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# THE ELECTROMAGNETIC-WAVE PROPAGATION THROUGH A STRATIFIED INHOMOGENEOUS ANISOTROPIC MEDIUM

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**Abstract**—The electromagnetic-wave propagation through a medium consisting of two dielectric half-spaces with a plate in between, has been investigated. The half-spaces are isotropic with their dielectric permittivity depending only on the $z$ coordinate. The plate is anisotropic, and the components of its dielectric permittivity tensor are also $z$-dependent. For the first time, the sufficient conditions allowing the transformation of the system of Maxwell's equations into two independent equations, are ascertained. For an arbitrary $z$-dependence of the dielectric permittivity, the plate's reflectance and transmittance coefficients are obtained, this result being a generalization of the Fresnel formulas. We have considered both determinate and random dependences of the dielectric permittivity on the $z$-coordinate, and the plate's full-transparency conditions are specified. For a statistically inhomogeneous plate, the conditions of its full opacity are formulated. The Faraday effect in such a medium is studied. The influence of the medium's inhomogeneity on the *temporal* rotation of the polarization plane of a propagating wave has been demonstrated.

## 1. INTRODUCTION

In Ref. [1], it has been shown that the medium's inhomogeneity coupled with its gyrotropy affects substantially the electromagnetic-wave propagation through the medium. This results in new effects, and changes — quantitatively and/or qualitatively — "ancient" ones, e.g., the Faraday effect. In this work [1], an electromagnetic wave propagates through a medium consisting of two isotropic homogeneous half-spaces parted with a homogeneous plane-parallel gyrotropic plate. In the present work, the medium includes two isotropic half-spaces with dielectric permittivities depending only on the $z$-coordinate, these half-spaces being divided by an anisotropic plate the dielectric-permittivity tensor of which is also $z$-dependent. The plate is confined with the $z = 0$ and $z = a$ plates, that is, $a$ is the plate's thickness. The wave propagates along the $z$-axis and so is incident normally to the plate's surfaces.

The paper consists of three sections.

In the first one, we consider the propagation of electromagnetic wave through and reflection from an inhomogeneous isotropic plate enclosed between two inhomogeneous isotropic half-spaces. In the second section, requirements are formulated which reduce the problem of wave propagation through anisotropic medium to a problem of propagation of two independent waves in isotropic medium, each of the waves having its own refractive index. And in the third section, the general results obtained in the two previous ones, are applied to a gyrotropic plate. It is to be noted that the medium studied in this work, allows to consider simultaneous influences of the interference, anisotropy and smooth inhomogeneity on the wave propagation.

## 2. ELECTROMAGNETIC-WAVE PROPAGATION IN ISOTROPIC INHOMOGENEOUS MEDIUM

The wave propagation in a medium consisting of two inhomogeneous isotropic half-spaces with an isotropic inhomogeneous plate between them, is described by the following equations

$$\begin{aligned} &\frac{d^2E}{dz^2} + k^2\varepsilon_l(z)E = 0, \quad z \leqslant 0. \\ &\frac{d^2E}{dz^2} + k^2\varepsilon(z)E = 0, \quad 0 \leqslant z \leqslant a. \\ &\frac{d^2E}{dz^2} + k^2\varepsilon_r(z)E = 0, \quad z \geqslant a. \end{aligned} \tag{1}$$

Here $E$ is the electrical field, $k = \frac{\omega}{c}$ with $\omega$ the frequency of the propagating wave, $c$ is the light velocity in vacuum, and $\varepsilon_l$, $\varepsilon$, $\varepsilon_r$ are the dielectric permittivities of, respectively, the left half-space, the plate, and the right half-space. The continuity conditions at the $z = 0$ and $z = a$ planes for the field $E$ and its derivative with respect to $z$ are:

$$\begin{aligned} E(-0) &= E(+0), \ \frac{\partial E(-0)}{\partial z} = \frac{\partial E(+0)}{\partial z}; \\ E(-a) &= E(+a), \ \frac{\partial E(-a)}{\partial z} = \frac{\partial E(+a)}{\partial z}. \end{aligned} \tag{2}$$

In addition, at $\pm\infty$, the radiation conditions have to be satisfied.

We consider first a problem of the wave reflecting and transmitting *with no plate* between the half-spaces, that is, assuming $a = 0$. The waves in the left half-space are described by the first of equations (1) and in the right half-space, by the third one. Let $U_{l1}(z)$ and $U_{l2}(z)$ be the fundamental solution-system of the first of equations (1) with the $U_{l1}(0) = U_{l2}(0) = 1$ condition. Also, in view of the radiation conditions, $U_{l1}(z)$ is the wave propagating to the right, i.e. the incident wave, and $U_{l2}(z)$ propagates to the left, i.e. the reflected wave. $U_r(z)$ is the wave propagating to the right (transmitted wave), $U_r(0) = 1$.

