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ELECTROSTATIC IMAGE THEORY FOR AN ANISOTROPIC BOUNDARY OF AN ANISOTROPIC HALF-SPACE

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Abstract—A novel image solution for the canonical electrostatic problem of a point charge in an anisotropic half-space bounded by an anisotropic surface is presented. The image source is obtained in operator form by using Fourier-transformed Maxwell equations and transmission line theory. After applying methods from Heaviside operator calculus, the image operator can be interpreted as a combination of a point charge and a line-charge-bounded sector of planar charge density. The new theory is shown to coincide with the previously known image solutions of less general anisotropic media. In addition to being applicable to any physically feasible anisotropic medium of electrostatics, the method can be used for steady-current conductivity problems via a duality transformation.

1. INTRODUCTION

The image principle is a general field-theoretical method that can be used to solve boundary-value problems by turning a difficult procedure — construction of a Green function — into a simple field calculation from sources in a geometry whose Green function is know beforehand. The classical example of the image method is the determination of electrostatic potential when a point charge is in front of an ideally conducting infinite plane. The "reflected" part of the total potential is obtained by placing a negative image charge to the mirror image point behind the plane, removing the plane, and using the free-space Green function to calculate the potential arising from the image source. Though the potential can be computed everywhere it is physically meaningful only in the half-space of the original source, this being a typical limitation of the image method.

Easing the field calculations is not the only benefit of the image method, however. Perhaps an even more remarkable but far too often overlooked feature of the method is that it gives insight into field theory: by inspecting the obtained image source it is possible to develop mental pictures and deeper understanding of the physics of the problem at hand. And this, in turn, makes it possible to derive alternative presentations or approximations for the solution.

In this paper we tackle one of the so far (to the authors' knowledge, at least) unsolved canonical image problems of electrostatics, namely the reflection image theory for a point charge in an anisotropic half-space bounded by an anisotropic surface. Over the years limited special solutions have been worked out, for example, for an anisotropic half-space over a perfectly electrically or magnetically conducting surface [1], for an isotropic half-space bounded by an anisotropic surface [2], and for a similarly anisotropic half-space and boundary [3]. Our proposed general solution, in contrast, will work with all physically justified medium parameters, and the known cases are built in the new theory, as will be shown.

It seems that a good part of the literature concerning static anisotropic problems deals with geophysics and current-based models (see, e.g., [4–9]). We nevertheless chose to develop the theory with electrostatic formulation due to its familiarity. The electrostatic and steady current approaches are related via the traditional duality transformations, so no applicability is lost.

The construction of the image starts, of course, from the Maxwell equations, the (dyadic) constitutive relation, and the boundary condition. These are spatially Fourier-transformed to obtain a transmission-line model, which, in turn, is used to write out the solution in a reflection coefficient-source formulation. Instead of inverse Fourier-transforming the intermediate result directly, we use Heaviside operator calculus to get the reflection image in operator form. The operator is then cast into a differential equation, whose solution finally is the "physical" image charge density. The paper ends with a comparison of the new solution with the known ones.

2. CONSTRUCTING THE EQUATIONS

2.1. Fields in an Anisotropic Medium

The Maxwell equations of electrostatics read

$$\nabla \times \bar{\mathbf{E}}(\bar{\mathbf{r}}) = \bar{\mathbf{0}}, \tag{1}$$

$$\nabla \cdot \bar{\mathbf{D}}(\bar{\mathbf{r}}) = \varrho(\bar{\mathbf{r}}), \tag{2}$$

$\varrho(\bar{\mathbf{r}})$ being the source of the fields. The constitutive relation of an anisotropic dielectric medium is

$$\bar{\mathbf{D}}(\bar{\mathbf{r}}) = \epsilon_0 \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\mathbf{E}}(\bar{\mathbf{r}}). \tag{3}$$

The most general relative permittivity dyadic $\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}$ of electrostatics is real, symmetric, and positive definite [10, §13 and §21], and it can be expressed in the right-handed basis formed from its orthonormal eigenvectors $\bar{\mathbf{u}}_u$, $\bar{\mathbf{u}}_v$, $\bar{\mathbf{u}}_w$, as

$$\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} = \bar{\mathbf{u}}_u \bar{\mathbf{u}}_u \epsilon_u + \bar{\mathbf{u}}_v \bar{\mathbf{u}}_v \epsilon_v + \bar{\mathbf{u}}_w \bar{\mathbf{u}}_w \epsilon_w. \tag{4}$$

The eigenvalues ϵ_u, ϵ_v, ϵ_w of $\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}$ are positive and, depending on the medium, there may be one, two, or three distinct values.

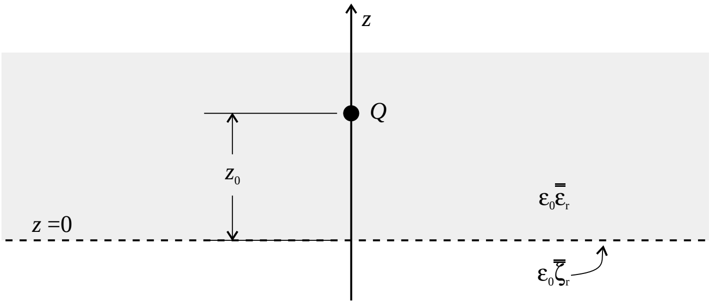

Figure 1. Geometry of the problem. The grey part depicts the anisotropic half-space.

The impedance boundary condition of the anisotropic surface ($z = 0$) of the anisotropic half-space ($z > 0$) is

$$\bar{\mathbf{n}} \cdot \bar{\mathbf{D}}(\bar{\boldsymbol{\rho}}, 0) = -\nabla_{\mathrm{t}} \cdot \bar{\bar{\boldsymbol{\Psi}}}_{\mathrm{s}}(\bar{\boldsymbol{\rho}}, 0) = -\epsilon_0 \nabla_{\mathrm{t}} \cdot \left(\bar{\bar{\boldsymbol{\zeta}}}_{\mathrm{r}} \cdot \bar{\mathbf{E}}_{\mathrm{t}}(\bar{\boldsymbol{\rho}}, 0)\right). \tag{5}$$

The surface impedance dyadic $\bar{\bar{\boldsymbol{\zeta}}}_{\mathrm{r}}$ is symmetric and positive definite (in the two-dimensional sense, see Appendix A2), and its dimension is length (i.e. $[\bar{\bar{\boldsymbol{\zeta}}}_{\mathrm{r}}] = \mathrm{m})^{\dagger}$. Furthermore,

$$\bar{\boldsymbol{\rho}} = \bar{\mathbf{u}}_x x + \bar{\mathbf{u}}_y y \tag{6}$$

is the transverse position vector, and subscript 't' denotes 'transverse' with respect to z. The quantity $\bar{\bar{\boldsymbol{\Psi}}}_{\mathrm{s}}(\bar{\boldsymbol{\rho}})$ is *the electric surface flux density*, $[\bar{\bar{\boldsymbol{\Psi}}}_{\mathrm{s}}] = \mathrm{As/m}$. The surface normal $\bar{\mathbf{n}}$ of the anisotropic boundary coincides with $\bar{\mathbf{u}}_z$.

2.2. The Fourier Transformation

The configuration of the problem is shown in Figure 1. In order to obtain the necessary transmission-line equations, we apply the two-dimensional Fourier transformation

$$f(\bar{\mathbf{K}}, z) = \int_{S_\rho} e^{j\bar{\mathbf{K}}\cdot\bar{\boldsymbol{\rho}}} f(\bar{\mathbf{r}}) dS_\rho \leftrightarrow f(\bar{\mathbf{r}}) = \frac{1}{(2\pi)^2} \int_{S_K} e^{-j\bar{\mathbf{K}}\cdot\bar{\boldsymbol{\rho}}} f(\bar{\mathbf{K}}, z) dS_K \tag{7}$$

† The quantity $\epsilon_0 \bar{\bar{\boldsymbol{\zeta}}}_{\mathrm{r}}$ should be called 'surface admittance', but the word 'impedance' is used by convention of the authors.

on every plane $z =$ constant to the relevant Maxwell equations, yielding

$$\begin{aligned} \nabla \times \bar{\mathbf{E}}(\bar{\mathbf{r}}) = (\nabla_{\mathrm{t}} + \bar{\mathbf{u}}_z\partial_z) \times \bar{\mathbf{E}}(\bar{\mathbf{r}}) &= \bar{\mathbf{0}} \\ \leftrightarrow -j\bar{\mathbf{K}} \times \bar{\mathbf{E}}(z) + \bar{\mathbf{u}}_z \times \partial_z\bar{\mathbf{E}}(z) &= \bar{\mathbf{0}}, \qquad (8) \\ \nabla \cdot \bar{\mathbf{D}}(\bar{\mathbf{r}}) = (\nabla_{\mathrm{t}} + \bar{\mathbf{u}}_z\partial_z) \cdot \bar{\mathbf{D}}(\bar{\mathbf{r}}) &= \varrho(\bar{\mathbf{r}}) \\ \leftrightarrow -j\bar{\mathbf{K}} \cdot \bar{\mathbf{D}}(z) + \bar{\mathbf{u}}_z \cdot \partial_z\bar{\mathbf{D}}(z) &= \varrho(z). \qquad (9) \end{aligned}$$

Here we have adopted the compact notation $f(z) \equiv f(\bar{\mathbf{K}}, z)$ for all Fourier-transformed quantities; also $\partial_z f \equiv \partial f/\partial z$. The transverse Fourier parameter vector

$$\bar{\mathbf{K}} = \bar{\mathbf{u}}_x K_x + \bar{\mathbf{u}}_y K_y \quad \text{and} \quad |\bar{\mathbf{K}}| = K. \qquad (10)$$

It is worth noting that the Fourier transformation multiplies the dimension of a quantity by a factor of length squared (or area), e.g., $[\bar{\mathbf{D}}(\bar{\mathbf{r}})] = \mathrm{As/m}^2$ but $[\bar{\mathbf{D}}(z)] = \mathrm{As}$.

Next we use $(\bar{\mathbf{u}}_z, \bar{\mathbf{K}}/K, \bar{\mathbf{u}}_z \times \bar{\mathbf{K}}/K)$ as our basis in the Fourier space and write out the transformed Maxwell equations in this basis. We are striving for equations compatible with the electric boundary conditions: the continuity of transversal (w.r.t. z) electric field and the z-component of $\bar{\mathbf{D}}$-field. Multiplying (8) by $\bar{\mathbf{u}}_z \cdot [\ ]$ (or by $\bar{\mathbf{K}} \cdot [\ ]$, the result will be the same) and by $(\bar{\mathbf{u}}_z \times \bar{\mathbf{K}}) \cdot [\ ]$, we obtain the two equations

$$-j\bar{\mathbf{u}}_z \cdot \bar{\mathbf{K}} \times \bar{\mathbf{E}}(z) + \bar{\mathbf{u}}_z \cdot \bar{\mathbf{u}}_z \times \partial_z\bar{\mathbf{E}}(z) = 0 \Rightarrow \bar{\mathbf{u}}_z \times \bar{\mathbf{K}} \cdot \bar{\mathbf{E}}(z) = 0 \qquad (11)$$

and

$$\begin{aligned} & -j(\bar{\mathbf{u}}_z \times \bar{\mathbf{K}}) \cdot (\bar{\mathbf{K}} \times \bar{\mathbf{E}}(z)) + (\bar{\mathbf{u}}_z \times \bar{\mathbf{K}}) \cdot (\bar{\mathbf{u}}_z \times \partial_z\bar{\mathbf{E}}(z)) \\ = & -j\bar{\mathbf{u}}_z \cdot \bar{\mathbf{K}} \times (\bar{\mathbf{K}} \times \bar{\mathbf{E}}(z)) + \bar{\mathbf{u}}_z \cdot \bar{\mathbf{K}} \times (\bar{\mathbf{u}}_z \times \partial_z\bar{\mathbf{E}}(z)) \\ = & \; jK^2\bar{\mathbf{u}}_z \cdot \bar{\mathbf{E}}(z) + \partial_z(\bar{\mathbf{K}} \cdot \bar{\mathbf{E}}(z)) = 0. \qquad (12) \end{aligned}$$

