Progress In Electromagnetics Research, PIER 48, 145–184, 2004

SPACE-TIME REVERSAL SYMMETRY PROPERTIES OF ELECTROMAGNETIC GREEN'S TENSORS FOR COMPLEX AND BIANISOTROPIC MEDIA

V. Dmitriev

Department of Electrical Engineering
University Federal of Para
Belem, Para, Brazil

Abstract—Space-Time reversal symmetry properties of free-Space electromagnetic Green's tensors for complex and bianisotropic homogeneous media are discussed. These properties are defined by symmetry of the medium under consideration, of the point sources and of the vector **S** connecting the source and the point of observation. The constraints imposed on Green's tensors by the restricted Time reversal, by the center and anticenter of symmetry are independent on the vector **S** orientation. Other Space-Time reversal operators lead to constraints on Green's tensors only for some special directions in Space. These directions are along the (anti)axes and (anti)planes and normal to the (anti)axes and (anti)planes. The full system of the continuous magnetic point groups for description of Space-Time reversal symmetry of Green's tensors is defined and a general group-theoretical method for calculation of simplified forms of Green's tensors is presented.

1. INTRODUCTION

Electromagnetic problems involving Maxwell's equations are formulated in terms of sources (their distribution and geometry), medium properties (in terms of constitutive relations or equation of motion) and boundary conditions with corresponding geometry. All these constituents of the problems may be considered from the point of view of Space-Time reversal symmetry. Space symmetry includes rotation-reflection and displacement symmetry operations, whilst Time reversal operation T presents changing the sign of Time ($t \to -t$). In some cases of magnetic structures, combined Space-Time reversal symmetry can exist. The resultant Space-Time reversal symmetry governs some general characteristics of the solutions of Maxwell's equations and in particular of the symmetry properties of Green's tensors.

Green's tensors (dyadics, functions) are a powerful tool for solving different types of linear differential equations found in electromagnetic radiation, scattering and diffraction. Green's tensors represent the solution to a given inhomogeneous equation with a point source. The problem of Green's tensor calculations is rather complicated. The complexity depends on the properties of material media and on the boundary conditions. In case of free-Space electromagnetic problems, boundaries are not present (in order to obtain unique solutions, one should use radiation conditions). However, even with this simplification, analytic solutions for Green's tensors for a general bianisotropic medium are not known. Examples of calculations of dyadic Green's tensors for complex and bianisotropic media can be found in publications of Cheng, Olyslager, Kong, Lindell, Tai, Weiglhofer and many others (see our list of references [1–3, 5, 7, 10–27] which is by no means exhaustive).

In the Space-Time domain $(\mathbf{r}, t)$, the 6×6 combined Green's tensor of second rank $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0; t, t_0)$ of the problem under consideration is defined by

$$\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0; t, t_0) = \begin{pmatrix} \bar{\mathbf{G}}^{ee}(\mathbf{r}, \mathbf{r}_0; t, t_0) & \bar{\mathbf{G}}^{em}(\mathbf{r}, \mathbf{r}_0; t, t_0) \\ \bar{\mathbf{G}}^{me}(\mathbf{r}, \mathbf{r}_0; t, t_0) & \bar{\mathbf{G}}^{mm}(\mathbf{r}, \mathbf{r}_0; t, t_0) \end{pmatrix}, \tag{1}$$

where $\bar{\mathbf{G}}^{ee}(\mathbf{r}, \mathbf{r}_0; t, t_0)$ and $\bar{\mathbf{G}}^{mm}(\mathbf{r}, \mathbf{r}_0; t, t_0)$ are the 3×3 Green tensors of second rank of the electric and magnetic type, respectively, $\bar{\mathbf{G}}^{em}(\mathbf{r}, \mathbf{r}_0; t, t_0)$ and $\bar{\mathbf{G}}^{me}(\mathbf{r}, \mathbf{r}_0; t, t_0)$ are the 3×3 Green's tensors of second rank of the mixed type.

Maxwell's equations with constitutive relations and corresponding differential equations for Green's tensors can possess different types of symmetry. The symmetry conditions force upon some restrictions on Green's tensors. For example, invariance of the problem with respect to arbitrary linear Space-Time displacements leads to some known general restrictions on Green's tensors [1]. Namely, Green's tensors for unbounded homogeneous stationary media depend only on the difference $(\mathbf{r} - \mathbf{r}_0)$ of the position vectors of the point of observation $\mathbf{r}$ and of the source point $\mathbf{r}_0$, and on the difference $(t - t_0)$, i.e.,

$$\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0; t, t_0) = \bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0; t - t_0). \tag{2}$$

Also, the duality principle, i.e., the internal symmetry of Maxwell's equations with respect to interchange of magnetic and electric quantities [2] gives some general relations. Duality leads to the following restrictions on Green's tensors (written in the Space-frequency domain $(\mathbf{r}, \omega)$):

$$\begin{aligned}\bar{\mathbf{G}}^{ee}(\,\bar{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\zeta}}, \mathbf{r}, \mathbf{r}_0\,) &= \bar{\mathbf{G}}^{mm}(\,\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\epsilon}}, -\bar{\boldsymbol{\zeta}}, -\bar{\boldsymbol{\xi}}, \mathbf{r}, \mathbf{r}_0\,),\\ \bar{\mathbf{G}}^{em}(\,\bar{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\zeta}}, \mathbf{r}, \mathbf{r}_0\,) &= -\,\bar{\mathbf{G}}^{me}(\,\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\epsilon}}, -\bar{\boldsymbol{\zeta}}, -\bar{\boldsymbol{\xi}}, \mathbf{r}, \mathbf{r}_0\,)\end{aligned} \tag{3}$$

where $\bar{\boldsymbol{\epsilon}}$, $\bar{\boldsymbol{\mu}}$, $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$ are the 3×3 constitutive tensors of bianisotropic media.

For reciprocal media, other important constraints on Green's tensors exist [2]:

$$\bar{\mathbf{G}}(\mathbf{r}_0 - \mathbf{r}) = (\bar{\mathbf{G}}^{\circ})^t(\mathbf{r} - \mathbf{r}_0). \tag{4}$$

where ^{t} denotes transposition, the symbol $^{\circ}$ stands for an operation which is called the adjugation. The adjugation changes the sign of the off-diagonal block tensors of the mixed type $\bar{\mathbf{G}}^{em}(\mathbf{r} - \mathbf{r}_0)$ and $\bar{\mathbf{G}}^{me}(\mathbf{r} - \mathbf{r}_0)$ (see Section 3.6).

Loss-free media possess a special type of symmetry which also manifests itself in Green's tensors. For lossless anisotropic media, the Green's tensors satisfy the following relation [1]:

$$\bar{\mathbf{G}}(\mathbf{r}_0 - \mathbf{r}) = -(\bar{\mathbf{G}}^{*})^t(\mathbf{r} - \mathbf{r}_0), \tag{5}$$

where the symbol * denotes complex conjugation.

It is possible to obtain some relations for Green's tensors which follow from a general Space-Time reversal symmetry consideration. Discussing this problem, Altman and Suchy [3] showed that for two regions of Space defined by orthogonal mapping $\mathbf{r}' = \bar{\mathbf{R}} \cdot \mathbf{r}$ and $\mathbf{r}'_0 = \bar{\mathbf{R}} \cdot \mathbf{r}_0$, where $\bar{\mathbf{R}}$ is a 3×3 mapping operator (see Appendix B), Green's tensor $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0)$ is mapped as follows:

$$\bar{\mathbf{G}}'(\mathbf{r}', \mathbf{r}'_0) = \bar{\mathbf{R}}_6 \cdot \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0) \cdot \bar{\mathbf{R}}_6^t, \tag{6}$$

and the adjoint Green's tensor as

$$\bar{\mathbf{G}}'(\mathbf{r}'_0, \mathbf{r}') = \bar{\mathbf{R}}_6 \cdot (\bar{\mathbf{G}})^t(\mathbf{r}, \mathbf{r}_0) \cdot \bar{\mathbf{R}}_6^t, \tag{7}$$

where 6×6 matrix $\bar{\mathbf{R}}_6$ is the mapping operator corresponding to $\bar{\mathbf{R}}$. The operator $\bar{\mathbf{R}}_6$ is defined below by Eq. (23). In the theory of Altman and Suchy, the vector $\mathbf{S} = (\mathbf{r} - \mathbf{r}_0)$ defined by the point of a source $\mathbf{r}_0$ and the point of observation $\mathbf{r}$ changes its position and orientation in Space after mapping so that the tensors $\bar{\mathbf{G}}'(\mathbf{r}', \mathbf{r}'_0)$ and $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0)$ correspond in general to different regions of Space (medium) and these regions possess different electromagnetic properties.

It is well-known that Space (rotation, reflection)-Time reversal symmetry of a medium leads to a simplified structure of constitutive tensors [6]. One can expect that this symmetry is reflected also in the structure of Green's tensors. However, Green's tensors are more complex mathematical objects as compared with constitutive tensors. Therefore, this problem requires a special consideration.

The main aim of this work is to demonstrate that using Space-Time reversal symmetry of media, one can simplify Green's tensors. The results of Altman and Suchy [3] are a basis for our work. Another basis is the theory of magnetic groups (see Appendix A).

We shall investigate symmetry properties of Green's tensors for the cases when the transformed Green's tensor is in the same environment (in the same medium) as the original (not transformed) tensor. This allows us to find some relations between the tensor elements. We shall treat in this paper the cases where the vector $\mathbf{S}$ is invariant with respect to the corresponding Space-Time reversal symmetry transformations or at most changes the sign preserving its orientation in Space. In cases of high symmetry of the medium (a spherical symmetry), the theory allows us to find some general restrictions on the tensor elements, and these restrictions do not depend on the vector $\mathbf{S}$ orientations. In cases of lower symmetries (for example, axial ones) we can define those directions in Space which lead to the simplest, canonical forms of Green's tensors and to calculate the structure of these tensors. The structure of the Green's tensors for these special directions (with the proper orientation of the coordinate system) is defined by symmetry of media.

We shall not be interested in the explicit expressions of Green's tensors in terms of $\mathbf{S}$ and medium parameters. Group-theoretical methods used in this paper do not allow one to obtain such expressions. Our interest lies in the structure of Green's tensors. This structure is defined by the equality of some of the entries of the tensors to zero (i.e., $G_{ij}(\mathbf{S}) = 0$) and equality of some of the entries among themselves (i.e., $G_{ij}(\mathbf{S}) = G_{ji}(\pm\mathbf{S})$ or $G_{ij}(\mathbf{S}) = -G_{ji}(\pm\mathbf{S})$). Thus, our problem is to obtain the information about Green's tensors which follow from the Space-Time reversal symmetry properties of the media. We shall illustrate the obtaining the simplified structure of Green's tensors with some examples.

Also, one of the main concerns of this paper is to introduce the full system of the continuous magnetic point groups for description of Space-Time reversal symmetry of Green's tensors. The full system of the continuous magnetic groups allows one to calculate and to catalogue all the admissible structures (forms) of Green's tensors for media with these symmetries.

Group-theoretical formalism adopted here is general and can be used in both the time and frequency domains. Here, the general ideas

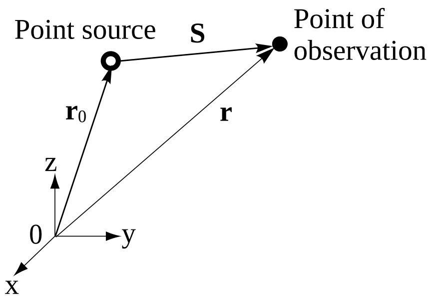

Figure 1. The position vectors $\mathbf{r}$ and $\mathbf{r}_0$ of Green's tensor $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$ and the vector $\mathbf{S}=(\mathbf{r}-\mathbf{r}_0)$.

will be demonstrated in the frequency domain (more exactly, in the $(\mathbf{r},\omega)$ domain). The results of the analysis are compared with some of the analytical solutions available in the literature.

2. SYMMETRY DESCRIPTION OF THE PROBLEM

We shall consider an unbounded homogeneous linear bianisotropic and in general lossy medium which can be described by a symmetry G_1. G_1 is a general notation of a magnetic group of the first, second or the third category (see Appendix A). The groups of the first category will be denoted by the usual letters, the groups of the second category by bold type and the groups of the third category by the symbol $G(H)$ where H is a unitary subgroup of the group G. Discussing the concrete magnetic groups, we shall use the Schoenflies notations [4].

The three-dimensional Dirac delta function $\delta(\mathbf{r}-\mathbf{r}_0)\bar{\mathbf{I}}$ ($\bar{\mathbf{I}}$ is the 3×3 unit matrix) used below for Green's tensor calculations belongs to the so-called generalized functions [5]. It describes an idealized physical object. From the point of view of symmetry, delta function has a peculiarity. From one side it presents a point object because it does not have a length or a volume and does not have directional properties. It means that we can consider delta function as a scalar. From the other side, its argument $\mathbf{S}=(\mathbf{r}-\mathbf{r}_0)$ is a vector (this vector in Cartesian coordinate system is shown in Fig. 1). Geometrically we can present the delta function as a small sphere with the vector $\mathbf{S}$ connecting the center of the sphere and a point in Space which is the point of observation. It is reasonable to consider the above two symmetry properties of the delta function separately.

Thus, Space-Time reversal properties of Green's tensors for a bianisotropic medium are defined by the following constituents:

- medium with a symmetry G_1,
- point source described by the group G_2,
- the vector $\mathbf{S} = (\mathbf{r} - \mathbf{r}_0)$ described by the group G_3.

The groups G_1, G_2 and G_3 will be defined below.

The resultant symmetry G_{res} of these three constituents can be defined by Curie's principle of symmetry superposition. This principle states that the symmetry G_{res} of our complex object consisting of a medium, the point source and the vector $\mathbf{S}$ is the highest common subgroup of the groups G_1, G_2, and G_3, i.e.,

$$G_{res} = G_1 \cap G_2 \cap G_3. \tag{8}$$

It is clear that because of presence of the vector $\mathbf{S}$, the complete symmetry with respect to full rotation group in this case is impossible, and we can consider at the most an axial symmetry.

Bianisotropic and complex media can have different magnetic and nonmagnetic symmetry G_1 described by the three categories of magnetic groups, from the highest group K_h of the first category corresponding to an isotropic achiral medium to the full absence of symmetry described by the group $\mathbf{C}_1$ of the second category (see Appendix A). In the case of an unbounded homogeneous medium, any point of it possesses the symmetry G_1. The symmetry G_1 of the medium is reflected in the structure of the constitutive tensors $\bar{\boldsymbol{\epsilon}}$, $\bar{\boldsymbol{\mu}}$, $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$ [6]. Some particular cases of a general bianisotropic medium under consideration are isotropic, chiral, anisotropic, double anisotropic and magnetized ferrite media, cold (ionospheric) magnetoplasma, moving media, many new artificial electromagnetic materials (such as for example, chiroferrites).