Desired solution of the problem has a form

$$E(z) = \begin{cases} U_{l1}(z) + R_0 U_{l2}(z) & \text{for} \quad z < 0, \\ T_0 U_r(z) & \text{for} \quad z > 0, \end{cases} \tag{3}$$

where $R_0$ and $T_0$ are the coefficients of reflection and transmission, respectively. $R_0$ and $T_0$ deduced from boundary conditions (2) are resulting in:

$$R_0 = \frac{U'_r(0) - U'_{l_1}(0)}{U'_{l_2}(0) - U'_r(0)}, \tag{4}$$

$$T_0 = \frac{U'_{l_2}(0) - U'_{l_1}(0)}{U'_{l_2}(0) - U'_r(0)}, \tag{5}$$

here and in the following equations, the tag means differentiation by $z$.

Now, consider the same problem, but *in the presence* of the inhomogeneous plate. Let $U_1(z)$ and $U_2(z)$ form the fundamental solution-system of the second of equations (1) with the $U_1(0) = U_2(0) = 1$ condition.

Then, the solution of the system (1) we are looking for, has a form

$$E(z) = \begin{cases} U_{l1}(z) + RU_{l2}(z) & \text{for} \quad z \leqslant 0, \\ A_1U_1(z) + A_2U_2(z) & \text{for} \quad 0 \leqslant z \leqslant a, \\ TU_r(z) & \text{for} \quad z \geqslant a. \end{cases} \quad . \tag{6}$$

Using boundary conditions (2) and solutions (6), a system of equations is obtained determining the reflection coefficient $R$, transmission coefficient $T$, and the $A_1$and $A_2$ coefficients. Then, solving this system gives the following expressions for $R$ and $T$:

$$R = \frac{\left[U_1'(a) - U_r(a)U_2'(a)\right]\left[U_2'(0) - U_{l_1}'(0)\right] + \left[U_2'(a) - U_2(a)U_r'(a)\right]\left[U_{l_1}'(0) - U_1'(0)\right]}{\left[U_1'(a) - U_1(a)U_r'(a)\right]\left[U_2'(0) - U_{l_1}'(0)\right] + \left[U_2'(a) - U_2(a)U_r'(a)\right]\left[U_{l_2}'(0) - U_1'(0)\right]}, \tag{7}$$

$$T = \frac{\left[U_2'(0) - U_{l_1}'(0)\right]\left[U_{l_2}'(0) - U_1'(0)\right] - \left[U_{l_1}'(0) - U_1'(0)\right]\left[U_2'(0) - U_{l_1}'(0)\right]}{\left[U_1'(a) - U_1(a)U_r(a)\right]\left[U_2'(0) - U_{l_1}'(0)\right] + \left[U_2'(a) - U_1(a)U_r'(a)\right]\left[U_{l_2}'(0) - U_1'(0)\right]} \times \frac{\left[U_1(a)U_2'(a) - U_1'(a)U_2(a)\right]}{U_2'(0) - U_1'(0)}. \tag{8}$$

From the equations (1), the following formulas are derived:

$$U_{l1,2}' = \pm i\sqrt{\varepsilon_l(0)}, \;\; U_{1,2}'(0) = \pm ik\sqrt{\varepsilon(0)}, \;\; U_r'(a) = ik\sqrt{\varepsilon(a)}. \tag{9}$$

Let us dwell on the concept of absolute transparency of a plate. For homogeneous plate, this question has been studied in detail in Ref. [2]. For a uniform isotropic plate, it has been shown in Ref. [2] that for a discrete set of frequencies fixed by:

$$k\sqrt{\varepsilon} = q\pi, \tag{10}$$

where $q$ is an integer, the wave reflected from the $l$-half-space/plate/$r$-half-space system behaves as if the plate is absent. This means that the plate can be referred to as completely transparent. The reflection and transmission coefficients coincide with, respectively, those for the $l$-half-space/$r$-half-space system determined by formulas (4) and (5). With no temporal dispersion and $\varepsilon$ $\omega$-independent, one obtains from (10)

$$\omega_q = \frac{q\pi c}{\sqrt{\varepsilon} a}. \tag{11}$$

The presence of temporal dispersion introduces serious difficulties in solving equation (10).

In the general case of inhomogeneous plate and half-spaces, the absolute-transparency conditions are apparently as follows:

$$R(a,\omega) = R_0(\omega), \quad T(a,\omega) = T_0(\omega), \tag{12}$$

whereof one can readily proceed to

$$[1 - U_1(a)]\left[U_2'(a) - U_r'(a)\right] - [1 - U_2(a)]\left[U_1'(a) - U_r'(a)\right] = 0$$

and then to

$$U_{1,2}(\omega, a) = U_{1,2}(\omega, 0) = 1 \tag{13}$$

or to

$$U_{1,2}'(\omega, a) = U_{1,2}'(\omega, 0). \tag{14}$$

Each of these correlations, (13) and (14), can be regarded as the equation for determining the frequencies and the plate's thicknesses for which the plate is transparent.

The wave propagation through an inhomogeneous plate with $\varepsilon$ specifically depending on $z$ has been considered previously [3, 4], but the question of the plate's transparency was not discussed in this work. As for an homogeneous plate, it is transparent for any thickness at the frequencies determined by (10).

For the following, we shall need an explicit form for the $U_{l1,2}$, $U_{1,2}$, and $U$ functions.