From (11) we see that the electric field has only two components and can thus be written as

$$\bar{\mathbf{E}}(z) = \left(\bar{\mathbf{K}}/K\right)\left(\bar{\mathbf{K}}/K\right) \cdot \bar{\mathbf{E}}(z) + \bar{\mathbf{u}}_z\bar{\mathbf{u}}_z \cdot \bar{\mathbf{E}}(z) = \bar{\mathbf{u}}_K E_K(z) + \bar{\mathbf{u}}_z E_z(z) \qquad (13)$$

if we denote $E_K(z) = (\bar{\mathbf{K}}/K) \cdot \bar{\mathbf{E}}(z) = \bar{\mathbf{u}}_K \cdot \bar{\mathbf{E}}(z)$ and $E_z(z) = \bar{\mathbf{u}}_z \cdot \bar{\mathbf{E}}(z)$. Now (12), in turn, gives a relation of the two components, namely

$$\partial_z E_K(z) = -jKE_z(z). \qquad (14)$$

Next we turn our attention to the $\bar{\mathbf{D}}(z)$-field. From the constitutive relation (3) we get

$$\begin{aligned} D_z(z) &= \bar{\mathbf{u}}_z \cdot \bar{\mathbf{D}}(z) = \bar{\mathbf{u}}_z \cdot \left(\epsilon_0 \bar{\bar{\boldsymbol{\epsilon}}}_r \cdot \bar{\mathbf{E}}(z)\right) \\ &= \epsilon_0 \bar{\mathbf{u}}_z \cdot [\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot (\bar{\mathbf{u}}_K E_K(z) + \bar{\mathbf{u}}_z E_z(z))] \qquad (15) \end{aligned}$$

whence

$$
\begin{aligned}
E_z(z) &= \frac{1}{\epsilon_0 \bar{\mathbf{u}}_z \cdot \bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r} \cdot \bar{\mathbf{u}}_z} \left[D_z(z) - \epsilon_0 \bar{\mathbf{u}}_z \cdot \bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r} \cdot \bar{\bar{\mathbf{I}}}_\mathrm{t} \cdot \bar{\mathbf{u}}_K E_K(z) \right] \\
&= \frac{1}{\epsilon_0 \epsilon_z} D_z(z) - \bar{\mathbf{a}} \cdot \bar{\mathbf{u}}_K E_K(z) \qquad (16)
\end{aligned}
$$

with

$$
\begin{aligned}
\epsilon_z &= \bar{\mathbf{u}}_z \cdot \bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r} \cdot \bar{\mathbf{u}}_z \quad \text{and} \qquad (17) \\
\bar{\mathbf{a}} &= \frac{1}{\epsilon_z} \bar{\mathbf{u}}_z \cdot \bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r} \cdot \bar{\bar{\mathbf{I}}}_\mathrm{t}. \qquad (18)
\end{aligned}
$$

Now (14) takes the form

$$
\begin{aligned}
&\partial_z E_K(z) + jK \left(\frac{1}{\epsilon_0 \epsilon_z} D_z(z) - \bar{\mathbf{a}} \cdot \bar{\mathbf{u}}_K E_K(z) \right) \\
&= \left(\partial_z - j \bar{\mathbf{a}} \cdot \bar{\mathbf{K}} \right) E_K(z) + \frac{jK}{\epsilon_0 \epsilon_z} D_z(z) = 0. \qquad (19)
\end{aligned}
$$

Vector $\bar{\mathbf{a}}$ is perpendicular to $\bar{\mathbf{u}}_z$, i.e., $\bar{\mathbf{a}} \cdot \bar{\mathbf{u}}_z = \bar{\mathbf{u}}_z \cdot \bar{\mathbf{a}} = 0$. The transverse unit dyadic $\bar{\bar{\mathbf{I}}}_\mathrm{t} = \bar{\bar{\mathbf{I}}} - \bar{\mathbf{u}}_z \bar{\mathbf{u}}_z$.

The K-component of $\bar{\mathbf{D}}(z)$ is

$$
\begin{aligned}
D_K(z) &= \bar{\mathbf{u}}_K \cdot \bar{\mathbf{D}}(z) = \epsilon_0 \bar{\mathbf{u}}_K \cdot \left(\bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r} \cdot \bar{\mathbf{E}}(z) \right) \\
&= \bar{\mathbf{u}}_K \cdot \left[\epsilon_0 \bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r} \cdot \left(\bar{\mathbf{u}}_K E_K(z) + \bar{\mathbf{u}}_z E_z(z) \right) \right] \\
&= \epsilon_0 \Bigg[\bar{\mathbf{u}}_K \cdot \bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r} \cdot \bar{\mathbf{u}}_K E_K(z) + \bar{\mathbf{u}}_K \cdot \bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r} \cdot \bar{\mathbf{u}}_z \\
&\qquad \cdot \left(\frac{1}{\epsilon_0 \epsilon_z} D_z(z) - \bar{\mathbf{a}} \cdot \bar{\mathbf{u}}_K E_K(z) \right) \Bigg]. \qquad (20)
\end{aligned}
$$

Applying this result to (9) we yield the equation

$$
\begin{aligned}
\partial_z D_z(z) - j \bar{\mathbf{K}} \cdot \bar{\mathbf{D}}(z) &= \partial_z D_z(z) - jK \bar{\mathbf{u}}_k \cdot \bar{\mathbf{D}}(z) \\
&= (\partial_z - j \bar{\mathbf{a}} \cdot \bar{\mathbf{K}}) D_z(z) - \frac{j\epsilon_0}{\epsilon_z K} \bar{\mathbf{K}} \cdot \bar{\bar{\mathbf{T}}} \cdot \bar{\mathbf{K}} E_K(z) \\
&= \varrho(z). \qquad (21)
\end{aligned}
$$

We will study in Appendix A2 the properties of the two-dimensional dyadic

$$
\bar{\bar{\mathbf{T}}} = \epsilon_z \bar{\bar{\mathbf{I}}}_\mathrm{t} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r} \cdot \bar{\bar{\mathbf{I}}}_\mathrm{t} - \epsilon_z^2 \bar{\mathbf{a}} \bar{\mathbf{a}}. \qquad (22)
$$

Finally, the boundary condition at $z = 0$ Fourier-transforms as

$$\begin{aligned}\bar{\mathbf{u}}_z \cdot \bar{\mathbf{D}}(\bar{\boldsymbol{\rho}}, 0) &= -\epsilon_0 \nabla_{\rm t} \cdot \left(\bar{\bar{\boldsymbol{\zeta}}}_{\rm r} \cdot \bar{\mathbf{E}}_{\rm t}(\bar{\boldsymbol{\rho}}, 0)\right) \leftrightarrow \\ D_z(0) &= j\epsilon_0 \bar{\mathbf{K}} \cdot \bar{\bar{\boldsymbol{\zeta}}}_{\rm r} \cdot \bar{\mathbf{E}}_{\rm t}(0) = \frac{j\epsilon_0}{K} \bar{\mathbf{K}} \cdot \bar{\bar{\boldsymbol{\zeta}}}_{\rm r} \cdot \bar{\mathbf{K}} E_K(0). \end{aligned} \quad (23)$$

2.3. The Transmission-Line Equations

After these steps we have obtained the pair of Equations (19) and (21), and the boundary condition ("loading impedance") (23), which we next cast into a form encountered in transmission-line theory. Writing (19) and (21) in matrix form

$$\begin{pmatrix} (\partial_z - j\bar{\mathbf{a}} \cdot \bar{\mathbf{K}}) & \dfrac{jK}{\epsilon_0 \epsilon_z} \\ -\dfrac{j\epsilon_0}{\epsilon_z K} \bar{\mathbf{K}} \cdot \bar{\bar{\mathbf{T}}} \cdot \bar{\mathbf{K}} & (\partial_z - j\bar{\mathbf{a}} \cdot \bar{\mathbf{K}}) \end{pmatrix} \begin{pmatrix} E_K(z) \\ D_z(z) \end{pmatrix} = \begin{pmatrix} 0 \\ \varrho(z) \end{pmatrix} \quad (24)$$

and multiplying the equation by a nonzero number $e^{-j\bar{\mathbf{a}} \cdot \bar{\mathbf{K}} z}$ we get

$$\begin{pmatrix} \partial_z & \dfrac{jK}{\epsilon_0 \epsilon_z} \\ -\dfrac{j\epsilon_0}{\epsilon_z K} \bar{\mathbf{K}} \cdot \bar{\bar{\mathbf{T}}} \cdot \bar{\mathbf{K}} & \partial_z \end{pmatrix} \begin{pmatrix} E_K(z) e^{-j\bar{\mathbf{a}} \cdot \bar{\mathbf{K}} z} \\ D_z(z) e^{-j\bar{\mathbf{a}} \cdot \bar{\mathbf{K}} z} \end{pmatrix} = \begin{pmatrix} 0 \\ \varrho(z) e^{-j\bar{\mathbf{a}} \cdot \bar{\mathbf{K}} z} \end{pmatrix}, \quad (25)$$

which resembles the standard transmission-line equation

$$\begin{pmatrix} \partial_z & \gamma Z \\ \gamma/Z & \partial_z \end{pmatrix} \begin{pmatrix} U(z) \\ I(z) \end{pmatrix} = \begin{pmatrix} u(z) \\ i(z) \end{pmatrix}. \quad (26)$$

The formal solution of this equation is

$$\begin{pmatrix} U(z) \\ I(z) \end{pmatrix} = \frac{1}{\partial_z^2 - \gamma^2} \begin{pmatrix} \partial_z & -\gamma Z \\ -\gamma/Z & \partial_z \end{pmatrix} \begin{pmatrix} u(z) \\ i(z) \end{pmatrix}, \quad (27)$$

and the corresponding telegraphy equation is

$$(\partial_z^2 - \gamma^2) \begin{pmatrix} U(z) \\ I(z) \end{pmatrix} = \begin{pmatrix} \partial_z & -\gamma Z \\ -\gamma/Z & \partial_z \end{pmatrix} \begin{pmatrix} u(z) \\ i(z) \end{pmatrix}. \quad (28)$$

Especially, if we define the (Fourier-transformed) voltage and current quantities

$$U(z) = -jKE_K(z) e^{-j\bar{\mathbf{a}} \cdot \bar{\mathbf{K}} z}, \quad [U(z)] = V, \quad \text{and} \quad (29)$$

$$I(z) = c_0 K D_z(z) e^{-j\bar{\mathbf{a}} \cdot \bar{\mathbf{K}} z}, \quad [I(z)] = A, \quad (30)$$

we can identify the transmission-line parameters propagation factor

$$\gamma = \frac{1}{\epsilon_z}\sqrt{\bar{\mathbf{K}}\cdot\bar{\bar{\mathbf{T}}}\cdot\bar{\mathbf{K}}}, \quad [\gamma] = \mathrm{rad/m}, \tag{31}$$

and characteristic impedance

$$Z = \frac{K\eta_0}{\sqrt{\bar{\mathbf{K}}\cdot\bar{\bar{\mathbf{T}}}\cdot\bar{\mathbf{K}}}} = \frac{K\eta_0}{\epsilon_z\gamma}, \quad [Z] = \Omega, \tag{32}$$

and the distributed sources

$$u(z) = 0, \qquad [u(z)] = \mathrm{V/m}, \tag{33}$$

$$i(z) = c_0 K\varrho(z)e^{-j\bar{\mathbf{a}}\cdot\bar{\mathbf{K}}z}, \quad [i(z)] = \mathrm{A/m}. \tag{34}$$

As usual, $c_0 = 1/\sqrt{\mu_0\epsilon_0}$ and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$. It is easy to see that the Fourier-transformed potential $\phi(z)$ is related to the voltage $U(z)$ as

$$\phi(z) = \frac{1}{K^2}U(z)e^{+j\bar{\mathbf{a}}\cdot\bar{\mathbf{K}}z}, \quad [\phi(z)] = \mathrm{Vm}^2. \tag{35}$$

Naturally $\bar{\mathbf{E}}(\bar{\mathbf{r}}) = -\nabla\phi(\bar{\mathbf{r}})$.

The boundary condition (23) will be represented by the loading impedance

$$Z_{\mathrm{L}} = -\frac{U(0)}{I(0)} = \frac{jE_K(0)}{c_0 D_z(0)} = \frac{K\eta_0}{\bar{\mathbf{K}}\cdot\bar{\bar{\zeta}}_{\mathrm{r}}\cdot\bar{\mathbf{K}}}. \tag{36}$$

The minus sign in front of the definition of Z_{L} is due to the choice of the positive direction of voltage and current on the transmission line.

3. IMAGE THEORY

3.1. The Image Sources

Let us consider a combination of voltage and current point sources with amplitudes U_0 and I_0, respectively, at $z = z_0$ on the transmission line. This basic source can be expressed as

$$\begin{pmatrix} u^{\mathrm{i}}(z) \\ i^{\mathrm{i}}(z) \end{pmatrix} = \begin{pmatrix} U_0 \\ I_0 \end{pmatrix}\delta(z - z_0), \tag{37}$$

and for this source the solution of (28) is

$$\begin{pmatrix} U(z) \\ I(z) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathrm{sgn}(z-z_0) & Z \\ 1/Z & \mathrm{sgn}(z-z_0) \end{pmatrix}\begin{pmatrix} U_0 \\ I_0 \end{pmatrix}e^{-\gamma|z-z_0|}. \tag{38}$$

From now on we require $z_0 > 0$, so the incident waves between $0 < z < z_0$ will be

$$\begin{pmatrix} U^{\rm i}(z) \\ I^{\rm i}(z) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & Z \\ 1/Z & -1 \end{pmatrix} \begin{pmatrix} U_0 \\ I_0 \end{pmatrix} e^{-\gamma|z-z_0|} = \begin{pmatrix} U^{\rm i}(0) \\ I^{\rm i}(0) \end{pmatrix} e^{+\gamma z}. \quad (39)$$

Because γ is purely real, we do not actually have propagating waves but exponentially decaying direct current and voltage as we recede from the source.