Now, we apply to the symmetry of the current point sources. One should make the following remark with respect to the delta function $\delta(\mathbf{r} - \mathbf{r}_0)\bar{\mathbf{I}}$ describing the point sources. The static electric field of a point electric charge has the spherical symmetry. Because of nonexistence of the magnetic monopoles, static magnetic field with spherical symmetry does not exist. As far as concerned the oscillating electromagnetic fields, symmetry analysis leads to the following statement: all spherically symmetric distributions do not radiate [8], or equivalently, electromagnetic sources with spherically symmetric radiation do not exist [28]. But nevertheless the idealized model of a point source with spherical symmetry which radiates isotropically is often used in theoretical investigations.

Finally we consider symmetry of the position vector $\mathbf{S}$. The vector $\mathbf{S} = (\mathbf{r}-\mathbf{r}_0)$ is a polar one which has the axis of infinite order C_∞ and an infinite number of planes of symmetry σ_v (the subscript v for vertical) passing through the axis C_∞. The vector **S** remains unaltered after the rotation by any angle around the axis C_∞ and after reflection in a plane σ_v. The elements C_∞ and σ_v will be used to find some relations between the entries of the tensor $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$.

There are other rotation-reflection operations which can give information about the properties of Green's tensors as well. The plane $\tilde{\sigma}_h$ (the subscript h for horizontal) which is perpendicular to C_∞ and bisects the vector is not a plane of symmetry because under reflection in this plane, **S** changes its sign. Under inversion $\tilde{i}$ with respect to the point of bisection, the vector **S** changes its sign as well. The same is valid for the rotations $\tilde{C}_2$ through angle π around the axis which is perpendicular to C_∞. The reflection $\tilde{\sigma}_h$, the inversion $\tilde{i}$ and the rotations $\tilde{C}_2$ which interchange the positions of the points **r** and $\mathbf{r}_0$ will be called the elements of additional symmetry. In order to distinguish them from the usual elements C_∞ and σ_v, we have denoted the elements of additional symmetry with a tilde.

Notice, that the transformation properties of the vector **S** under the operations $\tilde{\sigma}_h$, $\tilde{i}$ and $\tilde{C}_2$ are defined by a certain irreducible representation of the group $D_{\infty h}$ of the first category. But for sake of simplicity, we shall not refer here to the theory of representations. The elements of additional symmetry can be used in order to find some relations for the entries of the tensor $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$ and the tensor $\bar{\mathbf{G}}(\mathbf{r}_0-\mathbf{r})$, i.e., for the tensors with opposite orientation of the vector **S**.

Now, we consider the symmetry of the vector **S** under Time reversal. Strictly speaking the position vector **S** does not depend on the sign of Time, i.e., it is even in Time. But for our purposes, we shall associate it with the wave vector **k**. The Time reversal operator changes the sign of the wave vector **k**, i.e., the direction of wave motion. Associating **S** with **k** we should consider the vector **S** as being odd in Time. That is, the vector **S** changes its sign under Time reversal indicating the exchange the position of the source and the point of observation. This artifice will allow us in Section 3 to use the combined Space-Time reversal operators of the magnetic groups and to formalize the procedure of obtaining constraints to Green's tensors.

Thus, in addition to the elements C_∞ and σ_v, we can consider the antireflection $T\sigma_v$ which changes the sign of **S** (notice that the antirotation TC_∞ does not exist). Also, we can consider the antielements of symmetry $T\tilde{\sigma}_h$, $T\tilde{i}$ and $T\tilde{C}_2$. The element T changes the sign of **S**, but each of the elements $\tilde{\sigma}_h$, $\tilde{i}$ and $\tilde{C}_2$ changes the sign of **S** once more so that the antielements $T\tilde{\sigma}_h$, $T\tilde{i}$ and $T\tilde{C}_2$ preserve the sign of **S**. These antielements give some relations between the entries of $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$.

Summarizing we can consider the following elements, elements of additional symmetry and antielements of symmetry of the vector **S**:

- the elements of symmetry C_∞ and σ_v which do not change the vector **S**,
- the elements of additional symmetry $\tilde{\sigma}_h$, $\tilde{i}$ and $\tilde{C}_2$ which change the sign of **S**,
- the pure Time reversal T which changes the sign of the vector **S**,
- the antireflection $T\sigma_v$ which changes the sign of **S**,
- the antielements of symmetry $T\tilde{\sigma}_h$, $T\tilde{i}$ and $T\tilde{C}_2$ which preserve the vector **S**.

Notice that side by side with the discrete transformations σ_v, $\tilde{i}$, C_2, T, the admissible symmetries contain also an element of continuous symmetry C_∞.

The above analysis shows that the highest possible symmetry group of our problem is defined by the vector **S** and it is the group $D_{\infty h}$ of the first category. All the other possible continuous groups of symmetry are subgroups of $D_{\infty h}$ and they can be found from the subgroup decomposition depicted on Fig. 2.

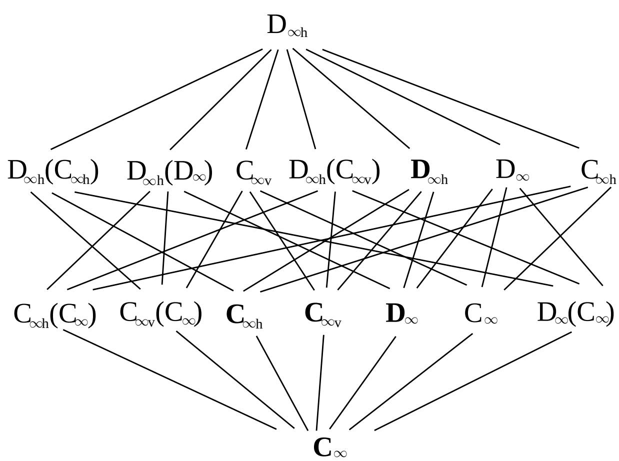

Figure 2. Subgroup decomposition of the continuous magnetic point group of the first category $D_{\infty h}$.

Thus, the point sources and the vector **S** discussed above have the fixed symmetries, but the material medium can possess different symmetries. The number of magnetic point groups which define symmetry of bianisotropic media is rather large. The constitutive tensors for the 122 crystallografic magnetic point groups and 21 magnetic continuous groups have been calculated in [9]. In this paper, we shall restrict ourselves by consideration of the *continuous* magnetic groups which are the leading groups for their *discrete* subgroups.

Having defined the symmetry of the constituents and the possible resultant symmetries of our problem, we can proceed to the group-theoretical formulation of the Space-Time reversal properties of Green's tensors.

3. SPACE-TIME REVERSAL SYMMETRY PROPERTIES OF GREEN'S TENSORS

3.1. Definition of Green's Tensors

We shall discuss in this paper Space-Time reversal symmetry properties of the Green's tensors. It is pertinent to remind the notion of symmetry of a tensor. Space-Time reversal symmetry of a tensor is its property to be invariant under Space-Time reversal transformations. The tensor has a symmetry element ((anti)axis, (anti)plane, (anti)center) if all its components are transformed into themselves under the transformation corresponding to this element. These symmetry properties lead to some restrictions on the tensor elements.

In order to compact mathematical description of our electromagnetic problem, we shall employ the six-vector notations [10]. The combined 6×6 Green's tensor of the problem is defined by Eq. (1). We consider unbounded homogeneous media, therefore Green's tensors depend only on the difference $(\mathbf{r}-\mathbf{r}_0)$, i.e., $\bar{\mathbf{G}}(\mathbf{r},\mathbf{r}_0)=\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$. Thus, in the $(\mathbf{r},\omega)$ domain our object of investigation is the Green's tensor

$$\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)=\begin{pmatrix}\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0) & \bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)\\ \bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0) & \bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)\end{pmatrix}. \tag{9}$$

If the Green's tensor $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$ is known, the solution of Maxwell's equations for a source $\mathbf{J}(\mathbf{r}_0)$ can be written as

$$\mathbf{F}(\mathbf{r})=i\omega\int_V[\,\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{J}(\mathbf{r}_0)\,]d^3r_0, \tag{10}$$

or, more explicitly

$$\mathbf{E}(\mathbf{r})=i\omega\int_V[\,\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{J}^e(\mathbf{r}_0)+\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{J}^m(\mathbf{r}_0)\,]d^3r_0, \tag{11}$$

$$\mathbf{H}(\mathbf{r})=i\omega\int_V[\,\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{J}^e(\mathbf{r}_0)+\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{J}^m(\mathbf{r}_0)\,]d^3r_0. \tag{12}$$

In Eq. (10), the six-vectors of the electromagnetic field $\mathbf{F}(\mathbf{r})$ and the electric-magnetic current density $\mathbf{J}(\mathbf{r}_0)$ are

$$\mathbf{F}(\mathbf{r}) = \begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix} \quad \text{and} \quad \mathbf{J}(\mathbf{r}_0) = \begin{pmatrix} \mathbf{J}^e(\mathbf{r}_0) \\ \mathbf{J}^m(\mathbf{r}_0) \end{pmatrix}. \tag{13}$$

Eq. (10) can be considered as a linear integral transform of the source $\mathbf{J}(\mathbf{r}_0)$ with the kernel $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$ into the field $\mathbf{F}(\mathbf{r})$.

3.2. Time Reversal and Space Inversion Transformations of Green's Tensors

In our considerations, two symmetry operations play a special role. These are Time reversal and Space inversion (see Section 4). The Time reversal and Space inversion transformation properties of the Green's tensors follow from defining equation (10). For example, the electric current density $\mathbf{J}^e(\mathbf{r}_0)$ changes its sign under Time reversal, i.e., it is odd in Time. The electric field $\mathbf{E}(\mathbf{r})$ retains its sign under Time reversal, i.e., it is even in Time. Taking into account that the Time reversal operator complex conjugates all the quantities, the multiplier $i\omega$ changes its sign as well. Thus, to preserve the form of equation (10) under Time reversal, the tensor $\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)$ should be considered as even in Time. Analogous consideration of three other tensors shows that $\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$ is also even in Time, but $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$ are odd in Time. Thus, we can write symbolically the transformation as follows:

$$\begin{aligned} \bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0) &\xrightarrow{T} (\bar{\mathbf{G}}^{ee})^*(\mathbf{r}-\mathbf{r}_0), \\ \bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0) &\xrightarrow{T} (\bar{\mathbf{G}}^{mm})^*(\mathbf{r}-\mathbf{r}_0), \\ \bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0) &\xrightarrow{T} -(\bar{\mathbf{G}}^{em})^*(\mathbf{r}-\mathbf{r}_0), \\ \bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0) &\xrightarrow{T} -(\bar{\mathbf{G}}^{me})^*(\mathbf{r}-\mathbf{r}_0). \end{aligned} \tag{14}$$

In the above transformation rules, we still did not use our convention of changing the sign of $(\mathbf{r}-\mathbf{r}_0)$ under Time reversal. It will be used in Section 3.6.

One can meet in literature (for example, in [3,13]) a defining equation for the Green's tensors (10) in a little bit different form, namely without the multiplier $i\omega$. In this case one should consider $\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$ as tensors which are odd in Time, but $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$ as tensors even in Time.

Comparing the Space inversion properties of currents $\mathbf{J}(\mathbf{r}_0)$ and fields $\mathbf{F}(\mathbf{r})$ in Eq. (10), we can also define Space inversion properties of $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$. For example, the electric current density $\mathbf{J}^e(\mathbf{r}_0)$ and the electric field $\mathbf{E}(\mathbf{r})$ are polar vectors. Therefore, the tensor $\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)$ must be a polar one. However, the magnetic current $\mathbf{J}^m(\mathbf{r}_0)$ is an axial vector, therefore the tensor $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$ which couples $\mathbf{J}^m(\mathbf{r}_0)$ and $\mathbf{E}(\mathbf{r})$ must be an axial tensor. Analogous consideration of two other tensors shows that $\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$ is a polar tensor, but $\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$ is an axial one. The Space inversion transformation (which is denoted by i) of the Green's tensors can be written symbolically as follows:

$$\begin{aligned}
\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0) &\xrightarrow{i} \bar{\mathbf{G}}^{ee}(\mathbf{r}_0-\mathbf{r}),\\
\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0) &\xrightarrow{i} \bar{\mathbf{G}}^{mm}(\mathbf{r}_0-\mathbf{r}),\\
\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0) &\xrightarrow{i} -\bar{\mathbf{G}}^{em}(\mathbf{r}_0-\mathbf{r}),\\
\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0) &\xrightarrow{i} -\bar{\mathbf{G}}^{me}(\mathbf{r}_0-\mathbf{r}).
\end{aligned} \tag{15}$$

Notice that under i, the signs of the arguments of the polar tensors $\bar{\mathbf{G}}^{ee}$ and $\bar{\mathbf{G}}^{mm}$ are changed. The axial tensors $\bar{\mathbf{G}}^{em}$ and $\bar{\mathbf{G}}^{me}$ themselves and their arguments change their signs under Space inversion (transformation properties of tensors are discussed in Appendix C).

3.3. Differential Equations for Green's Tensors in $(\mathbf{r}, \omega)$ Domain

Maxwell's equations (with time dependence in the form of $\exp(i\omega t)$ which is suppressed) combined with the constitutive relations can be written concisely as

$$\mathcal{M}\mathbf{F}(\mathbf{r}) \equiv \left(\bar{\mathcal{D}} + i\omega\bar{\mathbf{K}}(\mathbf{r})\right)\cdot\mathbf{F}(\mathbf{r}) = -\mathbf{J}(\mathbf{r}), \tag{16}$$

where $\mathcal{M}$ is Maxwell's operator which consists of the differential part $\bar{\mathcal{D}}$ and the algebraic part $i\omega\bar{\mathbf{K}}$ with $\bar{\mathbf{K}}$ being the medium six-tensor. The differential part of Maxwell's operator has the following form:

$$\bar{\mathcal{D}} = \begin{pmatrix} \bar{\mathbf{0}} & -\nabla\times\bar{\mathbf{I}} \\ \nabla\times\bar{\mathbf{I}} & \bar{\mathbf{0}} \end{pmatrix}, \quad \nabla\times\bar{\mathbf{I}} = \begin{pmatrix} 0 & -\partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & -\partial/\partial x \\ -\partial/\partial y & \partial/\partial x & 0 \end{pmatrix}. \tag{17}$$

The combined constitutive tensor $\bar{\mathbf{K}}(\mathbf{r})$ couples the flux-density vectors $\mathbf{D}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ to the field-intensity vectors $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$:

$$\begin{pmatrix} \mathbf{D}(\mathbf{r}) \\ \mathbf{B}(\mathbf{r}) \end{pmatrix} = \bar{\mathbf{K}}(\mathbf{r}) \cdot \begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix} \quad \text{with} \quad \bar{\mathbf{K}}(\mathbf{r}) = \begin{pmatrix} \bar{\boldsymbol{\epsilon}} & \bar{\boldsymbol{\xi}} \\ \bar{\boldsymbol{\zeta}} & \bar{\boldsymbol{\mu}} \end{pmatrix}, \tag{18}$$

where $\bar{\boldsymbol{\epsilon}}$, $\bar{\boldsymbol{\mu}}$, $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$ are the 3×3 constitutive tensors. For unbounded homogeneous media, the tensor $\bar{\mathbf{K}}(\mathbf{r})$ does not depend on $\mathbf{r}$.