Let the distances $b_l$, $b_r$, and $b$, at which the $\varepsilon_l(z)$, $\varepsilon_r(z)$, and $\varepsilon(z)$ functions substantially change, far exceed the wave-length, so that the WKB method is applicable. Note that

$$b_l^{-1} \sim \frac{1}{\varepsilon_l}\frac{d\varepsilon_l}{dz}, \; b_r^{-1} \sim \frac{1}{\varepsilon_r}\frac{d\varepsilon_r}{dz}, \; b^{-1} \sim \frac{1}{\varepsilon}\frac{d\varepsilon}{dz}.$$

The criteria of applicability of the WKB method and the explicit form of the corresponding solutions are well known [4]:

$$\frac{1}{k\sqrt{\varepsilon_l(z)}}\frac{d\varepsilon_l(z)}{dz} \ll 1, \; \frac{1}{k\sqrt{\varepsilon_r(z)}}\frac{d\varepsilon_r(z)}{dz} \ll 1, \; \frac{1}{k\sqrt{\varepsilon(z)}}\frac{d\varepsilon(z)}{dz} \ll 1, \tag{15}$$

which is the same as $k\sqrt{\varepsilon_l(z)}b_l, k\sqrt{\varepsilon_r(z)}b_r, k\sqrt{\varepsilon(z)}b \gg 1$, and

$$\begin{aligned} U_{l1,2}(z) &= \left(\frac{\varepsilon_l(0)}{\varepsilon_l(z)}\right)^{1/4} e^{\pm iS_l(z)}, \\ U_r(z) &= \left(\frac{\varepsilon_r(0)}{\varepsilon_r(z)}\right)^{1/4} e^{\pm iS_r(z)}, \\ U_{1,2}(z) &= \left(\frac{\varepsilon(0)}{\varepsilon(z)}\right)^{1/4} e^{\pm iS(z)}. \end{aligned} \tag{16}$$

Here, $S_l(z) = k\int_0^z \varepsilon_l^{1/2}(z)dz$, $S_r(z) = k\int_0^z \varepsilon_r^{1/2}(z)dz$, and $S(z) = k\int_0^z \varepsilon^{1/2}(z)dz$. Substituting (16) into (7) and (8) gives for the reflection, $R$, and transmission, $T$, coefficients:

$$R = \frac{R_1 + R_2 e^{2iS}}{1 + R_1 R_2 e^{2iS}}, \quad T = \left(\frac{\varepsilon(0)}{\varepsilon(a)}\right)^{1/4} \frac{T_1 T_2 e^{iS}}{1 + R_1 R_2 e^{2iS}}, \tag{17}$$

where

$$R_1 = \frac{\sqrt{\varepsilon(0)} - \sqrt{\varepsilon_l(0)}}{\sqrt{\varepsilon(0)} + \sqrt{\varepsilon_l(0)}}, \qquad R_2 = \frac{\sqrt{\varepsilon_r(a)} - \sqrt{\varepsilon(a)}}{\sqrt{\varepsilon_r(a)} + \sqrt{\varepsilon(a)}},$$
$$T_1 = \frac{2\sqrt{\varepsilon_l(0)}}{\sqrt{\varepsilon(0)} + \sqrt{\varepsilon_l(0)}}, \qquad T_2 = \frac{2\sqrt{\varepsilon(a)}}{\sqrt{\varepsilon_r(a)} + \sqrt{\varepsilon(a)}}.$$

Note that in the absence of the plate ($a = 0$), the reflection $R_{lr}$ and transmission $T_{lr}$ coefficients are:

$$R_{lr} = \frac{\sqrt{\varepsilon_r(0)} - \sqrt{\varepsilon_l(0)}}{\sqrt{\varepsilon_r(0)} + \sqrt{\varepsilon_l(0)}}, \qquad T_{lr} = \frac{2\sqrt{\varepsilon_l(0)}}{\sqrt{\varepsilon_r(0)} + \sqrt{\varepsilon_l(0)}}. \tag{18}$$

From (13), as well as from (14), the following criterions of transparency can be derived:

$$S(a) = \pi q, \quad \varepsilon(a) = \varepsilon(0), \tag{19}$$

where $q$ is an integer.

We consider some examples of applications for formulas (19).

Let $\varepsilon(z)$ be a periodical function of $z$ with the period $L$, that is, $\varepsilon(z+L) = \varepsilon(z)$. Then, from the second equation of (19), we obtain

$$a = pL, \tag{20}$$

where $p$ is an integer. Decomposing $\sqrt{\varepsilon(z)}$ into its Fourier expansion and isolating the zero term, gives

$$\sqrt{\varepsilon(z)} = \sum_{\alpha=-\infty}^{\infty} \eta_\alpha e^{\frac{2\pi i\alpha}{L}z} = \eta_0 + \sum_{\substack{\alpha=-\infty \\ \alpha\neq 0}}^{\infty} \eta_\alpha e^{\frac{2\pi i\alpha}{L}z}. \tag{21}$$

Substituting this expansion into the first formula of (19), the transparency condition for the plate's thickness is

$$k\eta_0 a = \pi q. \tag{22}$$

With no temporal dispersion, i.e., $\eta_0$ being non-dependent of $\omega$, the result is given by

$$\omega_q = \frac{q\pi c}{\eta_0 a}, \tag{23}$$

and, with account of (20):

$$\omega_q = \omega_{qp} = \frac{\pi c}{\eta_0 L}\frac{q}{p}. \tag{24}$$

It is seen that all the plates with thickness according to (20) and at frequencies defined by (24), are absolutely transparent. The cases of such a kind are infinite in number.