We next terminate the transmission line with the loading impedance $Z_{\rm L}$ at $z = 0$. From circuit theory we get the reflection coefficient R for the voltage wave in the form

$$R = \frac{Z_{\rm L} - Z}{Z_{\rm L} + Z} = \frac{1/Z - 1/Z_{\rm L}}{1/Z + 1/Z_{\rm L}}. \quad (40)$$

For the current wave the reflection coefficient is $-R$; this is due to the previously mentioned current-voltage sign convention. Because the reflection coefficient is a function of γ and $\bar{\mathbf{K}}$, we introduce the notation $R = R(\gamma, \bar{\mathbf{K}})$.

The reflected waves in $z > 0$ can thus be written as

$$\begin{aligned} \begin{pmatrix} U^{\rm r}(z) \\ I^{\rm r}(z) \end{pmatrix} &= \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \begin{pmatrix} U^{\rm i}(0) \\ I^{\rm i}(0) \end{pmatrix} e^{-\gamma z} \\ &= \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \begin{pmatrix} U^{\rm i}(-z) \\ I^{\rm i}(-z) \end{pmatrix}. \end{aligned} \quad (41)$$

For the purpose of the inverse Fourier transformation the reflection coefficient could be written as a Taylor series in powers of γ. If we substituted such a series to the middle term in the expression above, we would see that in every term of the series $\gamma^n e^{-\gamma z} = (-\partial_z)^n e^{-\gamma z}$. Thus we write

$$R(\gamma, \bar{\mathbf{K}}) e^{-\gamma z} = R(-\partial_z, \bar{\mathbf{K}}) e^{-\gamma z} \quad (42)$$

turning R into an operator. Then, noting the Fourier transformation $\nabla_{\rm t} \leftrightarrow -j\bar{\mathbf{K}}$, we further write

$$R(-\partial_z, \bar{\mathbf{K}}) \leftrightarrow R(-\partial_z, j\nabla_{\rm t}). \quad (43)$$

This is the formal inverse Fourier transformation of $R(-\partial_z, \bar{\mathbf{K}})$.

To obtain the sources of the reflected waves, we remove the loading impedance, extend the $z > 0$ transmission line to $z < 0$, and insert (41) to (26), yielding

$$\begin{aligned}
\begin{pmatrix} u^{\mathrm{r}}(z) \\ i^{\mathrm{r}}(z) \end{pmatrix} &= \begin{pmatrix} \partial_z & \gamma Z \\ \gamma/Z & \partial_z \end{pmatrix} \begin{pmatrix} U^{\mathrm{r}}(z) \\ I^{\mathrm{r}}(z) \end{pmatrix} \\
&= \begin{pmatrix} \partial_z & \gamma Z \\ \gamma/Z & \partial_z \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \begin{pmatrix} U^{\mathrm{i}}(-z) \\ I^{\mathrm{i}}(-z) \end{pmatrix} \\
&= \begin{pmatrix} -R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} -\partial_z & \gamma Z \\ \gamma/Z & -\partial_z \end{pmatrix} \begin{pmatrix} U^{\mathrm{i}}(-z) \\ I^{\mathrm{i}}(-z) \end{pmatrix} \\
&= \begin{pmatrix} -R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \partial_{z'} & \gamma Z \\ \gamma/Z & \partial_{z'} \end{pmatrix} \begin{pmatrix} U^{\mathrm{i}}(z') \\ I^{\mathrm{i}}(z') \end{pmatrix} \Bigg|_{z'=-z} \\
&= \begin{pmatrix} -R(-\partial_z, \bar{\mathbf{K}}) & 0 \\ 0 & R(-\partial_z, \bar{\mathbf{K}}) \end{pmatrix} \begin{pmatrix} u^{\mathrm{i}}(-z) \\ i^{\mathrm{i}}(-z) \end{pmatrix}. \qquad (44)
\end{aligned}$$

Extending the $z > 0$ transmission line to $z < 0$ means, of course, filling the lower (transmission- side) physical half-space with the anisotropic medium of the upper half-space.

3.2. The Image of a Point Charge

Now we are ready to return to our specific problem. Our charge density

$$\varrho(\bar{\mathbf{r}}) = Q\delta(\bar{\mathbf{r}} - \bar{\mathbf{u}}_z z_0), \quad z_0 > 0, \qquad (45)$$

representing a point charge in the anisotropic half-space. The distributed sources are, as per (33) and (34),

$$\begin{aligned}
u^{\mathrm{i}}(z) &= 0, & (46)\\
i^{\mathrm{i}}(z) &= Qc_0 K e^{-j\bar{\mathbf{a}}\cdot\bar{\mathbf{K}}z}\delta(z - z_0), & (47)
\end{aligned}$$

and it is straightforward to read the reflection sources from (44):

$$\begin{aligned}
\begin{pmatrix} u^{\mathrm{r}}(z) \\ i^{\mathrm{r}}(z) \end{pmatrix} &= \begin{pmatrix} -R(-\partial_z, \bar{\mathbf{K}}) u^{\mathrm{i}}(-z) \\ R(-\partial_z, \bar{\mathbf{K}}) i^{\mathrm{i}}(-z) \end{pmatrix} \\
&= \begin{pmatrix} -R(-\partial_z, \bar{\mathbf{K}}) \cdot 0 \\ Qc_0 K R(-\partial_z, \bar{\mathbf{K}}) e^{+j\bar{\mathbf{a}}\cdot\bar{\mathbf{K}}z}\delta(-z - z_0) \end{pmatrix}. \qquad (48)
\end{aligned}$$

So, we only have a current source

$$i^{\mathrm{r}}(z) = Qc_0 K R(-\partial_z, \bar{\mathbf{K}}) e^{-j\bar{\mathbf{a}}\cdot\bar{\mathbf{K}}z_0}\delta(z + z_0). \qquad (49)$$

The symmetry property $\delta(-x) = \delta(x)$ of the delta function has been used.

In the Fourier space the image charge corresponding to the original source (45) is thus

$$\begin{aligned} &\varrho^{\mathrm{r}}(z) \\ =\ & \frac{1}{c_0 K e^{-j\bar{\mathbf{a}}\cdot\bar{\mathbf{K}}z}} i^{\mathrm{r}}(z) = \frac{Q c_0 K}{c_0 K e^{-j\bar{\mathbf{a}}\cdot\bar{\mathbf{K}}z}} R(-\partial_z, \bar{\mathbf{K}}) e^{-j\bar{\mathbf{a}}\cdot\bar{\mathbf{K}}z_0} \delta(z+z_0) \\ =\ & Q e^{j\bar{\mathbf{a}}\cdot\bar{\mathbf{K}}(z-z_0)} R(-\partial_z, \bar{\mathbf{K}}) \delta(z+z_0), \end{aligned} \tag{50}$$

and the image charge in the anisotropic space is the inverse Fourier transformation of (50), being in operator form

$$\varrho^{\mathrm{r}}(\bar{\mathbf{r}}) = Q e^{-(z-z_0)\bar{\mathbf{a}}\cdot\nabla_{\mathrm{t}}} R(-\partial_z, j\nabla_{\mathrm{t}}) \delta(\bar{\boldsymbol{\rho}}) \delta(z+z_0). \tag{51}$$

The remaining task is to interpret this expression. The easy part is the exponential operator — it just shifts any function laterally (i.e., in the xy-plane):

$$e^{(z_0-z)\bar{\mathbf{a}}\cdot\nabla_{\mathrm{t}}} f(\bar{\boldsymbol{\rho}}) = f\left(\bar{\boldsymbol{\rho}} + \bar{\mathbf{a}}(z_0 - z)\right). \tag{52}$$

To interpret the reflection operator we shall study the reflection coefficient.

4. THE IMAGE PROPER

4.1. The Reflection Operator

We first define the transformation

$$\bar{\mathbf{K}}' = \frac{1}{\epsilon_z} \bar{\bar{\mathbf{T}}}^{1/2} \cdot \bar{\mathbf{K}}. \tag{53}$$

Taking the square root is permitted on the basis of what is said about $\bar{\bar{\mathbf{T}}}$ in Appendix A2. This changes (31) and (36) to read

$$\gamma = \frac{1}{\epsilon_z} \sqrt{\bar{\mathbf{K}} \cdot \bar{\bar{\mathbf{T}}} \cdot \bar{\mathbf{K}}} = \frac{1}{\epsilon_z} \sqrt{\bar{\mathbf{K}} \cdot \bar{\bar{\mathbf{T}}}^{1/2} \cdot \bar{\bar{\mathbf{T}}}^{1/2}} = \sqrt{\bar{\mathbf{K}}' \cdot \bar{\mathbf{K}}'} = K' \quad \text{and} \tag{54}$$

$$Z_{\mathrm{L}} = \frac{K\eta_0}{\bar{\mathbf{K}} \cdot \bar{\bar{\boldsymbol{\zeta}}}_{\mathrm{r}} \cdot \bar{\mathbf{K}}} = \frac{K\eta_0}{\epsilon_z^2 \bar{\mathbf{K}}' \cdot \bar{\bar{\mathbf{T}}}^{-1/2} \cdot \bar{\bar{\boldsymbol{\zeta}}}_{\mathrm{r}} \cdot \bar{\bar{\mathbf{T}}}^{-1/2} \cdot \bar{\mathbf{K}}'} = \frac{K\eta_0}{\epsilon_z \bar{\mathbf{K}}' \cdot \bar{\bar{\boldsymbol{\zeta}}}'_{\mathrm{r}} \cdot \bar{\mathbf{K}}'}, \tag{55}$$

respectively; with this transformation we turn the upper half-space isotropic. Here

$$\bar{\bar{\boldsymbol{\zeta}}}'_{\mathrm{r}} = \epsilon_z \bar{\bar{\mathbf{T}}}^{-1/2} \cdot \bar{\bar{\boldsymbol{\zeta}}}_{\mathrm{r}} \cdot \bar{\bar{\mathbf{T}}}^{-1/2} = \bar{\mathbf{u}}'_x \bar{\mathbf{u}}'_x \zeta'_x + \bar{\mathbf{u}}'_y \bar{\mathbf{u}}'_y \zeta'_y, \tag{56}$$

because all important properties of $\overline{\overline{\mathbf{T}}}$ and $\overline{\overline{\boldsymbol{\zeta}}}_{\rm r}$, namely the symmetry and positive-definiteness, are carried over to $\overline{\overline{\boldsymbol{\zeta}}}'_{\rm r}$. (For the details of $\overline{\overline{\boldsymbol{\zeta}}}'_{\rm r}$ see Appendix A3.) We have chosen direction $\bar{\mathbf{u}}'_x$ to correspond to the eigenvector of the larger eigenvalue of $\overline{\overline{\boldsymbol{\zeta}}}'_{\rm r}$. We next define

$$\zeta'_{\rm d} = \zeta'_x - \zeta'_y, \tag{57}$$

making $\zeta'_{\rm d} \geq 0$, and get

$$\begin{aligned}\bar{\mathbf{K}}'\cdot\overline{\overline{\boldsymbol{\zeta}}}'_{\rm r}\cdot\bar{\mathbf{K}}' &= (\bar{\mathbf{u}}_x K'_x+\bar{\mathbf{u}}'_y K'_y)\cdot(\bar{\mathbf{u}}'_x\bar{\mathbf{u}}'_x(\zeta'_y+\zeta'_{\rm d})+\bar{\mathbf{u}}'_y\bar{\mathbf{u}}'_y\zeta'_y)\cdot(\bar{\mathbf{u}}'_x K'_x+\bar{\mathbf{u}}'_y K'_y)\\ &= {K'_x}^2(\zeta'_y+\zeta'_{\rm d}) + {K'_y}^2\zeta'_y\\ &= \zeta'_y K'^2 + \zeta'_{\rm d}{K'_x}^2 = \zeta'_y\gamma^2 + \zeta'_{\rm d}{K'_x}^2.\end{aligned} \tag{58}$$

