Suffice it to say that in all these formulas coefficients symbolized by a will now be replaced by a corresponding A. The field in the internal domain will be constructed, similarly to (70), as a superposition of plane waves, however, the integration contour is now chosen to represent the nonsingular Bessel functions

$$\begin{aligned}
\boldsymbol{E}_{inT} &= \hat{\boldsymbol{x}} \Sigma_n e^{-i\omega_n t} \frac{1}{2\pi} \int E_{\alpha n T} d\alpha = \hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_t} \frac{1}{2\pi} \int e^{i \mathrm{K}_n C_{\varphi-\alpha}} \mathrm{A}'_{nm} e^{im\alpha} d\alpha \\
&= \hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_n t} i^m \mathrm{A}'_{nm} J_m(\mathrm{K}_n) e^{im\varphi},\ \int = \int_{\alpha=0}^{\alpha=2\pi} \qquad (99)
\end{aligned}$$

The analog of (71) now reads

$$\begin{aligned}
\boldsymbol{E}_{inT} &= \hat{\boldsymbol{x}} \Sigma_{nm} e^{-i\omega_n t} i^m e^{im\varphi} \Phi_{nm},\ \Lambda_n = \mathrm{K}_n (A^{(2)} - 1)/8 \\
\Phi_{nm} &= A_{nm} J_m - \beta_0^{(2)} \Sigma_{\nu=n\pm1; \mu=m\pm1} \mathrm{A}_{\nu\mu} J_m / 4 \qquad (100) \\
&\quad + \beta_0^{(1)} \Lambda_n (\Sigma_{\nu=n\pm1; \mu=m+1\pm1} \mathrm{A}_{\nu\mu} J_{m+1} - \Sigma_{\nu=n\pm1; \mu=m-1\pm1} \mathrm{A}_{\nu\mu} J_{m-1}) \\
J_m &= J_m(\mathrm{K}_n)
\end{aligned}$$

and the analog of (75) follows in an obvious manner.

Hence this outline demonstrates how the boundary-value problem for the cylindrical Fizeau experiment configuration is solved, along the lines of the corresponding boundary-value problem of the oscillating cylinder.

10. CONCLUDING REMARKS

The feasibility of deriving a consistent model for scattering by time-varying objects and media has been investigated. A quasi Lorentz transformation for time-varying coordinates has been introduced.

The model used here is relativistically exact to the first order in v/c and takes into account the Doppler frequency shifts and the change of the propagation velocity (phase velocity) in moving media, also called the Fresnel drag effect. The boundary conditions are based on the Lorentz force formulas, which to the first order in v/c agree with the Special Relativity field transformations.

The geometry chosen for plane interfaces and circular cylinders, as well as the harmonic oscillations and its direction of motion are conducive to less complicated problems, thus bringing out the essentials of this class of problems.

The most obvious effect is the creation of sidebands, due to the modulation produced by the motion. It is hoped that such multi-spectral scattering problems will lead to improved remote sensing techniques. Typically, the solutions involve interactions between scattering coefficients that are usually absent in velocity-independent problems.

The present treatment is analytic. It is hoped to have in the future simulations that will graphically reveal the effects of motion in specific scattering problems.

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