Now, let the function $\sqrt{\varepsilon(z)}$ be polynomial of $N$th power, which represents a general enough and interesting case because, on finite interval $[0, a]$, $\sqrt{\varepsilon(z)}$ can be approximated by a polynomial with any degree of accuracy. It is obvious that this polynomial can be written as

$$\sqrt{\varepsilon(z)} = \sqrt{\varepsilon(0)} + \frac{z}{b}\sum_{\zeta=1}^{N}\eta_\zeta \left(\frac{z}{b}\right)^{\zeta-1}, \tag{25}$$

here $b$ is the distance at which the $\varepsilon(z)$ function substantially changes. It follows from the second condition of (19) that the possible values of thickness $a$ are defined by the equation

$$\sum_{\zeta=1}^{N}\eta_\zeta \left(\frac{a}{b}\right)^{\zeta-1} = 0, \tag{26}$$

and the number of these values is equal to that of positive solutions of (25). It is evident that the possible number of these thicknesses is not more than $N-1$.

Consider some special cases.

a) $N = 1$. Equation (24) has no roots, which means that a plate whose $\sqrt{\varepsilon(z)}$ function changes linearly along the $z$ axis, is always non-transparent.

b) $N = 2$. $a = -b\frac{\eta_2}{\eta_1}$, the plate is transparent if $-\frac{\eta_2}{\eta_1} > 0$.

c) $N > 2$. These cases can be considered by analytically or numerically solving Eq. (25).

In the absence of temporal dispersion, from Eq. (25) an expression for the transparency frequencies is obtained:

$$\omega_{q\alpha} = \frac{q\pi c}{\sqrt{\varepsilon(0)}a_\alpha\left[1 + \frac{1}{\sqrt{\varepsilon(0)}}\sum_{\zeta=1}^{N'}\frac{\eta_\zeta}{\zeta+1}(\frac{a_\alpha}{b})^\zeta\right]}. \tag{27}$$

Here $\alpha$ denotes positive-root's number of equation (25) and $N'$ the number of these roots.

If the medium's dielectric permittivity changes at distances much shorter than the wavelength, then electrodynamic properties of the medium, as it was shown in Ref. [5], are described by a dielectric tensor with components non-dependent on $z$, that is the plate is homogeneous.

Note that all results obtained in this section hold also for the scalar field $\psi$ described by the Helmgoltz equation. The case of oblique incidence can also be considered with a solution in the following form:

$$\psi(x, z) = U(z)e^{ik\cos\theta x},$$

where $\theta$ is the incident angle and $U(z)$ is described, as before, by Eqs. (1), where the following substitutions have to be performed: $\varepsilon_l(z) \to \varepsilon_l(z) - \cos^2\theta$, $\varepsilon(z) \to \varepsilon(z) - \cos^2\theta$, and $\varepsilon_r(z) \to \varepsilon_r(z) - \cos^2\theta$.

We end this section by considering an inhomogeneous plate with an inhomogeneous half-space at each side of the plate, the dielectric permittivity of the latter being a random function of the $z$-coordinate:

$$\varepsilon(z) = \overline{\varepsilon(z)} + \delta\varepsilon(z), \tag{28}$$

the upper bar denotes ensemble averaging. Thus, $\overline{\varepsilon(z)}$ is the average value of the dielectric permittivity and $\delta\varepsilon(z)$ is the fluctuation of the latter. By definition, $\overline{\delta\varepsilon(z)} = 0$. Suppose as usual that $R_1$ and $R_2$ in formulas (17) for $R$ and $T$ can be regarded as determinate values, and the only random value here is the phase $S$:

$$S = k\int_0^a n(z)dz, \tag{29}$$

where $n(z) = \sqrt{\varepsilon(z)}$. Writing the phase as

$$S = \overline{S} + \delta S, \tag{30}$$

with $\overline{S}$ being the average value of the phase and $\delta S$ its fluctuation,

$$\overline{S} = k\int_0^a \overline{n(z)}dz \text{ and } \delta S = k\int_0^a \delta n(z)dz, (\overline{\delta S} = 0). \tag{31}$$

Average square of $\delta S$ is defined by:

$$\overline{\delta S^2} = k^2\int_0^a \overline{\delta n^2(z)} l_n(z)dz, \tag{32}$$

where

$$\overline{\delta n^2(z)} l_n(z) = \int_0^\infty \overline{\delta n(\xi + z)\delta n(z)} d\xi, \tag{33}$$

$l_n(z)$ being the mean correlation coefficient of fluctuations of the refractive-index. If the fluctuations of the dielectric permittivity $\delta\varepsilon$ are small compared to its mean value $\overline{\varepsilon}$, then

$$\delta n = \frac{\delta\varepsilon(z)}{2\sqrt{\overline{\varepsilon(z)}}}, \tag{34}$$

and for $\overline{(\delta S)^2}$, we have

$$\overline{(\delta S)^2} = \frac{k^2}{4} \int_0^a \frac{\overline{\delta\varepsilon^2(z)} l_\varepsilon(z)}{\overline{\varepsilon(z)}} dz. \tag{35}$$

Here $l_\varepsilon(z)$ stands for the effective correlation coefficient of fluctuations of the dielectric permittivity. Note that $l_n(z)$ and $l_\varepsilon(z)$ are of the order of $b$.