Substituting (32) and (36) into (40), we write

$$\begin{aligned} R(\gamma,\bar{\mathbf{K}}) &= \frac{\dfrac{K\eta_0}{\bar{\mathbf{K}}\cdot\overline{\overline{\boldsymbol{\zeta}}}_{\rm r}\cdot\bar{\mathbf{K}}} - \dfrac{K\eta_0}{\epsilon_z\gamma}}{\dfrac{K\eta_0}{\bar{\mathbf{K}}\cdot\overline{\overline{\boldsymbol{\zeta}}}_{\rm r}\cdot\bar{\mathbf{K}}} + \dfrac{K\eta_0}{\epsilon_z\gamma}} = \frac{\epsilon_z\gamma - \bar{\mathbf{K}}\cdot\overline{\overline{\boldsymbol{\zeta}}}_{\rm r}\cdot\bar{\mathbf{K}}}{\epsilon_z\gamma + \bar{\mathbf{K}}\cdot\overline{\overline{\boldsymbol{\zeta}}}_{\rm r}\cdot\bar{\mathbf{K}}}\\ &= -1 + \frac{2\epsilon_z\gamma}{\epsilon_z\gamma + \epsilon_z\bar{\mathbf{K}}'\cdot\overline{\overline{\boldsymbol{\zeta}}}'_{\rm r}\cdot\bar{\mathbf{K}}'}\\ &= -1 + \frac{2\gamma}{\gamma + \zeta'_y\gamma^2 + \zeta'_{\rm d}{K'_x}^2}.\end{aligned} \tag{59}$$

Before proceeding it is good to elaborate the $\zeta'_{\rm d}{K'_x}^2$ term a bit. In the inverse Fourier transformation we integrate over the $\bar{\mathbf{K}}$-plane, but here we have an expression containing $\bar{\mathbf{K}}'$. However, we see that

$$\begin{aligned}\zeta'_{\rm d}{K'_x}^2 &= \zeta'_{\rm d}(\bar{\mathbf{u}}'_x\cdot\bar{\mathbf{K}}')^2 = \zeta'_{\rm d}(\bar{\mathbf{u}}'_x\cdot\overline{\overline{\mathbf{T}}}^{-1/2}\cdot\bar{\mathbf{K}}/\epsilon_z)^2\\ &= \zeta''_{\rm d}(\bar{\mathbf{u}}''_x\cdot\bar{\mathbf{K}})^2 \leftrightarrow \zeta''_{\rm d}(j\bar{\mathbf{u}}''_x\cdot\nabla_{\rm t})^2 = -\zeta''_{\rm d}\partial^2_{x''}\end{aligned} \tag{60}$$

with

$$\zeta''_{\rm d} = (\zeta'_{\rm d}/\epsilon_z^2)\bar{\mathbf{u}}'_x\cdot\overline{\overline{\mathbf{T}}}\cdot\bar{\mathbf{u}}'_x \quad\text{and}\quad \bar{\mathbf{u}}''_x = \frac{\bar{\mathbf{u}}'_x\cdot\overline{\overline{\mathbf{T}}}^{1/2}}{\sqrt{\bar{\mathbf{u}}'_x\cdot\overline{\overline{\mathbf{T}}}\cdot\bar{\mathbf{u}}'_x}}. \tag{61}$$

The image expression in operational form is now

$$\begin{aligned}\varrho^r(\bar{\mathbf{r}}) &= Qe^{(z_0-z)\bar{\mathbf{a}}\cdot\nabla_{\rm t}}\left[-\delta(\bar{\boldsymbol{\rho}})\delta(z+z_0)+\frac{-2\partial}{-\partial_z+\zeta_y'\partial_z^2-\zeta_{\rm d}''\partial_{x''}^2}\delta(\bar{\boldsymbol{\rho}})\delta(z+z_0)\right]\\ &= -Q\delta\left(\bar{\boldsymbol{\rho}}+\bar{\mathbf{a}}(z_0-z)\right)\delta(z+z_0)+Qe^{(z_0-z)\bar{\mathbf{a}}\cdot(\bar{\mathbf{u}}_x''\partial_{x''}+\bar{\mathbf{u}}_y''\partial_{y''})}\\ &\quad\times\frac{-2\partial_z}{-\partial_z+\zeta_y'\partial_z^2-\zeta_{\rm d}''\partial_{x''}^2}\delta(x'')\delta(y'')\delta(z+z_0)\\ &= -Q\delta(x''+s_x'')\delta(y''+s_y'')\delta(z+z_0)\\ &\quad+2Q\delta\left(y''+a_y''(z_0-z)\right)\left(e^{(z_0-z)\bar{\mathbf{a}}\cdot\bar{\mathbf{u}}_x''\partial_{x''}}\left[\partial_z F_{\rm p}(x'',z)\right]\right)\end{aligned} \quad (62)$$

with

$$\bar{\mathbf{s}} = \bar{\mathbf{a}}2z_0, \quad s_x'' = \bar{\mathbf{u}}_x''\cdot\bar{\mathbf{s}} = 2z_0(\bar{\mathbf{u}}_x''\cdot\bar{\mathbf{a}}) = 2z_0a_x'',$$

$$\text{and} \quad s_y'' = \bar{\mathbf{u}}_y''\cdot\bar{\mathbf{s}} = 2z_0(\bar{\mathbf{u}}_y''\cdot\bar{\mathbf{a}}) = 2z_0a_y''; \quad (63)$$

$$\bar{\mathbf{u}}_y'' = \bar{\mathbf{u}}_z\times\bar{\mathbf{u}}_x'', \quad x'' = \bar{\mathbf{u}}_x''\cdot\bar{\mathbf{r}} \quad \text{and} \quad y'' = \bar{\mathbf{u}}_y''\cdot\bar{\mathbf{r}}. \quad (64)$$

So, our image consists of a point charge in the laterally shifted mirror image point and of a planar surface charge, the possibly slanted plane being $y'' = -a_y''(z_0-z)$.

4.2. The Planar Image Function

The planar image is contained in the function

$$F_{\rm p}(x'',z) = \frac{1}{\partial_z-\zeta_y'\partial_z^2+\zeta_{\rm d}''\partial_{x''}^2}\delta(x'')\delta(z+z_0), \quad (65)$$

which is similar to the function $F(x,z)$ encountered in [2] when solving the problem of a point charge in an isotropic half-space over an anisotropic impedance surface. Therefore it is not necessary to repeat the solution process of the resulting Klein-Gordon type equation — it is sufficient to compare the corresponding equation here and in the reference, and to write out the solution for $z+z_0\le 0$ immediately:

$$\begin{aligned}F_{\rm p}(x'',z) &= -\frac{e^{(z+z_0)/(2\zeta_y')}}{2\sqrt{\zeta_y'\zeta_{\rm d}''}}I_0\left(\sqrt{\frac{(z+z_0)^2}{4{\zeta_y'}^2}-\frac{x''^2}{4\zeta_y'\zeta_{\rm d}''}}\right)\\ &\quad\times\Theta\left(-\sqrt{\zeta_{\rm d}''/\zeta_y'}(z+z_0)-x''\right)\Theta\left(-\sqrt{\zeta_{\rm d}''/\zeta_y'}(z+z_0)+x''\right).\end{aligned} \quad (66)$$

The solution was required to vanish in the region $z + z_0 > 0$. Here $I_0(\xi)$ is the zeroth-order modified Bessel function.

The Heaviside unit step functions $\Theta(\xi)$ above implicate that the solution is discontinuous along the lines

$$x'' = \pm\sqrt{\zeta''_{\rm d}/\zeta'_y}(z + z_0) = \pm g(z), \tag{67}$$

and that the solution vanishes outside the sector $|x''| \leq -g(z)$.

4.3. The Solution

To take into account the last exponential operator, $e^{(z_0-z)\bar{\mathbf{a}}\cdot\bar{\mathbf{u}}''_x\partial_{x''}}$ (an x''-shift), we need to export it inside the ∂_z-differentiation. The planar part will read

$$\begin{aligned} & e^{(z_0-z)\bar{\mathbf{a}}\cdot\bar{\mathbf{u}}''_x\partial_{x''}}\left[\partial_z F_{\rm p}(x'', z)\right] \\ = \ & \partial_z\left[e^{(z_0-z)\bar{\mathbf{a}}\cdot\bar{\mathbf{u}}''_x\partial_{x''}} F_{\rm p}(x'', z)\right] - \partial_z\left[e^{(z_0-z)\bar{\mathbf{a}}\cdot\bar{\mathbf{u}}''_x\partial_{x''}}\right] F_{\rm p}(x'', z) \\ = \ & \partial_z\left[F_{\rm p}\left(x'' + a''_x(z_0 - z), z\right)\right] + a''_x\partial_{x''}\left[F_{\rm p}\left(x'' + a''_x(z_0 - z), z\right)\right]. \end{aligned} \tag{68}$$

We have finally arrived to the solution of our problem. The reflection image source is

$$\begin{aligned} \varrho^{\rm r}(\bar{\mathbf{r}}) = \ & -Q\delta(\bar{\mathbf{r}} + \bar{\mathbf{s}} + \bar{\mathbf{u}}_z z_0) - \frac{Q}{\sqrt{\zeta'_y\zeta''_{\rm d}}}\delta(y'' + a''_y(z_0 - z)) \\ & \times\left\{\partial_z\left[\exp\left(\frac{z+z_0}{2\zeta'_y}\right) I_0\left(\sqrt{\frac{(z+z_0)^2}{4{\zeta'_y}^2} - \frac{\xi^2}{4\zeta'_y\zeta''_{\rm d}}}\right)\right.\right. \\ & \left.\left.\times\Theta(-g(z) - \xi)\Theta(-g(z) + \xi)\right]\right\}_{\xi = x'' + a''_x(z_0 - z)} \\ = \ & -Q\delta(\bar{\mathbf{r}} + \bar{\mathbf{s}} + \bar{\mathbf{u}}_z z_0) \\ & -\frac{Q}{\sqrt{\zeta'_y\zeta''_{\rm d}}}\delta(y'' + a''_y(z_0 - z))(\partial_z + a''_x\partial_{x''}) \\ & \cdot\left[\exp\left(\frac{z+z_0}{2\zeta'_y}\right) I_0\left(\sqrt{\frac{(z+z_0)^2}{4{\zeta'_y}^2} - \frac{[x'' + a''_x(z_0 - z)]^2}{4\zeta'_y\zeta''_{\rm d}}}\right)\right. \\ & \times\Theta\left(-\sqrt{\zeta''_{\rm d}/\zeta'_y}(z + z_0) - x'' - a''_x(z_0 - z)\right) \end{aligned} \tag{69}$$

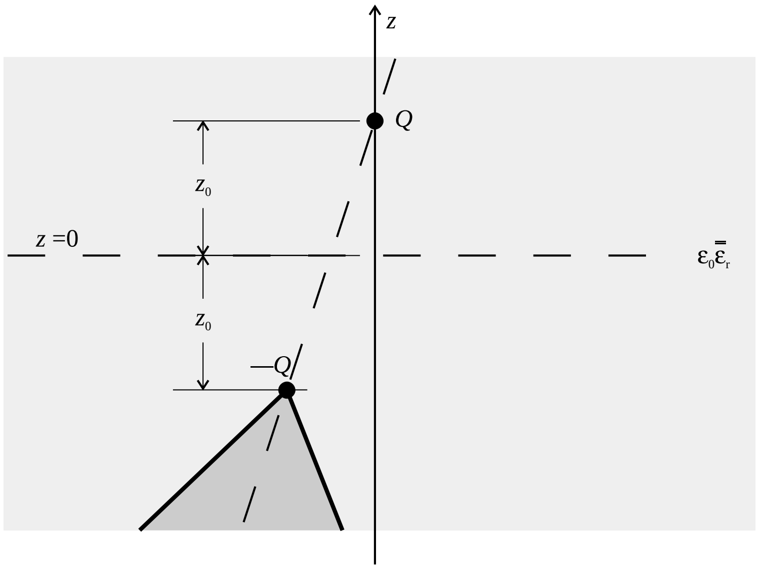

Figure 2. Geometry of the image charge in the anisotropic space. The planar part (darker grey) is laterally symmetric with respect to its centerline (the dashed oblique line). The plane of the image is generally not the plane of the Figure nor parallel with it.

$$\times\Theta\left(-\sqrt{\zeta_{\rm d}''/\zeta_y'}(z+z_0)+x''+a_x''(z_0-z)\right)\Bigg]. \qquad (70)$$

We have retained the differentiation ∂_z because it can be eliminated in a partial integration when calculating the fields from the electrostatic Green function of anisotropic space.

To summarize, we have a point charge in the laterally shifted mirror image point and a slanted, laterally skewed planar sector of surface charge, the tip of which is in the shifted mirror image point, Figure 2. As has been shown in [2], the planar part has concentrated line charges on its edges (due to the differentiation $\partial_z\Theta(z)$), and between the edges there is a smooth distribution of surface charge. Alternative expressions for the surface charge have been given in the same article.