The current density $\mathbf{J}(\mathbf{r})$ can be presented in the form

$$\mathbf{J}(\mathbf{r}) = \int_V \left[\delta(\mathbf{r} - \mathbf{r}_0)\bar{\mathbf{I}}_6 \cdot \mathbf{J}(\mathbf{r}_0) \right] d^3 r_0, \tag{19}$$

where

$$\delta(\mathbf{r} - \mathbf{r}_0)\bar{\mathbf{I}}_6 = \begin{pmatrix} \delta(\mathbf{r} - \mathbf{r}_0)\bar{\mathbf{I}} & \bar{\mathbf{0}} \\ \bar{\mathbf{0}} & \delta(\mathbf{r} - \mathbf{r}_0)\bar{\mathbf{I}} \end{pmatrix}, \tag{20}$$

$\bar{\mathbf{I}}_6$ is the unit 6×6 matrix, $\bar{\mathbf{I}}$ is the unit 3×3 matrix and $\bar{\mathbf{0}}$ is the 3×3 zero matrix.

Substituting Eq. (19) and Eq. (10) into Maxwell's equations (16), one can deduce the differential equation for the Time-harmonic Green's tensor $\bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0)$:

$$\mathcal{M}\bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0) \equiv \left(\bar{\mathcal{D}} + i\omega\bar{\mathbf{K}}(\mathbf{r})\right) \cdot \bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0) \left(\frac{1}{i\omega}\right) \bar{\mathbf{I}}_6. \tag{21}$$

In order to obtain a unique solution to these equations, one should use radiation conditions at infinity.

From Eq. (21), we can write the decoupled differential equations for the tensors $\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)$, $\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$, $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$. For example, for the tensor of electric type we have:

$$\left[(\boldsymbol{\nabla} \times \bar{\mathbf{I}} - i\omega\bar{\boldsymbol{\xi}}) \cdot \bar{\boldsymbol{\mu}}^{-1} \cdot (\nabla \times \bar{\mathbf{I}} + i\omega\bar{\boldsymbol{\zeta}}) - \omega^2\bar{\boldsymbol{\epsilon}}\right] \cdot \bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0) = -\delta(\mathbf{r}-\mathbf{r}_0)\bar{\mathbf{I}}. \tag{22}$$

Using Space inversion-Time reversal transformation properties of the constitutive tensors $\bar{\boldsymbol{\epsilon}}$, $\bar{\boldsymbol{\mu}}$, $\bar{\boldsymbol{\zeta}}$, $\bar{\boldsymbol{\xi}}$, of the operator $\boldsymbol{\nabla} \times \bar{\mathbf{I}}$ and of the Dirac delta function $\delta(\mathbf{r} - \mathbf{r}_0)\bar{\mathbf{I}}$ in Eq. (22), we can once more verify that $\bar{\mathbf{G}}^{ee}(\mathbf{r} - \mathbf{r}_0)$ is even in Time polar tensor.

3.4. Symmetry Operators

Let our object be invariant under a magnetic group of symmetry. It means that the medium, the point source and the vector $\mathbf{S} = (\mathbf{r} - \mathbf{r}_0)$ are invariant under the corresponding operations. In other words, any of the symmetry operations transforms our object into a new configuration which is not distinguishable from the original one. The only admissible change is the sign of the vector $\mathbf{S} = (\mathbf{r} - \mathbf{r}_0)$, i.e., the source and the point of observation can interchange their positions.

In the following discussion, we shall use the active transformation of the configuration space (in our case the configuration space is the medium in which electromagnetic sources and fields are defined). Thus, the configuration space is rotated or reflected while the coordinate system is kept fixed in Space.

Let us denote a discrete or continuous Space operator corresponding to rotation-reflection symmetry elements as $\mathcal{R}$ (three-dimensional matrix representations of these elements are given in Appendix B). Then, a combined Space-Time reversal operator can be written as $\mathcal{T}\mathcal{R}$, where $\mathcal{T}$ is the restricted Time reversal operator [3] corresponding to the Time reversal T.

The effect of a symmetry operator on equation (21) differs for the two cases:

(a) the symmetry operator does not contain the Time reversal operator $\mathcal{T}$,

(b) the symmetry operator is a combined operator $\mathcal{T}\mathcal{R}$ or it is the operator $\mathcal{T}$ itself.

Dealing with the six-vector formalism, we should replace the operator $\mathcal{T}\mathcal{R}$ by its 6×6 matrix representation $\bar{\mathbf{R}}_6$:

$$\bar{\mathbf{R}}_6 = \begin{pmatrix} \bar{\mathbf{R}} & \bar{\mathbf{0}} \\ \bar{\mathbf{0}} & \pm\det(\bar{\mathbf{R}})\,\bar{\mathbf{R}} \end{pmatrix}, \tag{23}$$

where $\det(\bar{\mathbf{R}})$ denotes the determinant of an orthogonal matrix $\bar{\mathbf{R}}$. $\bar{\mathbf{R}}$ is the three-dimensional representation of the rotation-reflection operator $\mathcal{R}$. Inclusion of $\det(\bar{\mathbf{R}})$ in formula (23) allows one to take into account the axial nature of the tensors $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$, $\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$, $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$. In formula (23), the signs $+$ and $-$ in front of $\det(\bar{\mathbf{R}})$ are for cases (a) and (b), respectively. The matrix $\bar{\mathbf{R}}_6$ has the property $\bar{\mathbf{R}}_6^t = \bar{\mathbf{R}}_6^{-1}$ where the superscript ^{t} denotes transposition.

The simplest particular cases of the matrix $\bar{\mathbf{R}}_6$ are

$$\bar{\mathbf{R}}_I = \bar{\mathbf{I}}_6 = \begin{pmatrix} \bar{\mathbf{I}} & \bar{\mathbf{0}} \\ \bar{\mathbf{0}} & \bar{\mathbf{I}} \end{pmatrix}, \quad \bar{\mathbf{R}}_T = \begin{pmatrix} \bar{\mathbf{I}} & \bar{\mathbf{0}} \\ \bar{\mathbf{0}} & -\bar{\mathbf{I}} \end{pmatrix}, \quad \bar{\mathbf{R}}_i = -\bar{\mathbf{R}}_T = \begin{pmatrix} -\bar{\mathbf{I}} & \bar{\mathbf{0}} \\ \bar{\mathbf{0}} & \bar{\mathbf{I}} \end{pmatrix}, \tag{24}$$

where $\bar{\mathbf{R}}_I$ is the unit 6×6 matrix which corresponds to the unit group element. The matrix $\bar{\mathbf{R}}_T$ is employed for Time reversal, and this matrix was called in [3] "the Poynting-vector reversing operator" because it reverses the sign of the magnetic field $\mathbf{H}$ and consequently, the sign of the Poynting vector. The matrix $\bar{\mathbf{R}}_i$ corresponds to the Space inversion operator.

3.5. Space (Rotation-Reflection) Symmetry of Green's Tensors

First, we consider case (a) which is simpler. Let us apply the operator $\mathcal{R}$ to Eq. (21):

$$\mathcal{R}\mathcal{M}\,\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0) = -\,\mathcal{R}\,\delta(\mathbf{r}-\mathbf{r}_0)\left(\frac{1}{i\omega}\right)\bar{\mathbf{I}}_6. \tag{25}$$

Substituting $\mathcal{R}$ by $\bar{\mathbf{R}}_6$ (with the sign + in front of $\det(\bar{\mathbf{R}})$ in formula (23)), multiplying Eq. (21) from the right by $\bar{\mathbf{R}}_6^{-1}$, we obtain the transformed Maxwell's equations for the transformed Space $\mathbf{r}' = \bar{\mathbf{R}}\cdot\mathbf{r}$, $\mathbf{r}'_0 = \bar{\mathbf{R}}\cdot\mathbf{r}_0$, $(\mathbf{r}'-\mathbf{r}'_0) = \bar{\mathbf{R}}\cdot(\mathbf{r}-\mathbf{r}_0)$ (see Appendix C):

$$\Big(\bar{\mathcal{D}}'(\mathbf{r}') + i\omega\bar{\mathbf{K}}'(\mathbf{r}')\Big)\cdot\mathbf{G}'(\mathbf{r}'-\mathbf{r}'_0) = -\,\delta(\mathbf{r}'-\mathbf{r}'_0)\left(\frac{1}{i\omega}\right)\bar{\mathbf{I}}_6, \tag{26}$$

where

$$\bar{\mathcal{D}}'(\mathbf{r}') = \bar{\mathbf{R}}_6\cdot\bar{\mathcal{D}}(\,\bar{\mathbf{R}}^{-1}\cdot\mathbf{r})\cdot\bar{\mathbf{R}}_6^{-1}, \qquad \bar{\mathbf{K}}'(\mathbf{r}') = \bar{\mathbf{R}}_6\cdot\bar{\mathbf{K}}(\,\bar{\mathbf{R}}^{-1}\cdot\mathbf{r})\cdot\bar{\mathbf{R}}_6^{-1}, \tag{27}$$

$$\mathbf{G}'(\mathbf{r}'-\mathbf{r}'_0) = \bar{\mathbf{R}}_6\cdot\mathbf{G}(\mathbf{r}'-\mathbf{r}'_0)\cdot\bar{\mathbf{R}}_6^{-1}, \quad \delta(\mathbf{r}'-\mathbf{r}'_0)\,\bar{\mathbf{R}}_6\cdot\bar{\mathbf{I}}_6\cdot\bar{\mathbf{R}}_6^{-1} = \delta(\mathbf{r}'-\mathbf{r}'_0)\bar{\mathbf{I}}_6. \tag{28}$$

The constitutive tensor $\bar{\mathbf{K}}(\mathbf{r})$ is invariant with respect to $\mathcal{R}$. It means that the transformed tensor $\bar{\mathbf{K}}'(\mathbf{r}')$ is equal to the original tensor $\bar{\mathbf{K}}(\mathbf{r}')$ at the transformed point $\mathbf{r}'$, i.e., $\bar{\mathbf{K}}'(\mathbf{r}') = \bar{\mathbf{K}}(\mathbf{r}')$. But for a homogeneous infinite medium, the tensor $\bar{\mathbf{K}}(\mathbf{r})$ is independent on $\mathbf{r}$, and the point $\mathbf{r}$ is indistinguishable from the point $\mathbf{r}' = \bar{\mathbf{R}}^{-1}\cdot\mathbf{r}$. The tensor $\bar{\mathbf{K}}(\mathbf{r})$ calculated for the points $\mathbf{r}$ and $\mathbf{r}'$ must have identical values. Thus, invariance of the medium with respect to $\mathcal{R}$ can be expressed as follows:

$$\bar{\mathbf{K}}'(\mathbf{r}') = \bar{\mathbf{R}}_6\cdot\bar{\mathbf{K}}(\mathbf{r})\cdot\bar{\mathbf{R}}_6^{-1} = \bar{\mathbf{K}}(\mathbf{r}). \tag{29}$$

It can also be shown [3] that for the rotation-reflection operations, the transformed differential operator $\bar{\mathcal{D}}'(\mathbf{r}')$ in Space $\mathbf{r}'$ has the same form as the original one $\bar{\mathcal{D}}(\mathbf{r})$ in Space $\mathbf{r}$. But the transformed Space is the same homogeneous unbounded medium. This allows us to write

$$\bar{\mathcal{D}}'(\mathbf{r}') = \bar{\mathbf{R}}_6\cdot\bar{\mathcal{D}}(\mathbf{r}')\cdot\bar{\mathbf{R}}_6^{-1} = \bar{\mathcal{D}}(\mathbf{r}). \tag{30}$$

Delta-function $\delta(\mathbf{r}-\mathbf{r}_0)\bar{\mathbf{I}}_6$ can at most change the sign of its argument $(\mathbf{r}-\mathbf{r}_0)$.

Thus, transformed Eq. (26) can be rewritten as follows:

$$(\,\bar{\mathcal{D}}(\mathbf{r}) + i\omega\bar{\mathbf{K}}(\mathbf{r})\,)\cdot\mathbf{G}'(\mathbf{r}'-\mathbf{r}'_0) = -\,\delta(\mathbf{r}'-\mathbf{r}'_0)\left(\frac{1}{i\omega}\right)\bar{\mathbf{I}}_6. \tag{31}$$

Comparing Eqs. (31) and (21) we see that the operator $\mathcal{R}$ preserves the form of the equations for Green's tensors, i.e., they are invariant under the corresponding symmetry transformation. Hence, the Green's tensor being the solution of these equations must be also invariant, i.e., $\mathbf{G}'(\mathbf{r}' - \mathbf{r}'_0) = \mathbf{G}(\mathbf{r}' - \mathbf{r}'_0)$, with possible changing the sign of $(\mathbf{r}' - \mathbf{r}'_0) = \pm(\mathbf{r} - \mathbf{r}_0)$, and this will be discussed below.

The vector $\mathbf{S} = (\mathbf{r} - \mathbf{r}_0)$ is not changed under transformations C_∞ and σ_v. Therefore, we can write down the condition of invariance of Green's tensor $\bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0)$ with respect to the operations C_∞ and σ_v:

$$\bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0) = \bar{\mathbf{R}}_6 \cdot \bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0) \cdot \bar{\mathbf{R}}_6^{-1}, \tag{32}$$

where $\bar{\mathbf{R}}_6$ is the matrix representation of the symmetry element C_∞ or σ_v. Relation (32) signifies that $\bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0)$ commutes with $\bar{\mathbf{R}}_6$.

Similarly, for the elements of additional symmetry $\tilde{\sigma}_h$, $\tilde{i}$ and $\tilde{C}_2$ we obtain:

$$\bar{\mathbf{G}}(\mathbf{r}_0 - \mathbf{r}) = \bar{\mathbf{R}}_6 \cdot \bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0) \cdot \bar{\mathbf{R}}_6^{-1}, \tag{33}$$

where $\bar{\mathbf{R}}_6$ is the matrix representation of the element $\tilde{\sigma}_h$, $\tilde{i}$ or $\tilde{C}_2$. Notice the interchange of the source and field point positions in Green's tensors on the left- and right-hand sides of Eq. (33).