We are interested in the mean values of the reflection $R$ and transmission $T$ coefficients, as well as in the average of their modules squared:

$$\begin{aligned} &\overline{R}, = \int R(S)\mathfrak{F}(S)dS, \ \overline{|R|^2} = \int |R(S)|^2 \mathfrak{F}(S)dS; \\ &\overline{T}, = \int T(S)\mathfrak{F}(S)dS, \ \overline{|T|^2} = \int |T(S)|^2 \mathfrak{F}(S)dS. \end{aligned} \tag{36}$$

Consider first a distribution function of the following type:

$$\mathfrak{F}(S) = \begin{cases} \dfrac{1}{2S_0}, & \text{for } \overline{S} - S_0 \leqslant S \leqslant \overline{S} + S_0 \\ 0, & \text{for } S \leqslant \overline{S} - S_0; \ S \geqslant \overline{S} + S_0, \end{cases} \tag{37}$$

where $S_0^2 = 3\overline{|\delta S|^2}$, i.e., the mean-square value, accurate to fixed factor, of the phase incursion. For the sought values, we get

$$\overline{R} = R_1 - \frac{iR_1^2}{4S_0} \ln \frac{1 + R_1 R_2 e^{2i(\overline{S}+S_0)}}{1 + R_1 R_2 e^{2i(\overline{S}-S_0)}}, \tag{38}$$

$$\overline{T} = \frac{\sqrt{\frac{\varepsilon(0)}{\varepsilon(a)}} T_1 T_2}{2S_0\sqrt{R_1 R_2}} \left[ \arctan \sqrt{R_1 R_2} e^{i(\overline{S}+S_0)} - \arctan \sqrt{R_1 R_2} e^{i(\overline{S}-S_0)} \right]. \tag{39}$$

Note that $S_0 = 2\pi q$, $\overline{R} = R$, and $\overline{T} = 0$; corresponding expressions for $\overline{|R|^2}$ and $\overline{|T|^2}$ are

$$\overline{|R|^2} = 1 - \frac{T_1^2 T_2^2}{2S_0\left(1 - R_1^2 R_2^2\right)} \left\{ \arctan\left[\frac{1 - R_1 R_2}{\left(1 + R_1 R_2\right)^2} \tan\left(\overline{S} + S_0\right)\right] - \arctan\left[\frac{1 - R_1 R_2}{\left(1 + R_1 R_2\right)^2} \tan\left(\overline{S} - S_0\right)\right]\right\}, \quad (40)$$

$$\overline{|T|^2} = \frac{\varepsilon(0)}{\varepsilon(a)} \left(1 - \overline{|R|^2}\right). \quad (41)$$

When $S_0 = 2\pi q$, $\overline{|R|^2} = 1$ and $\overline{|T|^2} = 0$, that is, the plate is completely opaque. This result is analogous to that of obtained in [6] for stochastic localization, but unlike the traditional stochastic localization, for which $\overline{|R|^2} = 1$ and $\overline{|T|^2} = 0$ only by $a = \infty$ and the correlation coefficient being small compared to the wave length, here the localization takes place by discrete set of the plate's thicknesses.

It should be mentioned that in Ref. [7], a problem of the electromagnetic wave propagation through a homogeneous plate confined by statistically rough surfaces, has been considered. In the present work, an analogous problem with Gaussian distribution of the wave phase, has been considered in detail.

## 3. EQUATIONS FOR ELECTROMAGNETIC WAVES PROPAGATING THROUGH ANISOTROPIC LAYERED MEDIA

The Maxwell equation system describing the electromagnetic wave propagation through an anisotropic layered medium is as follows:

$$\begin{aligned} &\frac{d^2 E_x}{dz^2} + k_0^2 \left[\varepsilon_{xx}(z)E_x + \varepsilon_{xy}(z)E_y + \varepsilon_{xz}(z)E_z\right] = 0 \\ &\frac{d^2 E_y}{dz^2} + k_0^2 \left[\varepsilon_{yx}(z)E_x + \varepsilon_{yy}(z)E_y + \varepsilon_{yz}(z)E_z\right] = 0 \\ &\varepsilon_{zx}(z)E_x + \varepsilon_{zy}(z)E_y + \varepsilon_{zz}(z)E_z = 0. \end{aligned} \quad (42)$$