Figures 3 and 4 sketch the charge density, excluding the line charges. A circle denotes the shifted mirror image point, i.e. the tip of the charge sector. The chosen parameters are shown with Figure 3; we have used the final, doubly primed coordinate system because it is difficult to construct media which give "nice" parameter values in the

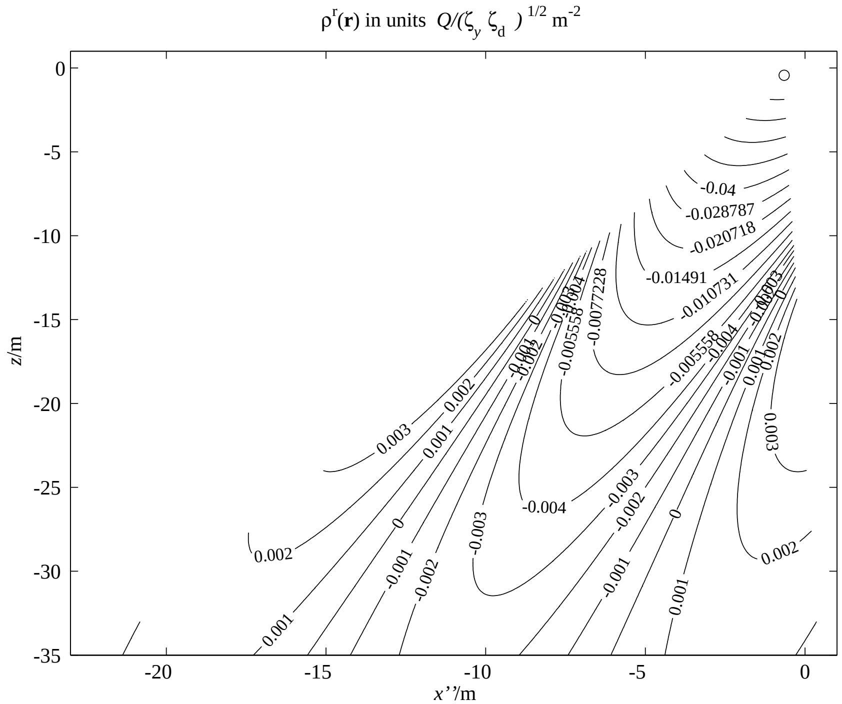

Figure 3. Equicharge lines of the surface charge for $\zeta'_y = 2.7$, $\zeta''_{\mathrm{d}} = 0.3$, $a''_x = 0.3$, $a''_y = 0.2$ and $z_0 = 1.1$. The circle is at the shifted mirror image point. The line charges are excluded.

final coordinates. The two coordinate transformations just stretch the image laterally and rotate it around the z-axis, so no generality is lost in the current way of representation.

To calculate the fields the electrostatic Green function for a homogeneous anisotropic medium is needed. For example, the reflected potential is (the primes now denoting the source region)

$$\phi^{\mathrm{r}}(\bar{\mathbf{r}}) = \int_{V'} \frac{\varrho^{\mathrm{r}}(\bar{\mathbf{r}}')}{4\pi\epsilon_0\sqrt{\det\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}}\sqrt{(\bar{\mathbf{r}}-\bar{\mathbf{r}}')\cdot\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{-1}\cdot(\bar{\mathbf{r}}-\bar{\mathbf{r}}')}}dV'. \tag{71}$$

Although this is computable everywhere (outside the source, at least), it has physical significance only in the region $z \geq 0$.

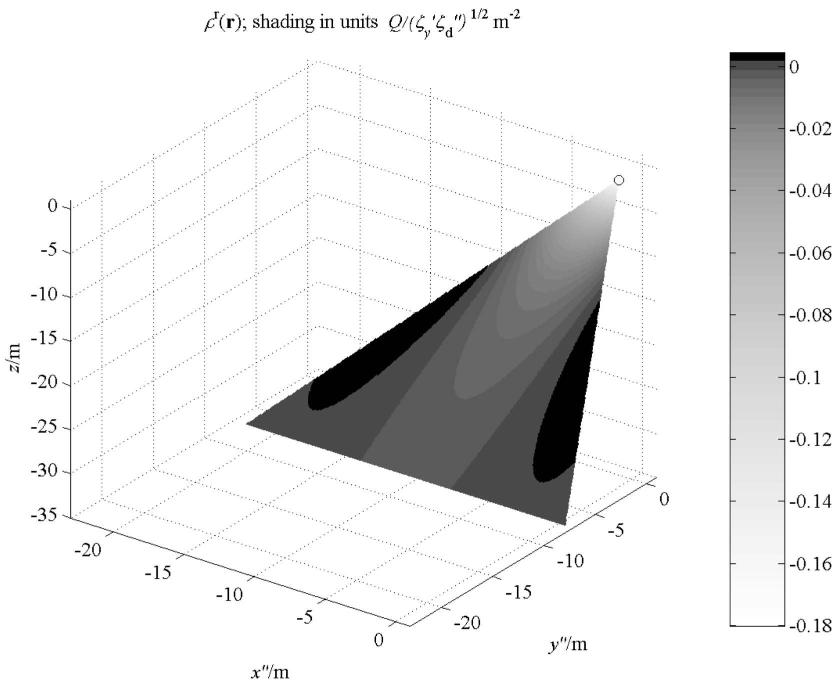

Figure 4. The location and orientation of the surface charge for the parameters of Figure 3. The circle is at the shifted mirror image point. The shading corresponds the charge density.

5. VERIFYING THE RESULT

5.1. An Isotropic Half-Space with an Anisotropic Boundary

As a first test we consider an isotropic half-space:

$$\bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r} = \epsilon_\mathrm{r}\bar{\bar{\mathbf{I}}} = \epsilon_\mathrm{r}(\bar{\bar{\mathbf{I}}}_\mathrm{t} + \bar{\mathbf{u}}_z\bar{\mathbf{u}}_z). \tag{72}$$

It is easy to see that $\epsilon_z = \epsilon_\mathrm{r}$, $\bar{\mathbf{a}} = \bar{\mathbf{0}}$, and $\bar{\bar{\mathbf{T}}} = \epsilon_\mathrm{r}^2\bar{\bar{\mathbf{I}}}_\mathrm{t}$. Because $\bar{\bar{\mathbf{T}}}$ is isotropic in the xy-plane, we can choose $\bar{\bar{\boldsymbol{\zeta}}}_\mathrm{r} = \bar{\mathbf{u}}_x\bar{\mathbf{u}}_x\zeta_x + \bar{\mathbf{u}}_y\bar{\mathbf{u}}_y\zeta_y$, $\zeta_x \geq \zeta_y$, without loss of generality. This leads to

$$\bar{\bar{\boldsymbol{\zeta}}}_\mathrm{r} = \epsilon_z\bar{\bar{\mathbf{T}}}^{-1/2} \cdot \bar{\bar{\boldsymbol{\zeta}}}_\mathrm{r} \cdot \bar{\bar{\mathbf{T}}}^{-1/2} = \epsilon_\mathrm{r}(\epsilon_\mathrm{r}^{-1}\bar{\bar{\mathbf{I}}}_\mathrm{t}) \cdot \bar{\bar{\boldsymbol{\zeta}}}_\mathrm{r} \cdot (\epsilon_\mathrm{r}^{-1}\bar{\bar{\mathbf{I}}}_\mathrm{t}) = \bar{\bar{\boldsymbol{\zeta}}}_\mathrm{r}/\epsilon_\mathrm{r}. \tag{73}$$

So, the unprimed and primed coordinate systems coincide, and $\zeta'_{x,y} = \zeta_{x,y}/\epsilon_\mathrm{r}$. Furthermore,

$$\begin{aligned} \zeta''_\mathrm{d} &= \left[(\zeta_x - \zeta_y)/\epsilon_\mathrm{r}^3\right] \bar{\mathbf{u}}_x \cdot \bar{\bar{\mathbf{T}}} \cdot \bar{\mathbf{u}}_x = \zeta_\mathrm{d}/\epsilon_\mathrm{r} \\ \text{and} \qquad \bar{\mathbf{u}}''_x &= \frac{\bar{\mathbf{u}}_x \cdot (\epsilon_\mathrm{r} \bar{\bar{\mathbf{I}}}_\mathrm{t})}{\sqrt{\bar{\mathbf{u}}_x \cdot (\epsilon_\mathrm{r}^2 \bar{\bar{\mathbf{I}}}_\mathrm{t}) \cdot \bar{\mathbf{u}}_x}} = \bar{\mathbf{u}}_x. \end{aligned} \tag{74}$$

This shows that the doubly primed coordinate system coincides with the unprimed system, too. Substituting these values to (70), the image source becomes

$$\begin{aligned} \varrho^\mathrm{r}(\bar{\mathbf{r}}) &= -Q\delta(\bar{\mathbf{r}} + \bar{\mathbf{u}}_z z_0) - \frac{Q\epsilon_\mathrm{r}}{\sqrt{\zeta_y \zeta_\mathrm{d}}} \delta(y)\partial_z \\ &\quad \cdot \left[\exp\left(\frac{\epsilon_\mathrm{r}(z+z_0)}{2\zeta_y} \right) I_0 \left(\epsilon_\mathrm{r} \sqrt{\frac{(z+z_0)^2}{4{\zeta_y}^2} - \frac{x^2}{4\zeta_y \zeta_\mathrm{d}}} \right) \right. \\ &\quad \left. \times \Theta\left(-\sqrt{\zeta_\mathrm{d}/\zeta_y}(z+z_0) - x \right) \Theta\left(-\sqrt{\zeta_\mathrm{d}/\zeta_y}(z+z_0) + x \right) \right]. \end{aligned} \tag{75}$$

This agrees with the previous result in [2] when $\epsilon_\mathrm{r} = 1$.

5.2. Similar Anisotropy of the Half-Space and the Boundary

In this context 'similar anisotropy' means that the modified surface impedance $\bar{\bar{\boldsymbol{\zeta}}}'_\mathrm{r}$ is isotropic, i.e.,

$$\begin{aligned} \bar{\bar{\boldsymbol{\zeta}}}'_\mathrm{r} = \zeta'_\mathrm{r} \bar{\bar{\mathbf{I}}}_\mathrm{t} = \epsilon_z \bar{\bar{\mathbf{T}}}^{-1/2} \cdot \bar{\bar{\boldsymbol{\zeta}}}_\mathrm{r} \cdot \bar{\bar{\mathbf{T}}}^{-1/2} \quad &\Leftrightarrow \quad \bar{\bar{\boldsymbol{\zeta}}}_\mathrm{r} = \frac{\zeta'_\mathrm{r}}{\epsilon_z} \bar{\bar{\mathbf{T}}} \\ \Leftrightarrow \quad \bar{\bar{\boldsymbol{\zeta}}}_\mathrm{r}^{-1} = \frac{\epsilon_z}{\zeta'_\mathrm{r}} \bar{\bar{\mathbf{T}}}^{-1} = \frac{1}{\zeta'_\mathrm{r}} \bar{\bar{\mathbf{I}}}_\mathrm{t} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r}^{-1} \cdot \bar{\bar{\mathbf{I}}}_\mathrm{t}. \end{aligned} \tag{76}$$

In this case $\zeta'_\mathrm{d} = \zeta''_\mathrm{d} = 0$ because $\zeta'_\mathrm{r} \equiv \zeta'_x \equiv \zeta'_y$. We look back to the reflection coefficient (59) and assume $0 < \zeta'_y < \infty$. The reflection coefficient is now

$$R(\gamma, \bar{\mathbf{K}}) = -1 + \frac{2}{1 + \zeta'_y \gamma}, \tag{77}$$

and the operational-form image source is

$$\varrho_0^\mathrm{r}(\bar{\mathbf{r}}) = Q\left(e^{(z_0 - z)\bar{\mathbf{a}} \cdot \nabla_\mathrm{t}} \delta(\bar{\boldsymbol{\rho}}) \right) \left(-\delta(z+z_0) + \frac{2}{1 - \zeta'_y \partial_z} \delta(z+z_0) \right). \tag{78}$$

The rightmost operator part, denoted $F_2(z)$, is interpretable through the differential equation

$$\left(1-\zeta_y'\partial_z\right)F_2(z)=2\delta(z+z_0) \tag{79}$$

with the requirement $F_2(z)=0$ for $z>z_0$. The solution is

$$F_2(z)=\frac{2}{\zeta_y'}e^{(z+z_0)/\zeta_y'}\Theta(-z-z_0), \tag{80}$$

giving us the image source

$$\varrho_0^{\rm r}(\bar{\mathbf{r}})=-Q\delta(\bar{\boldsymbol{\rho}}+\bar{\mathbf{s}})\delta(z+z_0)+\frac{2Q}{\zeta_y'}e^{(z+z_0)/\zeta_y'}\delta(\bar{\boldsymbol{\rho}}+\bar{\mathbf{a}}(z_0-z))\Theta(-z-z_0). \tag{81}$$

This equals the result obtained in [3], where an affine transformation was used to solve the problem‡. The reference deals with static currents, but the result is applicable to electrostatics just by changing the symbols: Q for I_0, $\bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}/\sqrt[3]{\det\bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}}$ for $\bar{\bar{\sigma}}_{\rm r}$ and $\epsilon_0/\sqrt[3]{\det\bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}}$ for σ_0 (by comparison of the Poisson equation of this problem and that of the reference); $\bar{\bar{\boldsymbol{\zeta}}}_{\rm r}^{-1}/\epsilon_0$ for $\bar{\bar{Z}}_{\rm s}$ (by the boundary conditions); z_0 for h, $-(z_0+z)\sqrt{\bar{\mathbf{u}}_z\cdot\bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}\cdot\bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}\cdot\bar{\mathbf{u}}_z}/\epsilon_z$, for ζ, and $\epsilon_z/(\zeta_y'\sqrt{\bar{\mathbf{u}}_z\cdot\bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}\cdot\bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}\cdot\bar{\mathbf{u}}_z})$ for τ/ν (by the role and transformation behaviour of each coordinate).