3.6. Combined Space (Rotation-Reflection)-Time Reversal Symmetry of Green's Tensors

In case (b), on acting with a combined symmetry operator $\mathcal{TR}$ on equation (25), we obtain:

$$\mathcal{TRM}\,\bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0) = -\,\mathcal{TR}\,\delta(\mathbf{r} - \mathbf{r}_0)\bar{\mathbf{I}}_6. \tag{34}$$

It is not difficult to show the following transformation property of the operator $\mathcal{T}$:

$$\mathcal{T}\left(\bar{\mathcal{D}} + i\omega\bar{\mathbf{K}}(\mathbf{r})\right) = -\left(\bar{\mathcal{D}} + i\omega\bar{\mathbf{K}}^\circ(\mathbf{r})\right)^t\mathcal{T}, \tag{35}$$

where the symbol $^\circ$ stands for an operation which is called the *adjugation* [10]. It is used in order to adjust even and odd in Time quantities in Eq. (21). This operation changes the sign of the off-diagonal block tensors and operators of this equation. In particular, it changes the sign of the off-diagonal block tensors entering into $\bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}_0)$, i.e., the signs of the Green's tensors of the mixed type $\bar{\mathbf{G}}^{em}(\mathbf{r} - \mathbf{r}_0)$ and $\bar{\mathbf{G}}^{me}(\mathbf{r} - \mathbf{r}_0)$. In mathematical description below, this changing is achieved by use of the matrix $\bar{\mathbf{I}}_T$.

Thus, employing the properties of the restricted Time reversal operator $\mathcal{T}$ described in Appendix C, substituting $\mathcal{R}$ by $\bar{\mathbf{R}}_6$ and multiplying from the right by $\bar{\mathbf{R}}_6^{-1}$, we can rewrite Eq. (34) as follows

$$\left(\bar{\mathcal{D}}''(\mathbf{r}') - i\omega\bar{\mathbf{K}}''(\mathbf{r}')\right)\cdot\mathbf{G}''(\mathbf{r}'-\mathbf{r}'_0) = \delta(\mathbf{r}'-\mathbf{r}'_0)\left(\frac{1}{i\omega}\right)\bar{\mathbf{I}}_6, \tag{36}$$

where

$$\bar{\mathcal{D}}''(\mathbf{r}') = \bar{\mathbf{R}}_6\cdot\bar{\mathcal{D}}^t(\mathbf{r}')\cdot\bar{\mathbf{R}}_6^{-1}, \qquad \bar{\mathbf{K}}''(\mathbf{r}') = \bar{\mathbf{R}}_6\cdot\bar{\mathbf{K}}^t(\mathbf{r}')\cdot\bar{\mathbf{R}}_6^{-1}, \tag{37}$$

$$\begin{aligned}\mathbf{G}''(\mathbf{r}'-\mathbf{r}'_0) &= \bar{\mathbf{R}}_6\cdot\mathbf{G}^t(\mathbf{r}'-\mathbf{r}'_0)\cdot\bar{\mathbf{R}}_6^{-1},\\ \delta(\mathbf{r}'-\mathbf{r}'_0)\bar{\mathbf{I}}_6 &= \delta(\mathbf{r}'-\mathbf{r}'_0)\,\bar{\mathbf{R}}_6\cdot\bar{\mathbf{I}}_6\cdot\bar{\mathbf{R}}_6^{-1} = \delta(\mathbf{r}'-\mathbf{r}'_0)\bar{\mathbf{I}}_6.\end{aligned} \tag{38}$$

As we agreed at the beginning of Section 3.4 the constitutive tensor $\bar{\mathbf{K}}(\mathbf{r})$ is invariant with respect to $\mathcal{TR}$, or equivalently, the medium is transformed into itself under $\mathcal{TR}$. This can be expressed as follows:

$$\bar{\mathbf{K}}''(\mathbf{r}') = \bar{\mathbf{R}}_6\cdot\bar{\mathbf{K}}^t(\mathbf{r})\cdot\bar{\mathbf{R}}_6^{-1} = \bar{\mathbf{K}}(\mathbf{r}). \tag{39}$$

Taking into account the properties of the differential operator $\bar{\mathcal{D}}(\mathbf{r}) = \bar{\mathcal{D}}^t(\mathbf{r})$ and $\bar{\mathcal{D}}^\circ(\mathbf{r}) = -\bar{\mathcal{D}}(\mathbf{r})$ we can write

$$\bar{\mathcal{D}}''(\mathbf{r}') = \bar{\mathbf{R}}_6\cdot\bar{\mathcal{D}}^t(\mathbf{r}')\cdot\bar{\mathbf{R}}_6^{-1} = -\bar{\mathcal{D}}(\mathbf{r}). \tag{40}$$

Under $\mathcal{TR}$, delta-function $\delta(\mathbf{r}-\mathbf{r}_0)\bar{\mathbf{I}}_6$ can at most change the sign of its argument $(\mathbf{r}-\mathbf{r}_0)$.

Thus, transformed Eq. (36) can be rewritten as follows:

$$\left(\bar{\mathcal{D}}(\mathbf{r}) + i\omega\bar{\mathbf{K}}(\mathbf{r})\right)\cdot\mathbf{G}''(\mathbf{r}'-\mathbf{r}'_0) = -\delta(\mathbf{r}'-\mathbf{r}'_0)\left(\frac{1}{i\omega}\right)\bar{\mathbf{I}}_6, \tag{41}$$

Depending on the operator $\mathcal{TR}$, the argument $(\mathbf{r}-\mathbf{r}_0)$ of the Green's function can preserve or change its sign, i.e., $(\mathbf{r}'-\mathbf{r}'_0) = \pm(\mathbf{r}-\mathbf{r}_0)$. Comparing Eqs. (41) and (21), we see that the differential equations for Green's tensors are invariant with respect to $\mathcal{TR}$.

To this point, it is pertinent to make a remark concerning application of the Time reversal in classical electrodynamics. In spite of invariance of Maxwell's equations under Time reversal, the solutions of the Time-reversed problems, strictly speaking, are "non-physical" because a lossy medium is transformed in an active one and therefore, attenuating waves are transformed into growing waves. Besides, the sources are transformed in sinks, retarded waves and Green's functions are transformed into non-causal advanced ones. Nevertheless, using the *restricted* Time reversal operator (which preserves a lossy or active character of the medium and consequently, attenuating or growing character of waves) and considering only the ray path between the point source and the point of observation, we can obtain the correct results in our calculations. An illuminating discussion of this problem can be found in [3].

Now, we can discuss restrictions imposed on $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$ by antielements of symmetry. First, we should justify our agreement that the Time reversal operator $\mathcal{T}$ changes the sign of $(\mathbf{r}-\mathbf{r}_0)$. The Green's tensors $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$ and the Lorentz-adjoint Green's tensors $\bar{\mathbf{G}}_L(\mathbf{r}-\mathbf{r}_0)$ (see Appendix D) are related as $\bar{\mathbf{G}}_L(\mathbf{r}-\mathbf{r}_0) = (\bar{\mathbf{G}}^\circ)^t(\mathbf{r}_0-\mathbf{r})$, i.e., with opposite signs of their arguments $(\mathbf{r}-\mathbf{r}_0)$. It was shown by Altman and Suchy [3] that the restricted Time reversed and the Lorentz-adjoint Maxwell's systems are identical. Thus, we can formally attribute to the Time reversal the property of changing the sign of $(\mathbf{r}-\mathbf{r}_0)$.

Let us consider the element of symmetry $T\sigma_v$, that is the antiplane of symmetry which is perpendicular to the vector $(\mathbf{r}-\mathbf{r}_0)$. The element σ_v does not change the sign of $(\mathbf{r}-\mathbf{r}_0)$. In accordance with our agreement, the Time reversal operator changes the sign of $(\mathbf{r}-\mathbf{r}_0)$. Thus, for the symmetry element $T\sigma_v$ we obtain the following relation:

$$\bar{\mathbf{G}}(\mathbf{r}_0-\mathbf{r}) \;=\; \bar{\mathbf{R}}_6 \cdot \bar{\mathbf{G}}^t(\mathbf{r}-\mathbf{r}_0) \cdot \bar{\mathbf{R}}_6^{-1} \tag{42}$$

with the matrix $\bar{\mathbf{R}}_6$ corresponding to $T\sigma_v$.

Analogously for the combined operators $T\tilde{\sigma}_h$, $T\tilde{i}$, and $T\tilde{C}_2$ which do not change the sign of $(\mathbf{r}-\mathbf{r}_0)$, we obtain:

$$\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0) \;=\; \bar{\mathbf{R}}_6 \cdot \bar{\mathbf{G}}^t(\mathbf{r}-\mathbf{r}_0) \cdot \bar{\mathbf{R}}_6^{-1}, \tag{43}$$

where $\bar{\mathbf{R}}_6$ corresponds to $T\tilde{\sigma}_h$, $T\tilde{i}$ or $T\tilde{C}_2$.

Relations (32), (33), (42) and (43) are a basis for the following calculations.

4. CONSTRAINTS ON GREEN'S TENSORS FOR MEDIA SYMMETRICAL WITH RESPECT TO THE RESTRICTED TIME REVERSAL AND FOR MEDIA WITH CENTER AND ANTICENTER OF SYMMETRY

Four important cases which deserve a special consideration are:

- media invariant with respect to the Time reversal T,
- media possessing the center of symmetry $\tilde{i}$,
- media with the anticenter of symmetry $T\tilde{i}$,
- media which possess simultaneously T and $\tilde{i}$.

Here, the time reversal T denotes the restricted operator $\mathcal{T}$.

Media invariant with respect to the Time reversal T.

The "pure" restricted Time reversal operator is a particular case of $\mathcal{T}\mathcal{R}$ where $\mathcal{R}$ is the unit operator of the group and $\bar{\mathbf{R}}_6$ is the unit matrix. Therefore, from Eq. (42) we have the symmetry conditions imposed by $\mathcal{T}$ upon $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$:

$$\bar{\mathbf{G}}(\mathbf{r}_0-\mathbf{r}) = (\bar{\mathbf{G}}^{\circ})^t(\mathbf{r}-\mathbf{r}_0). \tag{44}$$

Relation (44) coincides with the known restriction on Green's tensors for reciprocal bianisotropic media which has been obtained in [11] by an electrodynamical method.

It is noted in [3] that "Lorentz reciprocity is an expression of the invariance of Maxwell's equations under time reversal". Thus, invariance of a bianisotropic medium with respect to the restricted Time reversal operator, i.e., the condition $\bar{\mathbf{K}} = (\bar{\mathbf{K}}^{\circ})^t$ means its reciprocity and imposes restriction (44) on Green's tensors.

Relation (44) written in terms of $\bar{\mathbf{G}}^{mm}$, $\bar{\mathbf{G}}^{ee}$, $\bar{\mathbf{G}}^{em}$ and $\bar{\mathbf{G}}^{me}$ gives [2]:

$$\bar{\mathbf{G}}^{ee}(\mathbf{r}_0-\mathbf{r}) = (\bar{\mathbf{G}}^{ee})^t(\mathbf{r}-\mathbf{r}_0), \quad \bar{\mathbf{G}}^{mm}(\mathbf{r}_0-\mathbf{r}) = (\bar{\mathbf{G}}^{mm})^t(\mathbf{r}-\mathbf{r}_0), \tag{45}$$

$$\bar{\mathbf{G}}^{em}(\mathbf{r}_0-\mathbf{r}) = -(\bar{\mathbf{G}}^{me})^t(\mathbf{r}-\mathbf{r}_0). \tag{46}$$

Media possessing the center of symmetry $\tilde{i}$.

For the media with the center of symmetry $\tilde{i}$, the cross-coupling constitutive tensors $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$ are zero. The operation $\tilde{i}$ transforms the vector $\mathbf{S}$ into $-\mathbf{S}$. Using relation (33) we can find the following general restrictions on Green's tensors for such media:

$$\bar{\mathbf{G}}^{mm}(\mathbf{r}_0-\mathbf{r}) = \bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0), \quad \bar{\mathbf{G}}^{ee}(\mathbf{r}_0-\mathbf{r}) = \bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0), \tag{47}$$

that is $\bar{\mathbf{G}}^{mm}$ and $\bar{\mathbf{G}}^{ee}$ are even functions of $\mathbf{S}$. Also,

$$\bar{\mathbf{G}}^{me}(\mathbf{r}_0-\mathbf{r}) = -\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0), \quad \bar{\mathbf{G}}^{em}(\mathbf{r}_0-\mathbf{r}) = -\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0), \tag{48}$$

i.e., $\bar{\mathbf{G}}^{me}$ and also $\bar{\mathbf{G}}^{em}$ are odd functions of $\mathbf{S}$. A particular case of these symmetrical media, namely, the so–called diagonal medium has been considered in [10].

Media with the anticenter of symmetry $T\tilde{i}$.

The operation $T\tilde{i}$ preserves the direction of $\mathbf{S}$. Therefore, using relation (43) we obtain the following restrictions on Green's tensors for media with the anticenter of symmetry $T\tilde{i}$:

$$\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0) = (\bar{\mathbf{G}}^{mm})^t(\mathbf{r}-\mathbf{r}_0), \quad \bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0) = (\bar{\mathbf{G}}^{ee})^t(\mathbf{r}-\mathbf{r}_0), \quad (49)$$

i.e., the tensors $\bar{\mathbf{G}}^{mm}$ and $\bar{\mathbf{G}}^{ee}$ are symmetrical with respect to the main diagonal, and

$$\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0) = (\bar{\mathbf{G}}^{em})^t(\mathbf{r}-\mathbf{r}_0). \quad (50)$$

Media which is invariant with respect to both T and $\tilde{i}$.

From the theory of magnetic groups, it is known that if a group possesses T and $\tilde{i}$, it has also $T\tilde{i}$ as an element of symmetry. Hence, combining the above restrictions (45)–(50) we can obtain the symmetry conditions for Green's tensors for media with such a symmetry.

5. GREEN'S TENSORS FOR BIANISOTROPIC UNI-AXIAL MEDIA

The symmetry elements T, $\tilde{i}$ and $T\tilde{i}$ discussed above are in a sense "universal". They give information about Green's tensors for any orientation of the vector $\mathbf{S}$. Other symmetry elements such as an (anti)axes and/or (anti)planes, give restrictions of the tensor elements only for parallel or perpendicular orientation of $\mathbf{S}$ with respect to these (anti)axes and/or (anti)planes. Below, we shall discuss these symmetry elements in more detail.

Symmetry of a tensor is its intrinsic property which does not depend on the chosen coordinate system. But the structure of the tensor depends on the orientation of the coordinate axes with respect to the axes and planes of symmetry of the medium. In our case of Green's tensors, their symmetry depends also on the vector $\mathbf{S}$ orientation. In a general case with an arbitrary orientation of the vector $\mathbf{S}$ with respect of the coordinate axes and medium axes and planes, Green's tensors have all the elements different from zero. The only exclusion is the simplest case of an isotropic achiral medium which will be discussed in Section 6.