Here $E_i(z)$ are components of the electric field, $i = x, y, z$, and $\varepsilon_{ik}(z)$ are components of the dielectric permittivity tensor. Eliminating $E_z$

from the equation system (42), we get

$$\begin{aligned}\frac{d^2E_x}{dz^2}+k^2\left[\xi_{xx}(z)E_x+\xi_{xy}(z)E_y\right]=0,\\ \frac{d^2E_y}{dz^2}+k^2\left[\xi_{yx}(z)E_x+\xi_{yy}(z)E_y\right]=0,\end{aligned}\tag{43}$$

where

$$\begin{aligned}\xi_{xx}&=\frac{\varepsilon_{xx}\varepsilon_{zz}-\varepsilon_{zx}\varepsilon_{xz}}{\varepsilon_{zz}},\ \xi_{xy}=\frac{\varepsilon_{xy}\varepsilon_{zz}-\varepsilon_{zy}\varepsilon_{xz}}{\varepsilon_{zz}},\\ \xi_{yx}&=\frac{\varepsilon_{yx}\varepsilon_{zz}-\varepsilon_{yz}\varepsilon_{zx}}{\varepsilon_{zz}},\ \xi_{yy}=\frac{\varepsilon_{yy}\varepsilon_{zz}-\varepsilon_{zy}\varepsilon_{yz}}{\varepsilon_{zz}}.\end{aligned}\tag{44}$$

Hence, the electromagnetic wave propagation through layered anisotropic medium is described by a system of two second-order equations. The solving of this system is significantly simplified if it can be reduced to two independent equations. For getting the conditions of such a reduction, multiply the second equation of the system (43) by a constant $Q_\pm$, sum it up with the first one, factor out the coefficient by $E_x$, and coefficient by $E_y$ equate with $Q_\pm$, which gives an equation for deriving $Q_\pm$. As a result, we arrive at

$$\frac{d^2E_\pm}{dz^2}+k^2\varepsilon_\pm(z)E_\pm=0,\tag{45}$$

where

$$\begin{aligned}E_\pm&=E_x+Q_\pm E_y,\\ \varepsilon_\pm(z)&=\xi_{xx}(z)+Q_\pm\xi_{yx}(z)\\ &=\frac{\xi_{xx}+\xi_{yy}}{2}\pm\sqrt{\frac{(\xi_{xx}-\xi_{yy})^2}{4}+\xi_{yx}\xi_{xy}},\end{aligned}\tag{46}$$

and

$$Q_\pm=-\frac{\xi_{xx}-\xi_{yy}}{2\xi_{yx}}\pm\sqrt{\left(\frac{\xi_{xx}-\xi_{yy}}{2\xi_{yx}}\right)^2+\frac{\xi_{xy}}{\xi_{yx}}}.\tag{47}$$

For the values $Q_\pm$ to be constant, it is sufficient that the following combinations of the $\xi$-tensor components would be constant:

$$\frac{\xi_{yy}-\xi_{xx}}{\xi_{yx}}=Q_++Q_-,\ \frac{\xi_{xy}}{\xi_{yx}}=-Q_+Q_-.\tag{48}$$

Hereinafter we shall assume the conditions (48) are satisfied and that the equation system (43) can be represented in the form of (45).

Consider some important special cases. Let the diagonal components $\xi_{xx}$ and $\xi_{yy}$ of the $\xi$-tensor be equal. Then $Q_+ + Q_- = 0$, from which follows

$$Q_+ = -Q_-, \; \frac{\xi_{xy}}{\xi_{yx}} = Q_\pm^2. \tag{49}$$

If the $\xi$-tensor is symmetric, $\xi_{xy} = \xi_{yx}$, then $E_\pm = E_x \pm E_y$, $n_\pm^2 = \xi_{xx} \pm \xi_{yx}$, and $Q_\pm = \pm 1$. If this tensor is antisymmetric, $\xi_{xy} = -\xi_{yx}$, then $E_\pm = E_x \pm iE_y$, $n_\pm^2 = \xi_{xx} \pm i\xi_{yx}$, and $Q_\pm = \pm i$, refractive index $n_\pm(z) = \sqrt{\varepsilon_\pm(z)}$ being defined by (46).

The case of $\xi_{xy} = \xi_{yx} = 0$ is also of importance. Then Eqs. (43) can be separated into two independent equations

$$\frac{d^2 E_x}{dz^2} + k^2 \xi_{xx} E_x = 0 \tag{50}$$

and

$$\frac{d^2 E_y}{dz^2} + k^2 \xi_{yy} E_y = 0. \tag{51}$$

Now, substituting the expressions for the components of the tensor $\xi$ from (43) into (48) gives

$$\frac{(\varepsilon_{yy} - \varepsilon_{xx})\,\varepsilon_z + \varepsilon_{xz}\varepsilon_{zx} - \varepsilon_{zy}\varepsilon_{yz}}{\varepsilon_{zz}} = Q_+ + Q_-, \; \frac{\varepsilon_{xy}\varepsilon_{zz} - \varepsilon_{zy}\varepsilon_{xz}}{\varepsilon_{yx}\varepsilon_{zz} - \varepsilon_{zx}\varepsilon_{yz}} = -Q_+ Q_-, \tag{52}$$

that is, in order the three-equation system (42) to be split into two independent equations, the following condition must be fulfilled: two combinations composed from the nine components of the $\varepsilon_{ik}$ tensor have to be constants. A wide enough class of the dielectric permittivity tensors meet this requirement.