To see that (70) is reducible to the exponential line (81), we repeat the limiting process $\zeta_{\rm d}''\to 0$ for the complete solution. As $\zeta_{\rm d}''\to 0$, the planar image sector becomes just the semi-infinite line $\delta(\bar{\boldsymbol{\rho}}+\bar{\mathbf{a}}(z_0-z))\Theta(-z-z_0)$. Simultaneously the magnitude of the planar image grows, so we expect to obtain a delta function (a line source) and truly an image similar to (81). To see the amplitude of the delta function, we will integrate the non-differentiated part of the planar image over the x''-coordinate at constant z . We first take the identity [12, formula 2.15.2.6]

$$\begin{aligned}\int_0^s\frac{\xi I_0(t\xi)}{\sqrt{s^2-\xi^2}}d\xi &= \sqrt{\frac{s}{2t}}\Gamma\left(\frac{1}{2}\right)I_{1/2}(st)=\sqrt{\frac{s}{2t}}\sqrt{\pi}\sqrt{\frac{2}{\pi st}}\sinh(st)\\ &= \frac{\sinh(st)}{t},\end{aligned} \tag{82}$$

‡ The referenced article, [3], seems to be inaccurate at a few key points. The following list proposes some amendments.

change the variable ξ to $\sqrt{s^2-\psi^2}$, and get

$$\begin{aligned}\int_0^s \frac{\xi I_0(t\xi)}{\sqrt{s^2-\xi^2}}d\xi &= \int_s^0 \frac{\sqrt{s^2-\psi^2}I_0\left(t\sqrt{s^2-\psi^2}\right)}{\sqrt{s^2-(s^2-\psi^2)}}\frac{-\psi d\psi}{\sqrt{s^2-\psi^2}} \\ &= \int_0^s I_0\left(t\sqrt{s^2-\psi^2}\right)d\psi. \end{aligned} \tag{83}$$

With these results at our disposal, we integrate over the planar part:

$$\begin{aligned} &-\frac{Qe^{\frac{z+z_0}{2\zeta_y'}}}{\sqrt{\zeta_y'\zeta_{\mathrm{d}}''}}\Theta(-z-z_0)\int_{-\sqrt{\zeta_{\mathrm{d}}''/\zeta_y''}(z+z_0)-a_x''(z_0-z)}^{-\sqrt{\zeta_{\mathrm{d}}''/\zeta_y''}(z+z_0)-a_x''(z_0-z)} \\ &\cdot I_0\left(\frac{1}{2\zeta_y'}\sqrt{(z+z_0)^2-\frac{\zeta_y'}{\zeta_{\mathrm{d}}''}\left[x''+a_x''(z_0-z)\right]^2}\right)dx'' \\ =& -4\frac{Qe^{(z+z_0)/(2\zeta_y')}}{\sqrt{\zeta_y'\zeta_{\mathrm{d}}''}}\zeta_y'\sqrt{\frac{\zeta_{\mathrm{d}}''}{\zeta_y'}}\sinh\left(-\frac{1}{2\zeta_y'}(z+z_0)\right)\Theta(-z-z_0) \\ =& \; -2Qe^{\frac{z+z_0}{2\zeta_y'}}\left(e^{-\frac{z+z_0}{2\zeta_y'}}-e^{\frac{z+z_0}{2\zeta_y'}}\right)\Theta(-z-z_0) \\ =& \; -2Q\left(1-e^{(z-z_0)/\zeta_y'}\right)\Theta(-z-z_0). \end{aligned} \tag{84}$$

Now we can continue with the full planar part; after $\zeta_{\mathrm{d}}''\to 0$ we have the line source

$$\begin{aligned}\varrho_{1\mathrm{p}}^{\mathrm{r}}(\bar{\mathbf{r}}) &= -2Q\delta\left(y''+a_y''(z_0-z)\right)\left(\partial_z+a_x''\partial_x''\right) \\ &\quad\cdot\left[\left(1-e^{(z-z_0)/\zeta_y'}\right)\delta\left(x''+a_y''(z_0-z)\right)\Theta(-z-z_0)\right] \\ &= \frac{2Q}{\zeta_y'}e^{(z+z_0)/\zeta_y'}\delta\left(\bar{\boldsymbol{\rho}}+\bar{\mathbf{a}}(z_0-z)\right)\Theta(-z-z_0). \end{aligned} \tag{85}$$

This is clearly of the form (81) and is thus the image source in the case of similar anisotropy.

- Eq. (30): the right half: $\mathbf{r}\to\mathbf{r}'=\bar{\bar{\sigma}}_r^{-1/2}\cdot\mathbf{r}$
- Eq. (46): the unit step function: $U(\zeta')$
- Eq. (47): $\mathbf{r}_i(\zeta)=\bar{\bar{\sigma}}_r^{1/2}\cdot\mathbf{r}_i'(\zeta')=\bar{\bar{\sigma}}_r^{1/2}\cdot\mathbf{r}_i'-\bar{\bar{\sigma}}_r^{1/2}\cdot\bar{\mathbf{n}}'\zeta'=\mathbf{r}_i(0)-\mathbf{q}\zeta$
- Eq. (48): $\mathbf{q}=(\bar{\bar{\sigma}}_r\cdot\mathbf{u}_z)/\sqrt{\mathbf{u}_z\cdot\bar{\bar{\sigma}}_r\cdot\bar{\bar{\sigma}}_r\cdot\mathbf{u}_z}$, $\zeta=\nu\zeta'$, $\nu=\sqrt{\mathbf{u}_z\cdot\bar{\bar{\sigma}}_r\cdot\bar{\bar{\sigma}}_r\cdot\mathbf{u}_z}/\sqrt{\mathbf{u}_z\cdot\bar{\bar{\sigma}}_r\cdot\mathbf{u}_z}$

5.3. An Anisotropic Half-Space with a PEC or PMC Boundary

If a planar perfect electric (PEC) or magnetic (PMC) conductor is the boundary of the anisotropic half-space, the boundary impedance will be isotropic and can be defined as the limit

$$\bar{\bar{\boldsymbol{\zeta}}}_{\mathrm{r}}^{\mathrm{E}} = \lim_{\zeta_{\mathrm{E}}\to\infty} \zeta_{\mathrm{E}}\bar{\bar{\mathbf{I}}}_{\mathrm{t}} \quad \text{(PEC)} \quad \text{or} \tag{86}$$

$$\bar{\bar{\boldsymbol{\zeta}}}_{\mathrm{r}}^{\mathrm{M}} = \lim_{\zeta_{\mathrm{M}}\to\infty} \zeta_{\mathrm{M}}\bar{\bar{\mathbf{I}}}_{\mathrm{t}} \quad \text{(PMC)}. \tag{87}$$

In the basis $\bar{\mathbf{u}}_x$, $\bar{\mathbf{u}}_y$ dictated by $\bar{\bar{\mathbf{T}}}$ we have $\zeta_x = \zeta_y = \zeta_{\mathrm{E,M}}$ and can therefore denote this common eigenvalue with ζ. We also see that $\zeta_{\mathrm{c}} = 0$. So, the eigenvalues of $\bar{\bar{\boldsymbol{\zeta}}}_{\mathrm{r}}^{\mathrm{E}',\mathrm{M}'}$ are

$$\begin{aligned}\begin{pmatrix}\zeta'_x\\ \zeta'_y\end{pmatrix} &= \frac{1}{2}\epsilon_z\left(\frac{\zeta}{\tau_1(1+\nu_1)}+\frac{\zeta}{\tau_1}\pm\left[\frac{\zeta}{\tau_1}-\frac{\zeta}{\tau_1(1+\nu_1)}\right]\right)\\ &= \begin{pmatrix}\epsilon_z\zeta/\tau_1\\ \epsilon_z\zeta/[\tau_1(1+\nu_1)]\end{pmatrix},\end{aligned} \tag{88}$$

which can be obtained by a direct comparison of the last two forms of (110), too. The corresponding eigenvectors are, perhaps surprisingly, $\bar{\mathbf{u}}'_x = \bar{\mathbf{u}}_y$ and $\bar{\mathbf{u}}'_y = -\bar{\mathbf{u}}_x$ due to the requirement $\zeta'_x > \zeta'_y$

The next part of the “recipe” would give us the doubly primed quantities

$$\begin{aligned}\zeta''_{\mathrm{d}} &= \frac{1}{\epsilon_z^2}\left(\frac{\epsilon_z\zeta}{\tau_1}-\frac{\epsilon_z\zeta}{\tau_1(1+\nu_1)}\right)\bar{\mathbf{u}}_y\cdot(\bar{\mathbf{u}}_x\bar{\mathbf{u}}_x\tau_1(1+\nu_1)+\bar{\mathbf{u}}_y\bar{\mathbf{u}}_y\tau_1)\cdot\bar{\mathbf{u}}_y\\ &= \frac{\zeta\nu_1}{\epsilon_z(1+\nu_1)},\end{aligned} \tag{89}$$

$$\begin{aligned}\bar{\mathbf{u}}''_x &= \frac{\bar{\mathbf{u}}_y\cdot(\bar{\mathbf{u}}_x\bar{\mathbf{u}}_x\sqrt{\tau_1(1+\nu_1)}+\bar{\mathbf{u}}_y\bar{\mathbf{u}}_y\sqrt{\tau_1}}{\sqrt{\tau_1}} = \bar{\mathbf{u}}_y,\\ &\text{and}\quad \bar{\mathbf{u}}''_y = -\bar{\mathbf{u}}_x; \qquad x'' = y \qquad \text{and} \qquad y'' = -x.\end{aligned} \tag{90}$$

- Eq. (50): $I_i(\zeta) = I_0\left[-\delta(\zeta) + 2(\tau/\nu)e^{-(\tau/\nu)\zeta}U(\zeta)\right]$
- Three lines below Eq. (55): $1/\sigma_0 = 1/\sqrt[3]{\det(\bar{\bar{\sigma}})}$
- Eq. (56), the third term: $+\int_0^\infty \left(2(\tau/\nu)I_0e^{-(\tau/\nu)\zeta}\right)/\left(4\pi\sigma_0 D_\sigma(\mathbf{r}-\mathbf{r}_i(\zeta))\right)d\zeta$

The corrections should, of course, be propagated through the rest of the article correspondingly. These modifications (especially the introduction of the *unit* vector $\mathbf{q}$) are necessary because the original article does not sufficiently take into account changes of metric inside the field integral in the affine transformation. One must also keep an eye on the field integral (71) of this paper when comparing — or constructing — the image sources.

(We could have chosen $\bar{\mathbf{u}}'_x = -\bar{\mathbf{u}}_y$ and $\bar{\mathbf{u}}'_y = \bar{\mathbf{u}}_x$, causing a sign-change in the double-primed coordinates and unit vectors. Nevertheless, the image source would remain the same.) From this point on, we could write out the image function with these eigenquantities and study its behaviour on the limits $\zeta \to 0$ or ∞, but we just use the results from the last section because a similar anisotropy ($\zeta''_\mathrm{d} = 0$) produces an isotropic ideal surface impedance on these extreme limits. We therefore replace ζ with ζ'_y of the last section.

Letting $\zeta'_y \to \infty$ (the PEC case), we see that the planar part (now the exponential line charge) vanishes, and we only have a negative point charge shifted laterally from the mirror image point, as in [1]:

$$\varrho^\mathrm{r}_\mathrm{E}(\bar{\mathbf{r}}) = \lim_{\zeta'_y \to \infty} \sigma^\mathrm{r}_{1\mathrm{p}}(\bar{\mathbf{r}}) = -Q\delta(\bar{\boldsymbol{\rho}} + \bar{\mathbf{s}})\delta(z + z_0). \tag{91}$$

If, in addition, $\bar{\mathbf{a}} = \bar{\mathbf{0}}$ (in other words, if $\bar{\bar{\boldsymbol{\epsilon}}}_\mathrm{r}$ has $\bar{\mathbf{u}}_z$ as its eigenvector), we have the familiar PEC image theory of electrostatics (a negative point charge in the mirror image point).