With a special choice of the vector $\mathbf{S}$ orientation, we can obtain a simplification of Green's tensor structure. Such a choice for example, corresponds to orientation of this vector parallel the axes x, y or z (with orientation of the (anti)axes and (anti)planes of the medium also parallel or normal to these axes). This special choice of the vector **S** orientation is analogous to the transformation of a second-rank material tensor to its principal axes where the tensor acquires the simplest form [32].

In order to illustrate the method, we apply to media with uniaxial symmetries. Below, analyzing the symmetry properties of Green's tensors, we shall assume that the principal axis C_∞ of the media which is in every continuous group of Fig. 2 will be oriented along the axis z of a Cartesian coordinate system. Though we shall investigate in detail the case of the vector **S** and the axis C_∞ orientation along the axis z or, symbolically, $\mathbf{S} \parallel z \parallel C_\infty$ where the symbol $\parallel$ means parallel, this procedure can also be used for the cases $\mathbf{S} \parallel x \perp C_\infty$ and $\mathbf{S} \parallel y \perp C_\infty$ where the symbol $\perp$ stands for perpendicular.

From the point of view of symmetry, we can divide all the continuous groups of Fig. 2 into two blocks A and B.

Table 1. Structure of Green's tensors for media with the symmetry $\mathbf{C}_\infty$ for the case $\mathbf{S} \parallel z \parallel C_\infty$.

$\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$
$\begin{pmatrix} G_{11}^{mm} & G_{12}^{mm} & 0 \\ -G_{12}^{mm} & G_{11}^{mm} & 0 \\ 0 & 0 & G_{33}^{mm} \end{pmatrix}$	$\begin{pmatrix} G_{11}^{ee} & G_{12}^{ee} & 0 \\ -G_{12}^{ee} & G_{11}^{ee} & 0 \\ 0 & 0 & G_{33}^{ee} \end{pmatrix}$	$\begin{pmatrix} G_{11}^{em} & G_{12}^{em} & 0 \\ -G_{12}^{em} & G_{11}^{em} & 0 \\ 0 & 0 & G_{33}^{em} \end{pmatrix}$	$\begin{pmatrix} G_{11}^{me} & G_{12}^{me} & 0 \\ -G_{12}^{me} & G_{11}^{me} & 0 \\ 0 & 0 & G_{33}^{me} \end{pmatrix}$

A. The groups of block A contain the axis of symmetry C_∞ and do not contain the planes σ_v. These groups are $\mathbf{C}_\infty$, $C_{\infty h}(C_\infty)$, $C_{\infty v}(C_\infty)$, $\mathbf{C}_{\infty h}$, $\mathbf{D}_\infty$, C_∞, $D_\infty(C_\infty)$, $D_{\infty h}(C_{\infty h})$, $D_{\infty h}(D_\infty)$, $\mathbf{D}_\infty$, D_∞ and $C_{\infty h}$. Using relation (32) for the element of symmetry C_∞, we have calculated a general structure of Green's tensors for this block of groups. They are given in Table 1. Other elements of symmetry if they are present in the group give some additional information. Let us consider some of the groups of block A in more detail.

- The group $\mathbf{C}_\infty$ of the second category has only one axis which has been used for calculations of Green's tensors presented in Table 1. Since this group has no other elements of symmetry, we can not obtain any new information. In particular, relations between the tensors $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}(\mathbf{r}_0-\mathbf{r})$ remain indeterminate.
- The group C_∞ of the first category has additionally the Time reversal T. Therefore we can use formulas (45) and (46) in order to obtain relations between the tensors $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}(\mathbf{r}_0-\mathbf{r})$: $\bar{\mathbf{G}}^{mm}(\mathbf{r}_0-\mathbf{r}) = (\bar{\mathbf{G}}^{mm})^t(\mathbf{r}-\mathbf{r}_0)$, $\bar{\mathbf{G}}^{ee}(\mathbf{r}_0-\mathbf{r}) = (\bar{\mathbf{G}}^{ee})^t(\mathbf{r}-\mathbf{r}_0)$.

The tensors $\bar{\mathbf{G}}^{me}$ and $\bar{\mathbf{G}}^{em}$ become related to each other: $\bar{\mathbf{G}}^{em}(\mathbf{r}_0 - \mathbf{r}) = -(\bar{\mathbf{G}}^{me})^t(\mathbf{r} - \mathbf{r}_0)$.

- In the group $C_{\infty h}(C_\infty)$, there exists the element of symmetry $T\tilde{\sigma}_h$, so that using formula (43) we obtain additional relations: $G_{12}^{mm}(\mathbf{r} - \mathbf{r}_0) = 0$, $G_{12}^{ee}(\mathbf{r} - \mathbf{r}_0) = 0$, $\bar{\mathbf{G}}^{me}(\mathbf{r} - \mathbf{r}_0) = (\bar{\mathbf{G}}^{em})^t(\mathbf{r} - \mathbf{r}_0)$.
- The group $C_{\infty v}(C_\infty)$ contains along with C_∞ also the antielement $T\sigma_v$, and with the help of relation (42) we obtain the following relations: $\bar{\mathbf{G}}^{mm}(\mathbf{r}_0 - \mathbf{r}) = \bar{\mathbf{G}}^{mm}(\mathbf{r} - \mathbf{r}_0)$, $\bar{\mathbf{G}}^{ee}(\mathbf{r}_0 - \mathbf{r}) = \bar{\mathbf{G}}^{ee}(\mathbf{r} - \mathbf{r}_0)$, $\bar{\mathbf{G}}^{me}(\mathbf{r}_0 - \mathbf{r}) = \bar{\mathbf{G}}^{em}(\mathbf{r} - \mathbf{r}_0)$.
- Besides the axis C_∞, the group $D_\infty(C_\infty)$ has also the antielement $T\tilde{C}_2$ (C_2 is rotation by π around axis x or y) which gives $\bar{\mathbf{G}}^{em}(\mathbf{r} - \mathbf{r}_0) = -\bar{\mathbf{G}}^{me}(\mathbf{r} - \mathbf{r}_0)$.

Table 2. Structure of Green's tensors for media with the symmetry $\mathbf{C}_{\infty v}$ for the case $\mathbf{S} \parallel z \parallel C_\infty$.

$\bar{\mathbf{G}}^{mm}(\mathbf{r} - \mathbf{r}_0)$	$\bar{\mathbf{G}}^{ee}(\mathbf{r} - \mathbf{r}_0)$	$\bar{\mathbf{G}}^{em}(\mathbf{r} - \mathbf{r}_0)$	$\bar{\mathbf{G}}^{me}(\mathbf{r} - \mathbf{r}_0)$
$\begin{pmatrix} G_{11}^{mm} & 0 & 0 \\ 0 & G_{11}^{mm} & 0 \\ 0 & 0 & G_{33}^{mm} \end{pmatrix}$	$\begin{pmatrix} G_{11}^{ee} & 0 & 0 \\ 0 & G_{11}^{ee} & 0 \\ 0 & 0 & G_{33}^{ee} \end{pmatrix}$	$\begin{pmatrix} 0 & -G_{12}^{em} & 0 \\ G_{12}^{em} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -G_{12}^{me} & 0 \\ G_{12}^{me} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

B. Let us apply to the groups of block B. Besides the axis of symmetry C_∞, the groups of this block $\mathbf{C}_{\infty v}$, $C_{\infty v}$, $D_{\infty h}(C_{\infty v})$ and $D_{\infty h}$ have also an infinite number of planes of symmetry σ_v passing through the axis. The calculated Green's tensors for this case are presented in Table 2. These tensors have been calculated using only two elements of symmetry, namely, C_∞ and σ_v. Now we shall consider some of the groups listed in block B.

- The group $\mathbf{C}_{\infty v}$ of the second category does not contain other elements of symmetry except C_∞ and σ_v which have already been used. Thus, we can not say anything else about the structure of the Green's tensors.
- The group $C_{\infty v}$ of the first category contains also the Time reversal T. As a result of the use of this element, we have $\bar{\mathbf{G}}^{mm}(\mathbf{r}_0 - \mathbf{r}) = \bar{\mathbf{G}}^{mm}(\mathbf{r} - \mathbf{r}_0)$, $\bar{\mathbf{G}}^{ee}(\mathbf{r}_0 - \mathbf{r}) = \bar{\mathbf{G}}^{ee}(\mathbf{r} - \mathbf{r}_0)$, $G_{12}^{em}(\mathbf{r}_0 - \mathbf{r}) = G_{12}^{me}(\mathbf{r} - \mathbf{r}_0)$.
- The antielement $T\tilde{\sigma}_h$ of the group $D_{\infty h}(C_{\infty v})$ gives the following additional information: $G_{12}^{em}(\mathbf{r} - \mathbf{r}_0) = -G_{12}^{me}(\mathbf{r} - \mathbf{r}_0)$.

- In case of the group $D_{\infty h}$, the elements $\tilde{\sigma}_h$ and T lead to the following relations: $\bar{\mathbf{G}}^{mm}(\mathbf{r}_0 - \mathbf{r}) = \bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$, $\bar{\mathbf{G}}^{ee}(\mathbf{r}_0-\mathbf{r}) = \bar{\mathbf{G}}^{ee}(\mathbf{r} - \mathbf{r}_0)$, $G_{12}^{em}(\mathbf{r}_0 - \mathbf{r}) = -G_{12}^{em}(\mathbf{r} - \mathbf{r}_0)$, $G_{12}^{me}(\mathbf{r}_0 - \mathbf{r}) = -G_{12}^{me}(\mathbf{r} - \mathbf{r}_0)$, $G_{12}^{em}(\mathbf{r}_0 - \mathbf{r}) = G_{12}^{me}(\mathbf{r} - \mathbf{r}_0)$.

6. GREEN'S TENSORS FOR ISOTROPIC ACHIRAL MEDIA

Now, we shall compare our calculations with the known analytical solutions of Green's tensors for two important classes of media. These media are isotropic achiral and isotropic chiral ones. The constitutive tensors for media in both cases are degenerate to scalars.

First, we consider isotropic achiral media. This medium is characterized by two scalars: the permittivity ϵ and the permeability μ. Green's dyadics in the closed-form for this medium are written, for example in [7]:

$$\begin{aligned} \bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0) &= -i\omega\mu\left(\bar{\mathbf{I}}+\frac{\nabla\nabla}{k^2}\right)\frac{\exp(-ikr)}{4\pi r}, \\ \bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0) &= \left(\nabla\times\bar{\mathbf{I}}\right)\frac{\exp(-ikr)}{4\pi r}, \end{aligned} \tag{51}$$

$$\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0) = -\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0), \quad \bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0) = \frac{\epsilon}{\mu}\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0), \tag{52}$$

where $k = \omega\sqrt{\epsilon\mu}$, ω is the frequency, $r = |\mathbf{r} - \mathbf{r}_0| = |\mathbf{S}|$. In tensor notations, the structure of these Green's tensors is presented in Table 3. We see that the tensors $\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)$ are symmetrical about the main diagonal, but the tensors $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$ are antisymmetrical with all diagonal elements equal to zero. It is the unique case where Green's tensors contain zero elements in general (for any orientation of the vector $\mathbf{S}$).

Table 3. Structure of Green's tensors for isotropic achiral media for an arbitrary orientation of the vector $\mathbf{S}$ with respect to rectangular coordinate system.

$\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$
$\begin{pmatrix} G_{11}^{mm} & G_{12}^{mm} & G_{13}^{mm} \\ G_{12}^{mm} & G_{22}^{mm} & G_{23}^{mm} \\ G_{13}^{mm} & G_{23}^{mm} & G_{33}^{mm} \end{pmatrix}$	$\begin{pmatrix} G_{11}^{ee} & G_{12}^{ee} & G_{13}^{ee} \\ G_{12}^{ee} & G_{22}^{ee} & G_{23}^{ee} \\ G_{13}^{ee} & G_{23}^{ee} & G_{33}^{ee} \end{pmatrix}$	$\begin{pmatrix} 0 & -G_{12}^{em} & G_{13}^{em} \\ G_{12}^{em} & 0 & -G_{23}^{em} \\ -G_{13}^{em} & G_{23}^{em} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & G_{12}^{em} & -G_{13}^{em} \\ -G_{12}^{em} & 0 & G_{23}^{em} \\ G_{13}^{em} & -G_{23}^{em} & 0 \end{pmatrix}$

It is known that any antisymmetrical axial tensor of the second rank which has three independent parameters describes a polar vector. Thus, we can state that the symmetry of the Green's tensors $\bar{\mathbf{G}}^{me}$ and $\bar{\mathbf{G}}^{em}$ of Table 3 corresponds to the symmetry of a polar vector which is obviously the vector $\mathbf{S}$.

From the point of view of symmetry, the isotropic medium is described by the highest point magnetic group of the first category K_h (this group is often denoted as $O(3)$). The group K_h besides the rotations possesses also the center of symmetry $\tilde{i}$ and infinite number of planes of symmetry. The medium with this symmetry is reciprocal. Green's tensors for this medium have the symmetry $D_{\infty h}$. These tensors calculated by group-theoretical method coincide in the form with those obtained by electrodynamic methods (see Table 3).

If we choose the orientation of the vector $\mathbf{S}$ along the axis z, Green's tensors acquire the simplified form presented in Table 2. The tensors $\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)$ become diagonal. The tensors $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$ contain only one independent parameter, $G_{12}^{me}(\mathbf{r}-\mathbf{r}_0) = -G_{12}^{em}(\mathbf{r}-\mathbf{r}_0)$. This is the simplest form of the Green's tensors which is achievable only for isotropic media. In order to come back to a general form of the tensors we can make an arbitrary rotation of these tensors (or the rectangular coordinate system) around the origin. This rotation obviously will lead to a more complex structure of the tensors namely, to the form presented in Table 3. But any symmetrical tensor (in our case, $\bar{\mathbf{G}}^{mm}$ and $\bar{\mathbf{G}}^{ee}$) remains symmetrical after any rotation, and any antisymmetrical tensor ($\bar{\mathbf{G}}^{em}$ and $\bar{\mathbf{G}}^{me}$) remains antisymmetrical under this transformation.

7. GREEN'S TENSORS FOR ISOTROPIC CHIRAL MEDIA

Let us apply to an isotropic chiral medium which is the simplest example of bianisotropic media called often as biisotropic one. This medium is characterized by three scalars: ϵ and μ are the permittivity and permeability, respectively, and ζ is the chirality parameter. Green's tensors for this medium have been discussed by many authors (see for example, [12, 13]). These tensors are defined by the relations:

$$\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0) = -i\omega\mu\Big(\bar{\mathbf{G}}_1 + \bar{\mathbf{G}}_2\Big), \tag{53}$$

$$\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0) = -i\omega\epsilon\Big(\bar{\mathbf{G}}_1 + \bar{\mathbf{G}}_2\Big), \tag{54}$$

$$\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0) = -(k_1 - i\omega\zeta)\bar{\mathbf{G}}_1 + (k_2 + i\omega\zeta)\bar{\mathbf{G}}_2, \tag{55}$$

$$\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0) = -\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0), \tag{56}$$

where

$$\bar{\mathbf{G}}_{1,2}(\mathbf{r}-\mathbf{r}_0)=\left(k_{1,2}\bar{\mathbf{I}}_3+\frac{\nabla\nabla}{k_{1,2}}\pm(\,\nabla\times\bar{\mathbf{I}}_3)\right)\frac{\exp(-ik_{1,2}r)}{4\pi(k_1+k_2)r}, \tag{57}$$

$$k_{1,2}=\omega(\pm i\zeta+\sqrt{\epsilon\mu}). \tag{58}$$

Analysis of these expressions shows that in general, Green's tensors have all their elements different from zero. Green's tensors contain both symmetrical and antisymmetrical (with respect to the main diagonal) parts.