$E_x$ and $E_y$, as well as $D_x$ and $D_y$, can be expressed through $E_\pm$:

$$\begin{aligned} E_x &= \frac{Q_- E_+ - Q_+ E_-}{Q_- - Q_+}, \; E_y = \frac{E_- - E_+}{Q_- - Q_+}; \\ D_x &= \xi_{xx} E_x + \xi_{xy} E_y, \; D_y = \xi_{yx} E_x + \xi_{yy} E_y. \end{aligned} \tag{53}$$

## 4. FARADAY EFFECT IN INHOMOGENEOUS ANISOTROPIC MEDIA

To begin with, consider electromagnetic waves propagating through unbounded inhomogeneous anisotropic media. Through this whole section we assume that the electromagnetic field in a medium is described by Eqs. (45), that is, the components of the $\xi$ tensor satisfy conditions (48). Exact solution of Eqs. (45) can be obtained for a wide enough class of the $\varepsilon_{\pm}(z)$ functions (cf., f.e., [2]). We will not dwell on these solutions and content ourselves with results obtained using the WKB approach. Such a choice is favoured by the fact that by small values of $z$, that is, when one can assume $\varepsilon_{\pm}(z) = \varepsilon_{\pm}(0)$, then $E_{\pm} = e^{ik\sqrt{\varepsilon_{\pm}(0)}z}$ is derived from the formulas for the fields obtained by the WKB method, see (16). For large values of $z$, the WKB approach is an asymptotically exact one. Thus, for small and large values of $z$, the WKB method gives correct results. We expect that also for intermediate values of $z$, the fields obtained by the WKB method are good approximations.

Consider first the Faraday effect in unbounded medium. It is known [5], that this effect is characterized by the relation $D_y/D_x$ describing the rotation of the polarization plane of a wave propagating in an anisotropic medium:

$$\frac{D_y}{D_x} = \frac{\xi_{yx}\left(Q_- E_+ - Q_+ E_-\right) + \xi_{yy}\left(E_- - E_+\right)}{\xi_{xx}\left(Q_- E_+ - Q_+ E_-\right) + \xi_{xy}\left(E_- - E_+\right)}. \tag{54}$$

$E_{\pm}$ being calculated in the WKB approximation, we obtain

$$E_{\pm} = \sqrt{\frac{\varepsilon_{\pm}(0)}{\varepsilon_{\pm}(z)}}e^{iS_{\pm}}, \; S_{\pm} = \int_0^z \sqrt{\varepsilon_{\pm}(z)}dz, \tag{55}$$

and formula (54) turns into

$$\frac{D_y}{D_x} = \frac{\left(\xi_{yx}Q_- - \xi_{yy}\right)\sqrt[4]{\frac{\varepsilon_+(0)}{\varepsilon_+(z)}}e^{iS_+} - \left(\xi_{yx}Q_+ - \xi_{yy}\right)\sqrt[4]{\frac{\varepsilon_-(0)}{\varepsilon_-(z)}}e^{iS_-}}{\left(\xi_{xx}Q_- - \xi_{xy}\right)\sqrt[4]{\frac{\varepsilon_+(0)}{\varepsilon_+(z)}}e^{iS_+} - \left(\xi_{xx}Q_+ - \xi_{xy}\right)\sqrt[4]{\frac{\varepsilon_-(0)}{\varepsilon_-(z)}}e^{iS_-}}. \tag{56}$$

With the aid of expression (56), we explore the rotation of the polarization plane of a wave propagating in a gyrotropic medium. A medium is called gyrotropic if the electrical induction vector is related to the electrical field vector by the relation [5]

$$\vec{D} = \varepsilon\vec{E} + i\left[\vec{E}, \vec{g}\right], \tag{57}$$

where $\vec{g}$ is the gyration vector, $\varepsilon$ and $\vec{g}$ being slowly changing functions of $z$.

We will assume the $\vec{g}$-vector lies in the $xz$-plane, i.e., having only components $g_x$ and $g_z$. We assume also that the $\vec{g}$-vector is small in the sense that $|g| \ll \varepsilon$ and values of $g^2$ can be neglected. Using (46) and (47), and the smallness of $g$, we get

$$\frac{D_y}{D_x} = \tan\left(\frac{k_0}{2}\int_0^z \frac{g(z)}{\sqrt{\varepsilon(z)}}dz\right). \tag{58}$$

With $g_x$ and $g_z$ non-dependent on $z$, this formula turns into the well-known expression for the rotation of the polarization plane of the field in gyrotropic homogeneous medium.

Now, let the inhomogeneous medium be consisting of an inhomogeneous anisotropic plate with an inhomogeneous half-space on each of sides. Then, the rotation of the polarization plane for the reflected and transmitted fields, respectively are

$$\left(\frac{D_y}{D_x}\right)_R = \frac{\xi_{yx}(0)\left(Q_-R_+ - Q_+R_-\right) + \xi_{yy}(0)\left(R_- - R_+\right)}{\xi_{xx}(0)\left(Q_-R_+ - Q_+R_-\right) + \xi_{xy}(0)\left(R_- - R_+\right)} \tag{59}$$

and

$$\left(\frac{D_y}{D_x}\right)_T = \frac{\xi_{yx}(a)\left(Q_-T_+ - Q_+T_-\right) + \xi_{yy}(a)\left(T_- - T_+\right)}{\xi_{xx}(a)\left(Q_-T_+ - Q_+T_-\right) + \xi_{xy}(a)\left(T_- - T_+\right)}. \tag{60}$$

In the general case, $R_\pm$ and $T_\pm$ are determined by formulas (7) and (8), respectively, and using the WKB approach, by (17).