By utilizing the delta sequence $\lim_{\kappa\to 0}(1/\kappa)e^{-|x|/\kappa}\Theta(-x) = \delta(x)$ we get

$$\begin{aligned} \varrho^\mathrm{r}_\mathrm{M}(\bar{\mathbf{r}}) &= \lim_{\zeta'_y \to 0} \varrho^\mathrm{r}_{1\mathrm{p}}(\bar{\mathbf{r}}) \\ &= -Q\delta(\bar{\boldsymbol{\rho}} + \bar{\mathbf{s}})\delta(z + z_0) + 2Q\delta(z + z_0)\delta(\bar{\boldsymbol{\rho}} + \bar{\mathbf{a}}(z_0 - z)) \\ &= Q\delta(\bar{\boldsymbol{\rho}} + \bar{\mathbf{s}})\delta(z + z_0), \end{aligned} \tag{92}$$

which, in turn, coincides with the PMC result in [1].

6. CONCLUSION

We have constructed, via the steps described in the Introduction, an electrostatic reflection image theory for an anisotropic boundary of an anisotropic half-space. The image function has features from the previously known theories: a negative point charge on the classical mirror image depth, but laterally shifted; a planar sector charge (with line charges on the edges) as in the image theory of an anisotropic boundary of isotropic half-space, but laterally shifted and skewed; and, as a limiting case, a slanted exponential line charge when the boundary and the half-space have similar anisotropy. The known PEC and PMC boundary cases can also be obtained.

It is good to remember that there is no need for a transmission image: the fields behind the boundary are zero. It should also be recalled that this electrostatic theory is readily applicable to steady-current problems by a duality transformation — one form of which was used in the checks of the theory.

APPENDIX A. DYADIC FORMULAS FOR THE MEDIA

A.1. Eigenvalues and Eigenvectors of Dyadics

We first define some dyadic tools. The dyadic *double-cross product* $\times\!\!\times$, *double-dot product* : , *and double-cross square* $\bar{\bar{\mathbf{A}}}^{(2)}$ are

$$\begin{aligned}(\bar{\mathbf{a}}\bar{\mathbf{b}}) \;{}^{\times}_{\times}\; (\bar{\mathbf{c}}\bar{\mathbf{d}}) &= (\bar{\mathbf{a}}\times\bar{\mathbf{c}})(\bar{\mathbf{b}}\times\bar{\mathbf{d}}), \qquad (\bar{\mathbf{a}}\bar{\mathbf{b}}):(\bar{\mathbf{c}}\bar{\mathbf{d}}) = (\bar{\mathbf{a}}\cdot\bar{\mathbf{c}})(\bar{\mathbf{b}}\cdot\bar{\mathbf{d}}),\\ \text{and}\quad \bar{\bar{\mathbf{A}}}^{(2)} &= \frac{1}{2}\bar{\bar{\mathbf{A}}} \;{}^{\times}_{\times}\; \bar{\bar{\mathbf{A}}},\end{aligned} \tag{A1}$$

respectively. We also define

$$\mathrm{tr}\bar{\bar{\mathbf{A}}} = \bar{\bar{\mathbf{A}}}:\bar{\bar{\mathbf{I}}} \qquad \text{and} \qquad \mathrm{spm}\bar{\bar{\mathbf{A}}} = \frac{1}{2}\bar{\bar{\mathbf{A}}} \;{}^{\times}_{\times}\; \bar{\bar{\mathbf{A}}}:\bar{\bar{\mathbf{I}}} = \mathrm{tr}\bar{\bar{\mathbf{A}}}^{(2)}, \tag{A2}$$

which are the *trace* and the *sum of principal minors* of any dyadic $\bar{\bar{\mathbf{A}}}$, respectively. The eigenvectors $\bar{\mathbf{a}}_i$ and eigenvalues α_i of a *symmetric real* dyadic $\bar{\bar{\mathbf{A}}}$ satisfy the equation

$$\bar{\mathbf{a}}_i\cdot\bar{\bar{\mathbf{A}}} = \bar{\bar{\mathbf{A}}}\cdot\bar{\mathbf{a}}_i = \alpha_i\bar{\mathbf{a}}_i. \tag{A3}$$

The equation for solving the eigenvalues is [11, (2.107)]

$$-\det(\bar{\bar{\mathbf{A}}}-\alpha_i\bar{\bar{\mathbf{I}}}) = \alpha_i^3 - \alpha_i^2\mathrm{tr}\bar{\bar{\mathbf{A}}} + \alpha_i\mathrm{spm}\bar{\bar{\mathbf{A}}} - \det\bar{\bar{\mathbf{A}}} = 0. \tag{A4}$$

The *determinant* of a dyadic is defined as

$$\det\bar{\bar{\mathbf{A}}} = \frac{1}{6}\bar{\bar{\mathbf{A}}} \;{}^{\times}_{\times}\; \bar{\bar{\mathbf{A}}}:\bar{\bar{\mathbf{A}}}. \tag{A5}$$

If, in addition, $\bar{\bar{\mathbf{A}}}$ is *two-dimensional* ($\bar{\mathbf{u}}\cdot\bar{\bar{\mathbf{A}}} = \bar{\bar{\mathbf{A}}}\cdot\bar{\mathbf{u}} = 0$ for some real unit vector $\bar{\mathbf{u}}$), we have $\det\bar{\bar{\mathbf{A}}} = 0$, and there are at most two nonzero eigenvalues. (The third eigenvalue, say α_3, is zero.) These two eigenvalues are the solutions of the remaining quadratic equation

$$\alpha_i^2 - \alpha_i\mathrm{tr}\bar{\bar{\mathbf{A}}} + \mathrm{spm}\bar{\bar{\mathbf{A}}} = 0 \tag{A6}$$

and read

$$\alpha_{1,2} = \frac{1}{2}\left(\mathrm{tr}\bar{\bar{\mathbf{A}}} \pm \sqrt{(\mathrm{tr}\bar{\bar{\mathbf{A}}})^2 - 4\mathrm{spm}\bar{\bar{\mathbf{A}}}}\right) = \frac{1}{2}\left(\mathrm{tr}\bar{\bar{\mathbf{A}}} \pm \sqrt{(\mathrm{tr}\bar{\bar{\mathbf{A}}})^2 - 4\mathrm{tr}\bar{\bar{\mathbf{A}}}^{(2)}}\right). \tag{A7}$$

Eigenvectors corresponding to simple nonzero roots of (A6) are of the form

$$\bar{\mathbf{a}}_i = (\bar{\bar{\mathbf{A}}} - \alpha_i \bar{\bar{\mathbf{I}}}) \,{}^{\times}_{\times}\, (\bar{\bar{\mathbf{A}}} - \alpha_i \bar{\bar{\mathbf{I}}}) \cdot \bar{\mathbf{c}} \tag{A8}$$

with a suitably chosen $\bar{\mathbf{c}}$. if $\alpha_1 = \alpha_2 = \alpha \neq 0$, we have $\bar{\bar{\mathbf{A}}} = \alpha(\bar{\bar{\mathbf{I}}} - \bar{\mathbf{u}}\bar{\mathbf{u}})$, and any vector $\bar{\mathbf{a}}_{1,2}$ satisfying $\bar{\mathbf{u}} \cdot \bar{\mathbf{a}}_{1,2} = 0$ is an eigenvector. An eigenvector corresponding to a zero eigenvalue is any vector $\bar{\mathbf{a}}_i$ with the property $\bar{\bar{\mathbf{A}}} \cdot \bar{\mathbf{a}}_i = 0$.

A.2. The Anisotropic Half-Space

Let us look at the properties of the $\bar{\bar{\mathbf{T}}}$-dyadic (22). We can use the dyadic identity [11, (2.47)]

$$\bar{\bar{\mathbf{A}}}^{(2)} \,{}^{\times}_{\times}\, \bar{\bar{\mathbf{B}}} = \bar{\bar{\mathbf{A}}}(\bar{\bar{\mathbf{A}}} : \bar{\bar{\mathbf{B}}}) - \bar{\bar{\mathbf{A}}} \cdot \bar{\bar{\mathbf{B}}}^{\mathrm{T}} \cdot \bar{\bar{\mathbf{A}}} \tag{A9}$$

and subsequently write

$$\begin{aligned}
\bar{\mathbf{K}} \cdot \bar{\bar{\mathbf{T}}} \cdot \bar{\mathbf{K}} &= \bar{\mathbf{K}} \cdot \left(\epsilon_z \bar{\bar{\mathbf{I}}}_{\mathrm{t}} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\bar{\mathbf{I}}}_{\mathrm{t}} - \epsilon_z^2 \bar{\mathbf{a}}\bar{\mathbf{a}}\right) \cdot \bar{\mathbf{K}} \\
&= \bar{\mathbf{K}} \cdot \left(\epsilon_z \bar{\bar{\mathbf{I}}}_{\mathrm{t}} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\bar{\mathbf{I}}}_{\mathrm{t}} - \epsilon_z^2 \left(\frac{\bar{\mathbf{u}}_z \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\bar{\mathbf{I}}}_{\mathrm{t}}}{\epsilon_z}\right)\left(\frac{\bar{\mathbf{u}}_z \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\bar{\mathbf{I}}}_{\mathrm{t}}}{\epsilon_z}\right)\right) \cdot \bar{\mathbf{K}} \\
&= \bar{\mathbf{u}}_z \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\mathbf{u}}_z \bar{\mathbf{K}} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\mathbf{K}} - \bar{\mathbf{K}} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\mathbf{u}}_z \bar{\mathbf{u}}_z \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\mathbf{K}} \\
&= \bar{\mathbf{K}} \cdot [\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}(\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} : \bar{\mathbf{u}}_z \bar{\mathbf{u}}_z) - \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\mathbf{u}}_z \bar{\mathbf{u}}_z \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}] \cdot \bar{\mathbf{K}} \\
&= \bar{\mathbf{K}} \cdot \left[\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} \,{}^{\times}_{\times}\, \bar{\mathbf{u}}_z \bar{\mathbf{u}}_z\right] \cdot \bar{\mathbf{K}}.
\end{aligned} \tag{A10}$$

Thus, $\bar{\bar{\mathbf{T}}}$ can be written in the alternative forms

$$\bar{\bar{\mathbf{T}}} = \epsilon_z \bar{\bar{\mathbf{I}}}_{\mathrm{t}} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} \cdot \bar{\bar{\mathbf{I}}}_{\mathrm{t}} - \epsilon_z^2 \bar{\mathbf{a}}\bar{\mathbf{a}} = \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} \,{}^{\times}_{\times}\, \bar{\mathbf{u}}_z \bar{\mathbf{u}}_z = \left(\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{-1} \,{}^{\times}_{\times}\, \bar{\mathbf{u}}_z \bar{\mathbf{u}}_z\right) \det \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}, \tag{A11}$$

the last of which follows from the identity $\bar{\bar{\mathbf{A}}}^{-1} = \bar{\bar{\mathbf{A}}}^{(2)\mathrm{T}}/\det\bar{\bar{\mathbf{A}}}$ and the symmetry of $\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}$ (and $\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{-1}$). The positive-definiteness of $\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}$ ensures that the inverse $\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{-1}$ exists and that $\det\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}} > 0$.

We already saw, while defining $\bar{\bar{\mathbf{T}}}$, that it is symmetric and two-dimensional ($\bar{\mathbf{u}}_z \cdot \bar{\bar{\mathbf{T}}} = \bar{\bar{\mathbf{T}}} \cdot \bar{\mathbf{u}}_z = 0$); from (A11) we now also see that $\bar{\bar{\mathbf{T}}}$ is positive definite *in the two-dimensional sense*, i.e., for every nonzero $\bar{\mathbf{b}} \perp \bar{\mathbf{u}}_z$ we have $\bar{\bar{\mathbf{T}}} : \bar{\mathbf{b}}\bar{\mathbf{b}} > 0$. This, in turn, means that $\bar{\bar{\mathbf{T}}}$ has two positive eigenvalues and two orthogonal eigenvectors in the xy-plane. We therefore can choose the coordinate axes so that the unit (eigen)vector corresponding to the larger eigenvalue is $\bar{\mathbf{u}}_x$ and the other unit vector is $\bar{\mathbf{u}}_y = \bar{\mathbf{u}}_z \times \bar{\mathbf{u}}_x$. Taking, for example, two numbers $\tau(>0)$ and $\nu(\geq 0)$, we would arrive to the form

$$\bar{\bar{\mathbf{T}}} = \bar{\mathbf{u}}_x\bar{\mathbf{u}}_x\tau(1+\nu) + \bar{\mathbf{u}}_y\bar{\mathbf{u}}_y\tau, \tag{A12}$$

which satisfies all requirements for a meaningful $\bar{\bar{\mathbf{T}}}$.