Now, let us consider the structure of Green's tensors from the group-theoretical point of view. The chiral medium is described by the group of symmetry of the first category K (notice, that this non-Abelian continuous group of rotations in three dimensions is often denoted also as $SO(3)$). This medium possesses only rotation symmetry and it is reciprocal. Green's tensors for this medium have the symmetry D_∞.

For a particular case of the vector $\mathbf{S}$ orientation along the axis z, we obtain the simplified, canonical form of the tensors which are presented in Table 1. These tensors have been calculated using only the principal axis of symmetry C_∞. Making use also the element of symmetry $T\tilde{C}_2$ where $\tilde{C}_2$ is rotation by π around the axis x, we obtain $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)=-\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$ which coincides with Eq. (56). Time reversal T gives additional constraints (45) and (46) on these Green's tensors.

8. GREEN'S TENSORS FOR MAGNETIC CHIRAL MEDIA

In this Section, we shall demonstrate application of our method to a medium with a large number of independent parameters of the constitutive tensors. We shall investigate symmetry properties of the magnetic chiral media (chiroferrites). A chiroferrite can be conceived as a ferrite with randomly oriented small chiral elements embedded in it. The magnetic group of symmetry of such media can be obtained by symmetry superposition (see Section 2). An isotropic chiral medium is described by the group K. The dc magnetic field has the symmetry $D_{\infty h}(C_{\infty h})$. These two groups have in common one axis of infinite order C_∞ and besides, an infinite number of axes of the second order C_2 perpendicular to the main axis C_∞ are converted under dc magnetic field into the antiaxes TC_2. Therefore, the resulting group is $D_\infty(C_\infty)$. The constitutive tensors for such a group for magnetization along the z-axis are written in Table 4. The tensors contain 9 independent

Table 4. Constitutive tensors for magnetic chiral media.

$\bar{\boldsymbol{\mu}}$	$\bar{\boldsymbol{\epsilon}}$	$\bar{\boldsymbol{\xi}}$	$\bar{\boldsymbol{\zeta}}$
$\begin{pmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{pmatrix}$	$\begin{pmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ -\epsilon_{12} & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{pmatrix}$	$\begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{pmatrix}$	$\begin{pmatrix} -\xi_{11} & -\xi_{12} & 0 \\ \xi_{12} & -\xi_{11} & 0 \\ 0 & 0 & -\xi_{33} \end{pmatrix}$

parameters. To our knowledge, an analytical solution for Green's tensors for this medium is not known.

Two orientations of the vector $\mathbf{S}$ give simplified Green's tensor forms. The first one is $\mathbf{S} \parallel z$. The second orientation is $\mathbf{S} \perp z$, i.e., the vector $\mathbf{S}$ lies in the plane $x0y$. For the case $\mathbf{S} \parallel z$, the group of symmetry of Green's tensors coincides with the group of symmetry of the medium, namely it is $D_\infty(C_\infty)$. Any rotation about the z-axis preserves the sign of $\mathbf{S}$. In the antirotation $T\tilde{C}_2$, the operator $\tilde{C}_2$ changes the sign of $\mathbf{S}$ and the operator T restores its sign. The calculated Green's tensors are written in Table 5. Notice that the

Table 5. Structure of Green's tensors for magnetic chiral media for the case $\mathbf{S} \parallel z \parallel C_\infty$.

$\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$
$\begin{pmatrix} G^{mm}_{11} & G^{mm}_{12} & 0 \\ -G^{mm}_{12} & G^{mm}_{11} & 0 \\ 0 & 0 & G^{mm}_{33} \end{pmatrix}$	$\begin{pmatrix} G^{ee}_{11} & G^{ee}_{12} & 0 \\ -G^{ee}_{12} & G^{ee}_{11} & 0 \\ 0 & 0 & G^{ee}_{33} \end{pmatrix}$	$\begin{pmatrix} G^{em}_{11} & G^{em}_{12} & 0 \\ -G^{em}_{12} & G^{em}_{11} & 0 \\ 0 & 0 & G^{em}_{33} \end{pmatrix}$	$\begin{pmatrix} -G^{em}_{11} & -G^{em}_{12} & 0 \\ G^{em}_{12} & -G^{em}_{11} & 0 \\ 0 & 0 & -G^{em}_{33} \end{pmatrix}$

tensors $\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$ in this Table are expressed in terms of the tensors $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$. Comparing Table 4 and Table 5, we see that the structures of the constitutive tensors and Green's tensors coincide.

For the orientation $\mathbf{S} \perp z$, for example $\mathbf{S} \parallel x$, the symmetry of Green's tensors is described by the discrete magnetic group of the third category $D_2(C_2)$ which is a subgroup of $D_\infty(C_\infty)$. A rotation $\tilde{C}_2$ around the axis z by π changes the sign of $\mathbf{S}$. The antiaxis TC_2 coinciding with $\mathbf{S}$ changes the sign of $\mathbf{S}$ due to operator T. The calculations give the structure of the tensors presented in Table 6. In this case, the structures of the constitutive tensors and Green's tensors do not coincide, the Green's tensors have a more complex form. The

Table 6. Structure of Green's tensors for magnetic chiral media for the case $\mathbf{S} \parallel x \perp C_\infty$.

$\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$	$\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$
$\begin{pmatrix} G_{11}^{mm} & G_{12}^{mm} & G_{13}^{mm} \\ -G_{12}^{mm} & G_{22}^{mm} & G_{23}^{mm} \\ G_{13}^{mm} & -G_{23}^{mm} & G_{33}^{mm} \end{pmatrix}$	$\begin{pmatrix} G_{11}^{ee} & G_{12}^{ee} & G_{13}^{ee} \\ -G_{21}^{ee} & G_{22}^{ee} & G_{23}^{ee} \\ G_{13}^{ee} & -G_{23}^{ee} & G_{33}^{ee} \end{pmatrix}$	$\begin{pmatrix} G_{11}^{em} & G_{12}^{em} & G_{13}^{em} \\ G_{21}^{em} & G_{22}^{em} & G_{23}^{em} \\ G_{31}^{em} & G_{32}^{em} & G_{33}^{em} \end{pmatrix}$	$\begin{pmatrix} -G_{11}^{em} & G_{21}^{em} & G_{31}^{em} \\ G_{12}^{em} & -G_{22}^{em} & G_{32}^{em} \\ G_{13}^{em} & G_{23}^{em} & -G_{33}^{em} \end{pmatrix}$

tensor $\bar{\mathbf{G}}^{mm}(\mathbf{r}_0-\mathbf{r})$ is expressed in terms of $\bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0)$:

$$\bar{\mathbf{G}}^{mm}(\mathbf{r}_0-\mathbf{r}) = \begin{pmatrix} G_{11}^{mm}(\mathbf{r}-\mathbf{r}_0) & G_{12}^{mm}(\mathbf{r}-\mathbf{r}_0) & -G_{13}^{mm}(\mathbf{r}-\mathbf{r}_0) \\ -G_{12}^{mm}(\mathbf{r}-\mathbf{r}_0) & G_{22}^{mm}(\mathbf{r}-\mathbf{r}_0) & -G_{23}^{mm}(\mathbf{r}-\mathbf{r}_0) \\ -G_{13}^{mm}(\mathbf{r}-\mathbf{r}_0) & G_{23}^{mm}(\mathbf{r}-\mathbf{r}_0) & G_{33}^{mm}(\mathbf{r}-\mathbf{r}_0) \end{pmatrix}. \tag{59}$$

The tensor $\bar{\mathbf{G}}^{em}(\mathbf{r}_0-\mathbf{r})$ in terms of $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$ is:

$$\bar{\mathbf{G}}^{em}(\mathbf{r}_0-\mathbf{r}) = \begin{pmatrix} G_{11}^{em}(\mathbf{r}-\mathbf{r}_0) & G_{12}^{em}(\mathbf{r}-\mathbf{r}_0) & -G_{13}^{em}(\mathbf{r}-\mathbf{r}_0) \\ G_{21}^{em}(\mathbf{r}-\mathbf{r}_0) & G_{22}^{em}(\mathbf{r}-\mathbf{r}_0) & -G_{23}^{em}(\mathbf{r}-\mathbf{r}_0) \\ -G_{31}^{em}(\mathbf{r}-\mathbf{r}_0) & -G_{32}^{em}(\mathbf{r}-\mathbf{r}_0) & G_{33}^{em}(\mathbf{r}-\mathbf{r}_0) \end{pmatrix}. \tag{60}$$

For any other orientation of the vector $\mathbf{S}$ different from $\mathbf{S} \parallel z$ or $\mathbf{S} \perp z$, the theory of symmetry fails to give any information about Green's tensors. The tensors will have a general form.

9. DISCUSSION AND CONCLUSIONS

Using group theory one can obtain useful information of Green's tensors properties without solving the corresponding differential equation. In particular, group theory allows one to reduce the number of independent parameters of Green's tensors. This method is general and it can be applied to many physical problems where Space-Time reversal symmetry exists.

We have analysed in this paper Space-Time reversal symmetry properties of Green's tensors for complex and bianisotropic media described by continuous point magnetic groups. In the framework of these groups, we have defined all the possible symmetries of the problem. The method of tensor calculations suggested here is also valid for the media described by discrete point magnetic groups which are subgroups of the continuous ones. The information is important because it is exact and the results of the analysis do not depend on details of media and on frequency.

Using our methods, one can catalogue all the admissible structures (forms) of the Green's tensors in the same way as it was made for constitutive tensors of the second rank for complex and bianisotropic media [9].

In order to define the simplified structure of Green's tensors for certain directions in a medium (with corresponding orientation of the rectangular coordinate axes), we must know the group of symmetry of this medium. The symmetry is defined by the symmetry of particles constituting the medium, their arrangements in Space and by symmetry of possible external perturbations. The isotropic achiral medium has for example the symmetry K_h, the isotropic chiral medium possesses the symmetry K, the magnetized ferrite is described by the magnetic group $D_{\infty h}(C_{\infty h})$, the moving dielectric media has the symmetry $D_{\infty h}(C_{\infty v})$, etc. Knowing the symmetry it is an easy task to calculate the structure of Green's tensors.

It should be stressed that for the Time-reversal symmetry and for the center and anticenter of symmetry, the results of calculations are valid for any direction in media. For other symmetries, the structure of Green's tensors can be simplified only for some symmetrical directions in Space. In general, the structure of Green's tensors is different for different directions.

The next remark is as follows. Comparing transformation formulas of Green's tensors (32), (33), (42), (43) and of the constitutive tensors presented in [9] we can note a resemblance of them. In spite of this resemblance, the structures of the calculated Green's tensors and the constitutive tensors for the same medium in general do not coincide. The structure of Green's tensors is usually more complex. We can give here one illustrative example. Media with center of symmetry have zero cross-coupling tensors $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$, but the Green's tensors of the mixed type $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$ are different from zero. A comparison of symmetry properties of the constitutive tensors and of the Green's tensors for reciprocal bianisotropic media is given in Appendix E.

We can state that the group of symmetry describing Green's tensors for a medium can not be higher than the group of symmetry describing the corresponding medium. The reason of this is as follows. The constitutive tensors define relations between the fields $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{E}$, $\mathbf{H}$ in one and the same point of medium (local property) therefore the structure of the constitutive tensors depends only on symmetry of medium. Green's tensors define electromagnetic field in one point of medium produced by a source located in another point (nonlocal property) so that the structure of Green's tensors depends also on the orientation of the vector $\mathbf{S}$ in medium. The group of symmetry of the

Green's tensors is a subgroup of the group of symmetry of the medium that keeps the direction **S** invariant.

The results of the presented theory which is free of approximations can be used in the first place as exact references for calculations of Green's tensors by analytical and numerical methods. In the second place, if an actual source (more complex than the idealized point sources) and the medium have a certain resultant symmetry, the simplified Green's tensors can be used for analytical calculations of electromagnetic field for some special directions in medium. These directions are defined by symmetry operations which transfer a direction into its equivalent one and the vector **S** into itself or into $-\mathbf{S}$. The directions are:

- along the (anti)axes of symmetry and normal to these axes for rotations by π;
- in (anti)planes and in the directions normal to these (anti)planes.

The method can be used not only for free-Space Green's tensor analysis but also for calculations of Green's tensors for the electromagnetic wave radiation in complex and bianisotropic media with obstacles and scattering if the corresponding problem has a Space-Time reversal symmetry. In particular, many of the microwave and optical boundary-value problems possess such a symmetry. In these cases, we must also take into account symmetry of the geometry of the object and symmetry of the boundary conditions.

ACKNOWLEDGMENT

This work was supported by the Brazilian agency CNPq.

APPENDIX A. BRIEF DESCRIPTION OF POINT MAGNETIC GROUPS

Time reversal operator. For magnetic structures, it is necessary to include into consideration the Time reversal T as an element of magnetic groups and combinations of space symmetry operations with T. T changes the sign of time, i.e., $(t) \rightarrow -(t)$. The Time reversal T commutes with all the space elements. It has the property $TT = T^2 = e$ (e is the unit element). The Time reversal operator $\mathbf{T}$ corresponding to the group element T belongs to the so-called antilinear operators [4].

When we deal with electromagnetic processes in the frequency domain, the usual description of electromagnetic quantities is in terms of complex functions. The effect of the operator $\mathbf{T}$ on time-harmonic quantities is expressed as follows. First of all, the operator reverses the velocities and changes the current directions, the signs of electron spins, magnetic fluxes, magnetic fields and Poynting's vector. All these quantities are odd in Time. Secondly, it complex conjugates all the electromagnetic quantities. This property is verified easily by considering Fourier transformation of the Time-reversed quantities [3].

Strictly speaking there no exists Time reversal symmetry in physical processes. The main reason of this is causality, i.e., initial conditions impose asymmetry with respect to the past and future. In the presence of losses in a medium, the physical processes are not the same in a given and in the Time reversed medium. For example, the operator **T** converts a damping electromagnetic wave into a growing one and vice versa because the dissipative processes are not Time reversible.