If the inhomogeneous plate is gyrotropic, one has to considerate two cases: a totally transparent plate and a non-transparent plate. The results are presented with components of the $\xi_{ik}-$tensor expressed in terms of the $\vec{g}$-vector, using Eq. (57). For the transparent plate, omitting cumbersome elementary calculations, we get:

$$\left(\frac{D_y}{D_x}\right)_R = \frac{R_2\left(1-R_1^2\right)e^{2iS_a}\sin\sigma}{R_1 + \left(1+R_1^2\right)R_2 e^{2iS_a}\cos\sigma + R_1R_2^2e^{4iS_a}}, \tag{61}$$

$$\left(\frac{D_y}{D_x}\right)_T = \frac{\left(1-R_1R_2\right)e^{2iS_a}\tan\sigma}{1+R_1R_2e^{2iS_a}}, \tag{62}$$

where $R_{1,2}$ are the same as in (17), $S_a = k_0\int_0^a \sqrt{\varepsilon_\pm(z)}dz$, and $\sigma = \frac{k_0}{2}\int_0^z \frac{g(z)}{\sqrt{\varepsilon(z)}}dz$.

Analysis and simplification of formulas (61) and (62) are analogous to the ones carried out in Ref. [1].

The transparency conditions for both waves, as follows from (19), are:

$$S_+(a) = \pi q_+,\ \varepsilon_+(a) = \varepsilon_+(0),\ S_-(a) = \pi q_-,\ \varepsilon_-(a) = \varepsilon_-(0), \qquad (63)$$

where $q_+$ and $q_-$ are integers, which in general do not coincide. The solutions defining the plate's thicknesses and corresponding frequencies at which the plate is transparent for the reflected wave do not coincide with those for the transmitted wave. However, with small gyrotropy ($g \ll \varepsilon$) and a narrow enough incident wave impulse [1], the plate can be transparent for both waves. The gyrotropy results in a frequency shift that can be determined from Eqs. (63), for which aim $q_+ = q_- = q$ has to be put. If the gyrotropy is small (the criterion is given above), then Eqs. (63) can be solved using the convergence method, which gives the frequency shift

$$\delta\omega = \delta\omega_{+q} = -\delta\omega_{-q} = k\frac{g(0) - g(a)}{\frac{d\varepsilon(a)}{da}} - \frac{\frac{k}{2}\int_0^{a_0} \frac{g(z)}{\sqrt{\varepsilon(z)}}dz}{\int_0^{a_0} \frac{dz}{v_g(z)}}. \qquad (64)$$

Here $\frac{1}{v_g(z)} = \frac{d}{d\omega}k\sqrt{\varepsilon(z)}$, $k$ and $a_0$ are determined from (19).

Note that for homogeneous plates, the first item in (64) is absent.

For the fields $E_{R\pm}$ and $E_{T\pm}$ we have

$$\begin{aligned} E_{R\pm} &= R_{lr}e^{-i[\omega_q t + S_l]}e^{\pm i\delta\omega\left[t + \int_0^z \frac{dz}{v_{gl}(z)}\right]}, \\ E_{T\pm} &= T_{lr}e^{-i[\omega_q t - S_r]}e^{\pm i\delta\omega\left[t - \int_0^z \frac{dz}{v_{gr}(z)} \pm\varphi\right]} \end{aligned} \qquad (65)$$

with

$$\varphi = k\frac{g(a) - g(0)}{d\varepsilon/da}.$$

And, finally, the rotation of the polarization plane of the reflected and transmitted fields are, respectively,

$$\begin{aligned} \left(\frac{D_y}{D_x}\right)_R &= \tan\delta\omega\left[t + \int_0^z \frac{dz}{v_{gl}(z)}\right], \\ \left(\frac{D_y}{D_x}\right)_T &= \tan\delta\omega\left[t - \int_0^z \frac{dz}{v_{gr}(z)} + \varphi\right]. \end{aligned} \qquad (66)$$

One can see that here, like in the case of homogeneous plate [1], the rotation of the polarization plane has both spatial and temporal components.

We will not dwell on the situation when $\xi_{xy} = \xi_{yx}$, because for this case Eqs. (50), (51) are of the same structure as Eqs. (45), which allows to carry out corresponding analysis in an analogous way.

## 5. CONCLUSIONS

As a summing up, the Fresnel formulas are generalized for the reflection coefficients of an isotropic plate the dielectric permittivity of which, as well as the dielectric permittivities of enclosing half-spaces, are arbitrary functions of the coordinate, either determinate or stochastic. For the general case, the conditions for the plate's full transparency are formulated. As an example, a case is considered when the plate's dielectric permittivity is a slowly changing function of the coordinate, which justifies application of the WKB method. When the plate's dielectric permittivity is a stochastic function of the coordinate, a new effect appears: the full non-transparency (opacity) for a discrete set of the plate's thicknesses. For an anisotropic plate, sufficient conditions are formulated for the two- equation Maxwell system to be split into two independent equations, which simplifies substantially the calculations. All these results can be applied immediately for working out the needed parameters of the Fabry-Perot interferometers, filters, resonators.

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