Because the two nonzero eigenvalues of $\bar{\bar{\mathbf{T}}}$ are positive, we can immediately see the ordering of the eigenvalues (A7), namely

$$\begin{aligned}
\alpha_{1,2} &= \begin{pmatrix} \tau(1+\nu) \\ \tau \end{pmatrix} = \frac{1}{2}\left(\mathrm{tr}\bar{\bar{\mathbf{T}}} \pm \sqrt{\left(\mathrm{tr}\bar{\bar{\mathbf{T}}}\right)^2 - 4\mathrm{spm}\bar{\bar{\mathbf{T}}}}\right) \\
&= \frac{1}{2}\left(\mathrm{tr}\bar{\bar{\mathbf{T}}} \pm \sqrt{\left(\mathrm{tr}\bar{\bar{\mathbf{T}}}\right)^2 - 4\mathrm{tr}\bar{\bar{\mathbf{T}}}^{(2)}}\right) \\
&= \frac{1}{2}\left(\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} : \left[(\bar{\mathbf{u}}_z\bar{\mathbf{u}}_z : \bar{\bar{\mathbf{I}}})\bar{\bar{\mathbf{I}}} - \bar{\mathbf{u}}_z\bar{\mathbf{u}}_z\right]\right. \\
&\quad \left. \pm \sqrt{\left(\bar{\bar{\mathbf{T}}} : \bar{\bar{\mathbf{I}}}\right)^2 - 4\frac{1}{2}\left[\left(\left[\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} \overset{\times}{\times} \bar{\mathbf{u}}_z\bar{\mathbf{u}}_z\right] : \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)}\right)\bar{\mathbf{u}}_z\bar{\mathbf{u}}_z\right] : \bar{\bar{\mathbf{I}}}}\right) \\
&= \frac{1}{2}\left(\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} : \bar{\bar{\mathbf{I}}}_{\mathrm{t}} \pm \sqrt{\left(\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} : \bar{\bar{\mathbf{I}}}_{\mathrm{t}}\right)^2 - 4\bar{\mathbf{u}}_z\bar{\mathbf{u}}_z : \left(\frac{1}{2}\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} \overset{\times}{\times} \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)}\right)}\right) \\
&= \frac{1}{2}\left(\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} : \bar{\bar{\mathbf{I}}}_{\mathrm{t}} \pm \sqrt{\left(\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} : \bar{\bar{\mathbf{I}}}_{\mathrm{t}}\right)^2 - 4\epsilon_z \mathrm{det}\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}}\right)
\end{aligned} \tag{A13}$$

We use the identity $\bar{\bar{\mathbf{A}}}^{(2)} \times \bar{\mathbf{a}} = (\bar{\bar{\mathbf{A}}} \cdot \bar{\mathbf{a}}) \times \bar{\bar{\mathbf{A}}}$ and denote $\bar{\mathbf{c}} = \bar{\mathbf{u}}_z \times \bar{\mathbf{d}}$ (with an arbitrary scalar coefficient) to get a prototype for the eigenvectors of $\bar{\bar{\mathbf{T}}}$ from (A8):

$$\begin{aligned}
\bar{\mathbf{t}}_i &= \left(\bar{\bar{\mathbf{T}}} - \alpha_i\bar{\bar{\mathbf{I}}}\right) \overset{\times}{\times} \left(\bar{\bar{\mathbf{T}}} - \alpha_i\bar{\bar{\mathbf{I}}}\right) \cdot \bar{\mathbf{c}} = \left(\bar{\bar{\mathbf{T}}} - \alpha_i\bar{\bar{\mathbf{I}}}\right)^{(2)} \cdot (\bar{\mathbf{u}}_z \times \bar{\mathbf{d}}) \\
&= \left[\left(\bar{\bar{\mathbf{T}}} - \alpha\bar{\bar{\mathbf{I}}}\right)^{(2)} \times \bar{\mathbf{u}}_z\right] \cdot \bar{\mathbf{d}} = \left(\left[\left(\bar{\bar{\mathbf{T}}} - \alpha_i\bar{\bar{\mathbf{I}}}\right) \cdot \bar{\mathbf{u}}_z\right] \times \left(\bar{\bar{\mathbf{T}}} - \alpha_i\bar{\bar{\mathbf{I}}}\right)\right) \cdot \bar{\mathbf{d}} \\
&= -\alpha_i\bar{\mathbf{u}}_z \times \left[\left(\bar{\bar{\mathbf{T}}} - \alpha_i\bar{\bar{\mathbf{I}}}\right) \cdot \bar{\mathbf{d}}\right] = -\alpha_i\bar{\mathbf{u}}_z \times \left[\left(\bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} \overset{\times}{\times} \bar{\mathbf{u}}_z\bar{\mathbf{u}}_z - \alpha_i\bar{\bar{\mathbf{I}}}\right) \cdot \bar{\mathbf{d}}\right] \\
&= \alpha_i\left(\bar{\mathbf{u}}_z\bar{\mathbf{u}}_z \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} - \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} + \alpha_i\bar{\bar{\mathbf{I}}}\right) \cdot \left(\bar{\mathbf{u}}_z \times \bar{\mathbf{d}}\right) \\
&= \left(\alpha_i\bar{\bar{\mathbf{I}}}_{\mathrm{t}} - \bar{\bar{\mathbf{I}}}_{\mathrm{t}} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\mathrm{r}}^{(2)} \cdot \bar{\bar{\mathbf{I}}}_{\mathrm{t}}\right) \cdot \bar{\mathbf{c}}.
\end{aligned} \tag{A14}$$

The final $\bar{\mathbf{c}}$ is any vector such that $\bar{\mathbf{t}}_i \neq \bar{\mathbf{0}}$. It is good to remember that the eigenvectors actually define "eigenorientations" and that the goal is to construct two orthonormal unit vectors $\bar{\mathbf{u}}_y = \bar{\mathbf{u}}_z \times \bar{\mathbf{u}}_x$, $\bar{\mathbf{u}}_x$ corresponding to the larger eigenvalue $(1+\nu)\tau$ in (A12). Should the eigenvalues be equal, the procedure above is not necessary — see the discussion below (A8). Sometimes the eigenvectors and eigenvalues can be seen directly from the expression of $\bar{\bar{\mathbf{T}}}$. Examples are given in the main body of the text.

If we choose $\bar{\mathbf{c}} = \bar{\mathbf{t}}_i$, we see that

$$\bar{\mathbf{t}}_i = \left(\alpha_i \bar{\bar{\mathbf{I}}}_{\rm t} - \bar{\bar{\mathbf{I}}}_{\rm t} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}^{(2)} \cdot \bar{\bar{\mathbf{I}}}_{\rm t}\right) \cdot \bar{\mathbf{t}}_i \Leftrightarrow (\alpha_i - 1)\bar{\mathbf{t}}_i = \left(\bar{\bar{\mathbf{I}}}_{\rm t} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}^{(2)} \cdot \bar{\bar{\mathbf{I}}}_{\rm t}\right) \cdot \bar{\mathbf{t}}_i. \tag{A15}$$

This means that $\alpha_i - 1$ is an eigenvalue of $\bar{\bar{\mathbf{I}}}_{\rm t} \cdot \bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}^{(2)} \cdot \bar{\bar{\mathbf{I}}}_{\rm t}$ (the planarized $\bar{\bar{\boldsymbol{\epsilon}}}_{\rm r}^{(2)}$) and that $\bar{\mathbf{t}}_i$ is the corresponding eigenvector.

One important property of $\bar{\bar{\mathbf{T}}}$ shall be mentioned last. Because of the (two-dimensional) positive-definiteness, $\bar{\bar{\mathbf{T}}}$ has a *two-dimensional inverse* $\bar{\bar{\mathbf{T}}}^{-1}$, which behaves very much like the normal inverse: $\bar{\bar{\mathbf{T}}}^{-1} \cdot \bar{\bar{\mathbf{T}}} = \bar{\bar{\mathbf{T}}} \cdot \bar{\bar{\mathbf{T}}}^{-1} = \bar{\bar{\mathbf{I}}}_{\rm t}$. No confusion should arise from the multipurpose use of one inversion symbol.

A.3. The Surface Impedance

After finding the eigenvectors of $\bar{\bar{\mathbf{T}}}$ as in Appendix A2 we can write the surface impedance dyadic in this basis as

$$\bar{\bar{\boldsymbol{\zeta}}}_{\rm r} = \bar{\mathbf{u}}_x\bar{\mathbf{u}}_x\zeta_x + (\bar{\mathbf{u}}_x\bar{\mathbf{u}}_y + \bar{\mathbf{u}}_y\bar{\mathbf{u}}_x)\,\zeta_{\rm c} + \bar{\mathbf{u}}_y\bar{\mathbf{u}}_y\zeta_y; \tag{A16}$$

$$\begin{aligned} \zeta_x &= \bar{\mathbf{u}}_x \cdot \bar{\bar{\boldsymbol{\zeta}}}_{\rm r} \cdot \bar{\mathbf{u}}_x, \qquad \zeta_{\rm c} = \bar{\mathbf{u}}_x \cdot \bar{\bar{\boldsymbol{\zeta}}}_{\rm r} \cdot \bar{\mathbf{u}}_y = \bar{\mathbf{u}}_y \cdot \bar{\bar{\boldsymbol{\zeta}}}_{\rm r} \cdot \bar{\mathbf{u}}_x, \\ \text{and} \qquad \zeta_y &= \bar{\mathbf{u}}_y \cdot \bar{\bar{\boldsymbol{\zeta}}}_{\rm r} \cdot \bar{\mathbf{u}}_y. \end{aligned} \tag{A17}$$

Then the modified surface impedance dyadic will be

$$\begin{aligned} \bar{\bar{\boldsymbol{\zeta}}}'_{\rm r} &= \epsilon_z \bar{\bar{\mathbf{T}}}^{-1/2} \cdot \bar{\bar{\boldsymbol{\zeta}}}_{\rm r} \cdot \bar{\bar{\mathbf{T}}}^{-1/2} \\ &= \epsilon_z \left(\frac{\bar{\mathbf{u}}_x\bar{\mathbf{u}}_x}{\sqrt{\tau_1}\sqrt{1+\nu_1}} + \frac{\bar{\mathbf{u}}_y\bar{\mathbf{u}}_y}{\sqrt{\tau_1}}\right) \cdot [\bar{\mathbf{u}}_x\bar{\mathbf{u}}_x\zeta_x + (\bar{\mathbf{u}}_x\bar{\mathbf{u}}_y + \bar{\mathbf{u}}_y\bar{\mathbf{u}}_x)\,\zeta_{\rm c} \\ &\quad +\bar{\mathbf{u}}_y\bar{\mathbf{u}}_y\zeta_y] \cdot \left(\frac{\bar{\mathbf{u}}_x\bar{\mathbf{u}}_x}{\sqrt{\tau_1}\sqrt{1+\nu_1}} + \frac{\bar{\mathbf{u}}_y\bar{\mathbf{u}}_y}{\sqrt{\tau_1}}\right) \\ &= \epsilon_z \left(\bar{\mathbf{u}}_x\bar{\mathbf{u}}_x \frac{\zeta_x}{\tau_1(1+\nu_1)} + (\bar{\mathbf{u}}_x\bar{\mathbf{u}}_y + \bar{\mathbf{u}}_y\bar{\mathbf{u}}_x)\frac{\zeta_{\rm c}}{\tau_1\sqrt{1+\nu_1}} + \bar{\mathbf{u}}_x\bar{\mathbf{u}}_x\frac{\zeta_y}{\tau_1}\right) \\ &= \bar{\mathbf{u}}'_x\bar{\mathbf{u}}'_x\zeta'_x + \bar{\mathbf{u}}'_y\bar{\mathbf{u}}'_y\zeta'_y. \end{aligned} \tag{A18}$$

Applying (A7), we get the eigenvalues

$$\begin{aligned}\alpha_{1,2} &= \begin{pmatrix} \zeta_x' \\ \zeta_y' \end{pmatrix} = \frac{1}{2}\left(\mathrm{tr}\overline{\overline{\boldsymbol{\zeta}}}_{\mathrm{r}}' \pm \sqrt{\left(\mathrm{tr}\overline{\overline{\boldsymbol{\zeta}}}_{\mathrm{r}}'\right)^2 - 4\mathrm{spm}\overline{\overline{\boldsymbol{\zeta}}}_{\mathrm{r}}'}\right) \\ &= \frac{1}{2}\epsilon_z\left(\frac{\zeta_x}{\tau_1(1+\nu_1)} + \frac{\zeta_y}{\tau_1} \pm \sqrt{\left[\frac{\zeta_x}{\tau_1(1+\nu_1)} - \frac{\zeta_y}{\tau_1}\right] + \frac{4\zeta_{\mathrm{c}}^2}{\tau_1^2(1+\nu_1)}}\right). \end{aligned} \tag{A19}$$

If the eigenvalues are equal, which happens when $\zeta_x = \zeta_y(1+\nu_1)$ *and* $\zeta_{\mathrm{c}} = 0$, we can choose $\bar{\mathbf{u}}_x' = \bar{\mathbf{u}}_x$ and $\bar{\mathbf{u}}_y' = \bar{\mathbf{u}}_y$. Otherwise we need the procedure of Appendix A1 for the determination of the eigenvectors.

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