Altman and Suchy [3] suggested to use along with **T** another operator which they called the *restricted* Time reversal operator. Their operator $\mathcal{T}$ fulfills the same functions as T with one exception: it is not applied to the imaginary dissipative terms of the electromagnetic quantities. This preserves the damping or growing character of the wave under the Time reversal.

Categories of magnetic groups. There exist three categories of discrete and continuous point magnetic groups. The group of the first category G consists of a unitary subgroup H (in our case, it contains the usual rotation-reflection elements) and products of T with all the elements of H. The full group is then $H + TH$ including $T = Te$. Sometimes, these groups are called nonmagnetic ones.

In the case of magnetic groups of the second category $\boldsymbol{G}$, there is no Space elements combined with the Time reversal T, and T itself is not an element of the groups. The nomenclature and the notations of the groups of the first (nonmagnetic) category and that of the second (magnetic) category coincide. In order to distinguish them, we use bold-face type for the groups of the second category.

The magnetic groups of the third category $G(H)$ contain in addition to the rotation-reflection elements of the unitary subgroup H, an equal number of antiunitary elements which are the product of T and the usual geometrical symmetry elements. These combined elements form a conjugate class TH' of the subgroup H and cause the existence of antiaxes, antiplanes and anticenter of symmetry. The full group is $H + TH'$ without T. Notice that the elements of H' are distinguished from those of H.

The unitary elements of a magnetic group of the third category form a unitary subgroup of index 2. It means that in every group of the third category there are equal number of elements with and without

T. In contrast to the groups of the first category, the operator T itself is not an element of the magnetic groups of the third category. The content of the three categories of magnetic groups is presented in Table A1.

Table A1. Content of magnetic groups of symmetry.

First category	Second category	Third category
$G = H + TH$ including T	$\boldsymbol{G}$ without T	$G(H) = H + TH', \quad H' \neq H$ T only in combination with rotation-reflections

APPENDIX B. MATRIX REPRESENTATIONS OF 3D POINT SYMMETRY OPERATORS

In order to describe symmetry operations in 3D space such as rotations and reflections, we use 3D matrix representations of the point groups. Each element of a group corresponding to a point symmetry can be presented by a 3×3 square orthonormal ($\bar{\mathbf{R}}^{-1} = \bar{\mathbf{R}}^{t}, \det \bar{\mathbf{R}} = \pm 1$) real matrix $\bar{\mathbf{R}}$. The unit element of the group has the unit 3×3 matrix as a representation. The matrices $\bar{\mathbf{R}}$ fulfilling rotations through an angle α about the axis x, y, and z are

$$\begin{aligned}
&\bar{\mathbf{R}}_{C_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad \bar{\mathbf{R}}_{C_y} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \\
&\bar{\mathbf{R}}_{C_z} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{B1}$$

respectively.

The 3D matrix representations for reflections in the planes $x = 0$, $y = 0$ and $z = 0$ are written respectively as

$$\bar{\mathbf{R}}_{\sigma_x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{\mathbf{R}}_{\sigma_y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{\mathbf{R}}_{\sigma_z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{B2}$$

and the matrix representing inversion i (the center of symmetry) is

$$\bar{\mathbf{R}}_i = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{B3}$$

The determinant of $\bar{\mathbf{R}}$ for rotations (B1) is $+1$ but it is equal to -1 for reflections (B2) and inversion (B3).

APPENDIX C. SPATIAL AND TIME-REVERSAL TRANSFORMATION PROPERTIES OF SCALAR AND VECTOR FUNCTIONS, TENSORS AND THE CURL OPERATOR

Maxwell's equations relate fields and their sources in 3D Space and Time. Let us consider the Space-Time reversal transformation properties of the quantities which are in Maxwell's equations. Let $\mathcal{R}$ be a rotation, reflection or inversion operator in Space which corresponds to a group element R, and $\mathcal{T}$ is the restricted Time reversal operator which corresponds to the Time reversal T (see Appendix A).

The position vector $\mathbf{r}$ *and the differential operator* $\partial/\partial\mathbf{r}$

Under a rotation-reflection operator $\mathcal{R}$, the position vector $\mathbf{r}$ is transformed in a new vector $\mathbf{r}'$. Symbolically, we can describe this operation as $\mathbf{r}' = \mathcal{R}\mathbf{r}$. Using the 3D representation matrix $\bar{\mathbf{R}}$ of the operator $\mathcal{R}$ (Appendix B), we can write this transformation as follows:

$$\mathbf{r}' = \bar{\mathbf{R}} \cdot \mathbf{r} = \mathbf{r} \cdot \bar{\mathbf{R}}^t, \tag{C1}$$

$\bar{\mathbf{R}}^t = \bar{\mathbf{R}}^{-1}$, the superscript ^{t} means matrix transposition, the superscript $^{-1}$ denotes the inverse matrix. The transformation of the differential operator $\partial/\partial\mathbf{r}$ is

$$\partial/\partial\mathbf{r}' = \bar{\mathbf{R}} \cdot \partial/\partial\mathbf{r} = \partial/\partial\mathbf{r} \cdot \bar{\mathbf{R}}^t. \tag{C2}$$

Obviously, the position vector $\mathbf{r}$ and the derivative are invariant with respect to the Time reversal operator $\mathcal{T}$.

Scalar and pseudoscalar functions

In this paper, we discuss homogeneous unbounded media. Thus, all the possible scalar functions (for example, the permittivity $\epsilon(\mathbf{r})$ in an isotropic media) and pseudoscalar functions (such as a chirality parameter in an isotropic chiral media) are independent of the position vector $\mathbf{r}$. However, our method is applicable also to inhomogeneous and bounded media with symmetry. Therefore, for the sake of generality and as a starting point for the following discussion of vectors and tensors, we shall describe briefly the transformation properties of scalar and pseudoscalar functions.

Under $\mathcal{R}$, a scalar function $f(\mathbf{r})$ is transformed into a new function $f'(\mathbf{r}) = \mathcal{R}f(\mathbf{r})$. The value of the new function f' at the transformed point $\mathbf{r}'$ must be equal to the value of the original function f at a given point $\mathbf{r}$, i.e.,

$$f'(\mathbf{r}') = f(\mathbf{r}), \tag{C3}$$

or

$$f'(\mathbf{r}) = f(\mathcal{R}^{-1}\mathbf{r}) = f(\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}), \tag{C4}$$

where $\mathcal{R}^{-1}$ is the inverse operator of $\mathcal{R}$ and $\bar{\mathbf{R}}^{-1}$ is the inverse matrix of $\bar{\mathbf{R}}$. Thus, on acting with $\mathcal{R}$ on $f(\mathbf{r})$, the argument $\mathbf{r}$ of the scalar function $f(\mathbf{r})$ is transformed to $\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}$.

The scalar function can be odd or even under Time reversal, i.e., it changes or does not change its sign under transformation $(t) \to -(t)$.

A pseudoscalar function $F(\mathbf{r})$ is transformed as follows:

$$F'(\mathbf{r}) = \det(\bar{\mathbf{R}})F(\mathcal{R}^{-1}\mathbf{r}) = \det(\bar{\mathbf{R}})F(\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}), \tag{C5}$$

where det means determinant. Thus, in contrast to a scalar function which is invariant under rotation-reflections, a pseudoscalar function preserves its sign under pure rotations but changes it under improper rotations, reflections and inversion. As well as in the case of scalar functions, a pseudoscalar function can be odd or even in Time.

Vector functions

Vectors are the simplest directional quantities. They may be considered as tensors of the first rank. There exist two types of the vectors: polar vectors (real ones) which will be denoted by $\mathbf{a}(\mathbf{r})$ and axial vectors (pseudovectors) denoted further as $\mathbf{A}(\mathbf{r})$. The position vector $\mathbf{r}$, electric current $\mathbf{J}^e(\mathbf{r})$, electric flux $\mathbf{D}(\mathbf{r})$, electric field $\mathbf{E}(\mathbf{r})$ are polar vectors, but magnetic current $\mathbf{J}^m(\mathbf{r})$, magnetic flux $\mathbf{B}(\mathbf{r})$, magnetic field $\mathbf{H}(\mathbf{r})$ are axial ones.

Spatial transformations $\mathbf{a}'(\mathbf{r}) = \mathcal{R}\mathbf{a}(\mathbf{r})$ of a polar vector $\mathbf{a}(\mathbf{r})$ are defined according to

$$\mathbf{a}'(\mathbf{r}) = \bar{\mathbf{R}} \cdot \mathbf{a}(\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}), \tag{C6}$$

where $\bar{\mathbf{R}}^{-1}$ is the inverse matrix of $\bar{\mathbf{R}}$. Thus, one should first transform $\mathbf{a}$ by $\mathcal{R}$ with $\mathbf{r}$ fixed and then transform $\mathbf{r}$ to $\mathbf{r}' = \mathcal{R}^{-1}\mathbf{r}$. In contrast to the scalar functions, both the vector function $\mathbf{a}$ itself and its argument (the position vector $\mathbf{r}$) are changed in the transformation.

The transformation $\mathbf{A}'(\mathbf{r}) = \mathcal{R}\mathbf{A}(\mathbf{r})$ for axial vectors under rotation-reflections is written as follows:

$$\mathbf{A}'(\mathbf{r}) = \det(\bar{\mathbf{R}})\,\bar{\mathbf{R}} \cdot \mathbf{A}(\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}). \tag{C7}$$

Comparing Eq. (C6) and Eq. (C7), we see that a distinction between the polar and axial vectors lies in their behavior under improper rotations, reflections and inversion where $\det(\bar{\mathbf{R}}) = -1$.

For the combined Space-Time reversal operators $\mathcal{TR}$ corresponding to antiaxes, antiplanes and anticenter of symmetry, the transformations $\mathbf{a}'(\mathbf{r}) = \mathcal{TR}\mathbf{a}(\mathbf{r})$ and $\mathbf{A}'(\mathbf{r}) = \mathcal{TR}\mathbf{A}(\mathbf{r})$ are written as

$$\mathbf{a}'(\mathbf{r}) = \pm\,\bar{\mathbf{R}} \cdot \mathbf{a}(\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}), \tag{C8}$$

and

$$\mathbf{A}'(\mathbf{r}) = \pm\det(\bar{\mathbf{R}})\,\bar{\mathbf{R}} \cdot \mathbf{A}(\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}), \tag{C9}$$

respectively, where the sign $+$ is used for the vectors even in Time (for example, $\mathbf{J}^m$, $\mathbf{E}$, $\mathbf{D}$), and the sign $-$ for the vectors which are odd in Time (for example, $\mathbf{J}^e$, $\mathbf{H}$, $\mathbf{B}$).

Tensors of the second rank

Tensors of the second rank are more complex directional quantities than vectors. Here, we must distinguish between the polar tensors (for example, $\bar{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\mu}}, \bar{\mathbf{G}}^{ee}, \bar{\mathbf{G}}^{mm}$) and the axial ones (for example, $\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\zeta}}, \bar{\mathbf{G}}^{em}, \bar{\mathbf{G}}^{me}$). The polar tensors $\bar{\mathbf{a}}$ define a linear relation between two polar vectors or between two axial ones, but the axial tensors $\bar{\mathbf{A}}$ determine a relation between an axial vector and a polar one. As in the case of vectors, the laws of spatial transformations are different for the polar tensors and for the axial ones. For the polar tensors, it is

$$\bar{\mathbf{a}}'(\mathbf{r}) = \bar{\mathbf{R}} \cdot \bar{\mathbf{a}}(\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}) \cdot \bar{\mathbf{R}}^{-1}, \tag{C10}$$

and for axial ones

$$\bar{\mathbf{A}}'(\mathbf{r}) = \det(\bar{\mathbf{R}})\,\bar{\mathbf{R}} \cdot \bar{\mathbf{A}}(\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}) \cdot \bar{\mathbf{R}}^{-1}. \tag{C11}$$

Here, as well as in the case of vectors, under rotations, the axial and polar tensors transform equally, but under reflections and inversion they transform in different way.

The combined Space-Time reversal operators $\mathcal{TR}$ provide the following laws of the tensor transformations. For polar tensors:

$$\bar{\mathbf{a}}'(\mathbf{r}) = \pm\,\bar{\mathbf{R}} \cdot \bar{\mathbf{a}}^t(\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}) \cdot \bar{\mathbf{R}}^{-1}, \tag{C12}$$

and for axial ones:

$$\bar{\mathbf{A}}'(\mathbf{r}) = \pm\,\det(\bar{\mathbf{R}})\,\bar{\mathbf{R}} \cdot \bar{\mathbf{A}}^t(\bar{\mathbf{R}}^{-1} \cdot \mathbf{r}) \cdot \bar{\mathbf{R}}^{-1}, \tag{C13}$$

where the signs + and − correspond, respectively, to the tensors even in Time (for example, $\bar{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\mu}}, \bar{\mathbf{G}}^{ee}, \bar{\mathbf{G}}^{mm}$) and odd in Time (for example, $\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\zeta}}, \bar{\mathbf{G}}^{em}, \bar{\mathbf{G}}^{me}$).

Curl operator

Finally, we shall describe transformation properties of the differential curl operator $(\nabla \times \bar{\mathbf{I}})$ of Eq. (21) under $\mathcal{R}$. Using the transformation properties of the operator $\partial/\partial\mathbf{r}$ defined by Eq. (C1) above, we can find:

$$\mathcal{R}(\nabla \times \bar{\mathbf{I}}) = (\nabla' \times \bar{\mathbf{I}}) = \det(\bar{\mathbf{R}})\,\bar{\mathbf{R}} \cdot (\nabla \times \bar{\mathbf{I}}) \cdot \bar{\mathbf{R}}^{-1}. \tag{C14}$$

The transformed operator $(\nabla' \times \bar{\mathbf{I}})$ applied in the transformed coordinates $\mathbf{r}'$ has the same form as the original operator $(\nabla \times \bar{\mathbf{I}})$ acting in the $\mathbf{r}$ coordinates [3]. The operator $(\nabla \times \bar{\mathbf{I}})$ is invariant under Time reversal.

APPENDIX D. TIME REVERSAL, FORMALLY ADJOINT AND LORENTZ ADJOINT MAXWELL'S EQUATIONS

We give below a brief description of different Maxwell's systems which are used in classical electrodynamics. A detailed discussion of these systems and their applications can be found in [3].

Maxwell's equations in the $(\mathbf{r}, \omega)$ domain are

$$(\bar{\mathcal{D}} + i\omega\bar{\mathbf{K}}) \cdot \mathbf{F}(\mathbf{r}) = \mathbf{J}(\mathbf{r}), \tag{D1}$$

where $\bar{\mathbf{K}}$ is medium six-tensor (18), $\mathbf{F}(\mathbf{r})$ and $\mathbf{J}(\mathbf{r})$ the six-vectors of the electromagnetic field and the electric-magnetic currents (13), respectively. The differential part of Maxwell's operator $\bar{\mathcal{D}}$ is defined by Eq. (17).

Complex conjugation of the Maxwell's equations and multiplication them from the left by the matrix $(-\bar{\mathbf{R}}_T)$ where

$$\bar{\mathbf{R}}_T = \begin{pmatrix} \bar{\mathbf{I}} & \bar{\mathbf{0}} \\ \bar{\mathbf{0}} & -\bar{\mathbf{I}} \end{pmatrix}, \tag{D2}$$

$\bar{\mathbf{R}}_T = \bar{\mathbf{R}}_T^{-1}$, $\bar{\mathbf{I}}$ is 3×3 unit matrix and $\bar{\mathbf{0}}$ is 3×3 zero matrix, leads to the Time reversed Maxwell's system:

$$(\bar{\mathcal{D}} + i\omega\bar{\mathbf{K}}_T) \cdot \mathbf{F}_T(\mathbf{r}) = \mathbf{J}_T(\mathbf{r}), \tag{D3}$$

where

$$\bar{\mathbf{K}}_T = \bar{\mathbf{R}}_T \cdot \bar{\mathbf{K}}^* \cdot \bar{\mathbf{R}}_T = (\bar{\mathbf{K}}^\circ)^*, \quad \mathbf{F}_T(\mathbf{r}) = \bar{\mathbf{R}}_T \cdot \mathbf{F}^*(\mathbf{r}), \quad \mathbf{J}_T(\mathbf{r}) = -\bar{\mathbf{R}}_T \cdot \mathbf{J}^*(\mathbf{r}). \tag{D4}$$

Comparing equations (D1) and (D3) we see that Maxwell's equations are invariant under Time reversal. Notice, that the matrix $\bar{\mathbf{R}}_T$ which was called in [3] the temporal mapping operator (and this is noted here by the subscript $_T$) is a particular case of matrix (23).

The formally adjoint to the Maxwell's system (D1) is obtained by transposing all the matrix operators and changing the sign of the differential operator $\bar{\mathcal{D}}$ [3]:

$$(-\bar{\mathcal{D}}^t + i\omega\bar{\mathbf{K}}^t) \cdot \tilde{\mathbf{F}}(\mathbf{r}) = \tilde{\mathbf{J}}(\mathbf{r}), \tag{D5}$$

where $\bar{\mathcal{D}}^t = \bar{\mathcal{D}}$. The system (D5) with the source $\tilde{\mathbf{J}}(\mathbf{r})$ and the field $\tilde{\mathbf{F}}(\mathbf{r})$ is nonphysical, nevertheless it is used in theoretical investigations (see, for example, [1] and [3]).

The Lorentz adjoint of the Maxwell's equations can be deduced premultiplying Eq. (D4) by $\bar{\mathbf{R}}_T$ with the result:

$$(\bar{\mathcal{D}} + i\omega\bar{\mathbf{K}}_L) \cdot \mathbf{F}_L(\mathbf{r}) = \mathbf{J}_L(\mathbf{r}), \tag{D6}$$

where

$$\bar{\mathbf{K}}_L = \bar{\mathbf{R}}_T \cdot \bar{\mathbf{K}}^t \cdot \bar{\mathbf{R}}_T = (\bar{\mathbf{K}}^\circ)^t, \quad \mathbf{F}_L(\mathbf{r}) = \bar{\mathbf{R}}_T \cdot \tilde{\mathbf{F}}(\mathbf{r}), \quad \mathbf{J}_L(\mathbf{r}) = -\bar{\mathbf{R}}_T \cdot \tilde{\mathbf{J}}(\mathbf{r}). \tag{D7}$$

Lorentz adjoint system (D6) is physical. It is employed in the Lorentz reciprocity theorem. The Lorentz adjoint system can also be obtained by applying the restricted Time reversal operator to Maxwell's system (D1), that is the Lorentz adjoint system and the restricted Time reversed system are identical. This circumstance is used in our paper for calculations of the Green's tensors for media possessing combined Space-Time reversal symmetries.

Below, we write down relations between the Green's tensors $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$, the adjoint Green's tensors $\tilde{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$ and the Lorentz-adjoint Green's tensors $\bar{\mathbf{G}}_L(\mathbf{r}-\mathbf{r}_0)$ [3]:

$$\tilde{\mathbf{G}}(\mathbf{r}_0-\mathbf{r}) = \bar{\mathbf{G}}^t(\mathbf{r}-\mathbf{r}_0), \ \bar{\mathbf{G}}_L(\mathbf{r}-\mathbf{r}_0) = \tilde{\mathbf{G}}^\circ(\mathbf{r}-\mathbf{r}_0), \ \bar{\mathbf{G}}_L(\mathbf{r}-\mathbf{r}_0) = (\bar{\mathbf{G}}^\circ)^t(\mathbf{r}_0-\mathbf{r}). \tag{D8}$$

APPENDIX E. COMPARISON OF SYMMETRY PROPERTIES OF THE CONSTITUTIVE TENSORS AND GREEN'S TENSORS FOR RECIPROCAL BIANISOTROPIC MEDIA

For comparison, we write down in Table E1 the restrictions imposed by reciprocity of a medium on the constitutive tensors $\bar{\mathbf{K}}$ and on Green's tensors $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$. Notice that for unbounded homogeneous media, the tensor $\bar{\mathbf{K}}$ does not depend on $\mathbf{r}$ but the tensor $\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0)$ depends on the vector $\mathbf{S} = (\mathbf{r}-\mathbf{r}_0)$.

The superscript ^{t} in Table E1 denotes transposition. The adjugation $^\circ$ changes the sign of the constitutive tensors $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$, and of the Green's tensors of the mixed type $\bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0)$ and $\bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0)$.

Table E1. Restrictions imposed by reciprocity on the constitutive and Green's tensors.

Constitutive tensors	Green's tensors
$\bar{\mathbf{K}} = \begin{pmatrix} \bar{\boldsymbol{\epsilon}} & \bar{\boldsymbol{\xi}} \\ \bar{\boldsymbol{\zeta}} & \bar{\boldsymbol{\mu}} \end{pmatrix}$	$\bar{\mathbf{G}}(\mathbf{r}-\mathbf{r}_0) = \begin{pmatrix} \bar{\mathbf{G}}^{ee}(\mathbf{r}-\mathbf{r}_0) & \bar{\mathbf{G}}^{em}(\mathbf{r}-\mathbf{r}_0) \\ \bar{\mathbf{G}}^{me}(\mathbf{r}-\mathbf{r}_0) & \bar{\mathbf{G}}^{mm}(\mathbf{r}-\mathbf{r}_0) \end{pmatrix}$
Reciprocity of medium	
$\bar{\mathbf{K}} = (\bar{\mathbf{K}}^\circ)^t$	$\bar{\mathbf{G}}(\mathbf{r}_0-\mathbf{r}) = (\bar{\mathbf{G}}^\circ)^t(\mathbf{r}-\mathbf{r}_0)$
$\bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}^{\,t}$, $\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}^{\,t}$, $\bar{\boldsymbol{\xi}} = -\bar{\boldsymbol{\zeta}}^{\,t}$.	$\bar{\mathbf{G}}^{ee}(\mathbf{r}_0-\mathbf{r}) = (\bar{\mathbf{G}}^{ee})^t(\mathbf{r}-\mathbf{r}_0)$, $\bar{\mathbf{G}}^{mm}(\mathbf{r}_0-\mathbf{r}) = (\bar{\mathbf{G}}^{mm})^t(\mathbf{r}-\mathbf{r}_0)$, $\bar{\mathbf{G}}^{em}(\mathbf{r}_0-\mathbf{r}) = -(\bar{\mathbf{G}}^{me})^t(\mathbf{r}-\mathbf{r}_0)$.

We see from Table E1, that for a reciprocal medium, the constitutive tensor $\bar{\mathbf{K}}$ is invariant with respect to transposition and adjugation, i.e., the nonreciprocal part of the tensor $\bar{\mathbf{K}}$ is equal to zero. Reciprocity of media imposes constraints on Green's tensors which give some relations between the elements of the tensors with opposite signs of the argument $\mathbf{S} = (\mathbf{r}-\mathbf{r}_0)$.

REFERENCES

1. Felsen, L. B. and N. Marcuvitz, *Radiation and Scattering of Waves*, IEEE Press, New York, 1994.

2. Kong, J. A., "Theorems of bianisotropic media," *Proc. IEEE*, Vol. 60, 1036–1046, 1972.
3. Altman, C. and K. Suchy, "Reciprocity," *Spatial Mapping and Time Reversal in Electromagnetics*, Kluwer, Dordrecht, 1991.
4. Hamermesh, M., *Group Theory and its Application to Physical Problems*, Addison-Wesley, Reading, MA, 1962.
5. Chew, W. C., *Waves and Fields in Inhomogeneous Media*, IEEE Press, New York, 1995.
6. Barybin, A. A. and V. A. Dmitriev, *Modern Electrodynamics and Coupled-Mode Theory: Application to Guided-Wave Optics*, Rinton Press, Princeton, USA, 2002.
7. Tai, C.-T., *Dyadic Green's Functions in Electromagnetic Theory*, IEEE Press, New York, 1994.
8. Kritikos, H. N., "Rotational symmetries of electromagnetic radiation fields," *IEEE Trans. Antennas Propag.*, Vol. 31, 377–382, 1983.
9. Dmitriev, V., "Tables of the second rank constitutive tensors for linear homogeneous media described by the point magnetic group of symmetry," *Progress in Electromagnetics Research*, J. A. Kong (ed.), Vol. 28, 47–99, EMW Publishing, Cambridge, MA, 2000.
10. Lindell, I. V., A. H. Sihvola, and K. Suchy, "Six-vector formalism in electromagnetics of bi-anisotropic media," *J. Electromagn. Waves Applic.*, Vol. 9, 887–903, 1995.
11. Kong, J. A., *Electromagnetic Wave Theory*, EMW Publishing, Cambridge, Massachusets, 1998.
12. Gvozdev, V. V. and A. N. Serdjukov, "Green's function and fields of the moving charges in a gyrotropic medium," *Optics and Spectr.*, Vol. 47, 544–548, 1979 (in Russian).
13. Bassiri, S., N. Engheta, and C. H. Papas, "Dyadic Green's function and dipole radiation in chiral media," *Alta Freq.*, Vol. LV, 83–88, 1986.
14. Sakoda, K., *Optical Properties of Photonic Crystals*, Springer-Verlag, Berlin, 2001.
15. Chen, H. C., *Theory of Electromagnetic Waves*, McGraw-Hill, New York, 1983.
16. Cheng, D., "Green dyadic and dipole radiation in triaxial omega medium," *Electron. Lett.*, Vol. 32, 529–530, 1996.
17. Cheng, D., "Green's dyadics in reciprocal uniaxial bianisotropic media by cylindrical vector wave functions," *Phys. Rev. E*, Vol. 54, 2917–2924, 1996.
18. Cottis, P. G. and G. D. Kondylis, "Properties of the dyadic

Green's function for an unbounded anisotropic medium," *IEEE Trans. Antennas Propag.*, Vol. AP-43, 154–161, 1995.

19. Lindell, I. V. and F. Olyslager, "Analytic Green dyadic for a class of nonreciprocal anisotropic media," *IEEE Trans. Antennas Propag.*, Vol. AP-45, 1563–1565, 1997.
20. Lindell, I. V. and W. S. Weiglhofer, "Green dyadic for a uniaxial bianisotropic medium," *IEEE Trans. Antennas Propag.*, Vol. 42, 1013–1016, 1994.
21. Olyslager, F., "Time-harmonic two- and three-dimensional Green's dyadics for general uniaxial bianisotropic media," *IEEE Trans. Antennas Propag.*, Vol. AP-43, 430–434, 1995.
22. Olyslager, F. and B. Jakoby, "Time-harmonic two- and three-dimensional Green dyadics for a special class of gyrotropic bianisotropic media," *IEE Proc.-Microw. Antennas Propag.*, Vol. 143, 413–416, 1996.
23. Olyslager, F. and I. V. Lindell, "Closed-form Green's dyadics for a class of media with axial bi-anisotropy," *IEEE Trans. Antennas Propag.*, Vol. AP-46, 1888–1890, 1998.
24. Olyslager, F. and I. V. Lindell, "Green's dyadics for a class of bi-anisotropic media with nonsymmetric bi-anisotropic dyadics," *Int. J. Electron. Commun.*, Vol. 52, 32–36, 1998.
25. Tan, E. L., "Vector wave function expansion of dyadic Green's functions for bianisotropic media," *IEE Proc.-Microw. Antennas Propag.*, Vol. 149, 57–63, 2002.
26. Weiglhofer, W. S., "Analytic methods and free-space dyadic Green's functions," *Radio Science*, Vol. 28, 847–857, 1993.
27. Weiglhofer, W. S., "Dyadic Green function for unbounded general uniaxial bianisotropic medium," *Int. J. Electronics*, Vol. 77, 105–115, 1994.
28. Comay, E., "The problem of spherically symmetric electromagnetic radiation," *Am. J. Phys.*, Vol. 70, 715–716, 2002.
29. Suchy K., C. Altman, and A. Schatzberg, "Orthogonal mappings of time-harmonic electromagnetic fields in inhomogeneous (bi)anisotropic media," *Radio Science*, Vol. 20, 149–160, 1985.
30. *Electromagnetic Symmetry*, Baum C. E. and N. H. Kritikos (eds.), Taylor & Francis, Washington, 1995.
31. Morse, P. M. and H. Feshbach, *Methods of Theoretical Physics*, p.I, McGraw-Hill, New York, 1953.
32. Nye, J. F., *Physical Properties of Crystals*, Oxford University Press, New York, 1993.

Victor A. Dmitriev was born in Russia in 1947. He received the M.Eng. degree in electrical and electronic engineering and the D.Sc. degree from the Moscow State Technical University (MSTU), named after Bauman, in 1971 and 1977, respectively. From 1973 to 1977 he was a researcher with the Ferrite laboratory at MSTU. From 1978 to 1994, he was an Associate Professor at MSTU. In 1989, he was a visiting researcher at Manchester University, England. In 1995–1997 he was a visiting professor at the Army Polytechnical School (ESPE), Quito, Ecuador. Currently, he is a visiting professor at the University Federal of Para, Brazil. His main research interests include microwave and high frequency circuits, theory of symmetry, application of group theory to electromagnetic problems, complex and composite media. He has authored and coauthored more than 100 scientific papers and inventor in 7 patents in the field of microwave devices. He is coauthor of the books *Computer Aided Design of Microstrip Ferrite Devices* (Moscow: Moscow High Technical School Publication, 1986), *Nonreciprocal Microwave Junctions Using Ferrite Resonators* (Moscow: Radio y svyaz, 1989) and others. Dr. V. Dmitriev received the golden medal from the USSR Ministry of Education in 1965, the medal "For Professional Distinction" from the USSR Government in 1980, and the medal "For Contribution to Cosmic Programs" from the USSR Federation of Cosmonautics in 1984.