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NON-RELATIVISTIC SCATTERING IN THE PRESENCE OF MOVING OBJECTS: THE MIE PROBLEM FOR A MOVING SPHERE

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Abstract—Recently non-relativistic boundary conditions based on the Lorentz force formulas have been introduced. It was shown that to the first order in the relative velocity v/c the results are in agreement with the exact relativistic formalism. Specific boundary value problems have been solved to get concrete results and demonstrate the feasibility of implementing the formalism. These included examples involving plane and cylindrical interfaces.

Presently the velocity-dependent Mie problem, *viz.* scattering of a plane wave by a moving sphere, is investigated. The sphere is assumed to move in a material medium without mechanically affecting the medium. The analysis follows closely the solution for the cylindrical case, given before. The mathematics here (involving spherical vector waves and harmonics) is more complicated, and therefore sufficient detail and references are provided.

The interesting feature emerging from the present analysis is that the velocity-dependent effects induce higher order multipoles, which are not present in the classical Mie solution for scattering by a sphere at rest. The formalism is sufficiently general to deal with arbitrary moving objects.

1. INTRODUCTION AND ROADMAP

Recently a new non-relativistic treatment for scattering in velocity-dependent systems has been proposed [1]. Examples for scattering by material objects in free space have been analyzed. Subsequently the question of solving cylindrical problems for velocity-dependent scattering for objects immersed in material media has been considered [2]. The proposed formalism [1] is based on boundary conditions derived from the Lorentz force formulas.

Consider an electric charge $q_e^{(1)}$ in medium "*1*". In this medium a boundary is in motion at a velocity $\boldsymbol{v}^{(b\mapsto 1)}(\boldsymbol{r}, t)$, which can be a function of space coordinates $\boldsymbol{r}$ and time t. Following both classical and relativistic formalisms, the charge is considered an invariant, i.e., $q_e^{(1)} = q_e^{(b)}$ is the same for observers in "*1*" and when co-moving with the boundary coordinates "*b*". The force $\boldsymbol{f}_e^{(1)}$, acting on a charge on the boundary, is given by the Lorentz force formula

$$\boldsymbol{f}_e^{(1)} = q_e^{(b)}(\boldsymbol{E}^{(1)} + \boldsymbol{v}^{(b\mapsto 1)} \times \boldsymbol{B}^{(1)}) = q_e^{(b)}\boldsymbol{E}_{eff}^{(1)} \tag{1}$$

where $\boldsymbol{E}_{eff}^{(1)}$ denotes a new effective field. If on the other side of the boundary we have another medium "*2*", then similarly we have

$$\boldsymbol{f}_e^{(2)} = q_e^{(b)}(\boldsymbol{E}^{(2)} + \boldsymbol{v}^{(b\mapsto 2)} \times \boldsymbol{B}^{(2)}) = q_e^{(b)}\boldsymbol{E}_{eff}^{(2)} \tag{2}$$

For a rigid object moving within the medium "*1*", we take in (2) $\boldsymbol{v}^{(b\mapsto 2)} = 0$. From (1, 2) and relativistic transformations for fields, it was suggested [2] that if magnetic sources were existent, we would be able to measure magnetic Lorentz forces, and corresponding to (1), (2), we would have

$$\boldsymbol{f}_m^{(1)} = q_m^{(b)}(\boldsymbol{H}^{(1)} - \boldsymbol{v}^{(b\mapsto 1)} \times \boldsymbol{D}^{(1)}) = q_m^{(b)}\boldsymbol{H}_{eff}^{(1)} \tag{3}$$

$$\boldsymbol{f}_m^{(2)} = q_m^{(b)}(\boldsymbol{H}^{(2)} - \boldsymbol{v}^{(b\mapsto 2)} \times \boldsymbol{D}^{(2)}) = q_m^{(b)}\boldsymbol{H}_{eff}^{(2)} \tag{4}$$

Zero net energy is dissipated on the boundary itself, and therefore no additional forces are created on it. It follows that at the boundary

(1)–(4) prescribe the equilibrium conditions

$$\hat{\boldsymbol{n}}^{(b)} \times \left(\boldsymbol{f}_e^{(1)} - \boldsymbol{f}_e^{(2)}\right) = 0 \tag{5}$$

$$\hat{\boldsymbol{n}}^{(b)} \times \left(\boldsymbol{f}_m^{(1)} - \boldsymbol{f}_m^{(2)}\right) = 0 \tag{6}$$

i.e., the tangential components of $\boldsymbol{f}_e, \boldsymbol{f}_m$ are continuous across the boundary.

Thusly (5), (6) provide the necessary boundary conditions for this class of problems

$$\hat{\boldsymbol{n}}^{(b)} \times \left(\boldsymbol{E}_1^{(1)} + \boldsymbol{v}^{(b\mapsto 1)} \times \boldsymbol{B}_1^{(1)} - \boldsymbol{E}_2^{(2)} - \boldsymbol{v}^{(b\mapsto 2)} \times \boldsymbol{B}_2^{(2)}\right) = 0 \tag{7}$$

$$\hat{\boldsymbol{n}}^{(b)} \times \left(\boldsymbol{H}_1^{(1)} - \boldsymbol{v}^{(b\mapsto 1)} \times \boldsymbol{D}_1^{(1)} - \boldsymbol{H}_2^{(2)} + \boldsymbol{v}^{(b\mapsto 2)} \times \boldsymbol{D}_2^{(2)}\right) = 0 \tag{8}$$

The Lorentz force and the relativistically exact boundary conditions [1] agree to the first order in ν/c only. This indicates that one should use (7), (8) only for relatively moderate velocities. Most practical problems are of this nature. On the other hand, the Lorentz force formula does not assume a constant velocity, hence in (7), (8) $\boldsymbol{v}(\boldsymbol{r}, t)$ may be any function of space and time.

The mathematical tools presented in the next section are quite general, in the sense that they facilitate the analysis of various complicated geometries, e.g., velocity and incident wave given in arbitrary directions. However, the aim of the present analysis is to demonstrate the feasibility of solving such problems in principle, rather than choose the most general, hence very complicated, cases. Accordingly, we analyze here the Mie problem of scattering of a plane electromagnetic wave by a sphere, and choose a case of high symmetry, where the velocity is along the polar axis, as explained below. This considerably simplifies the mathematical detail.

From here on we need a roadmap to understand the method employed for solving the scattering problem: We start with a mathematical overview which puts at our disposal the tools used later, and introduce notational conventions, with some of the frequently used abbreviations collected in a list at a subsequent section.

The analysis leans heavily on the plane wave representations (essentially Sommerfeld-type integrals in the complex domain) for the vector spherical waves. The boundary value problem for Mie scattering by a moving sphere is developed. The manipulation of the coefficients in the complex integrals requires the application of various formulas given below, some taken from the literature, others developed here. The end product of this procedure facilitates representations in terms of series of vector spherical wave functions, but the associated coefficients are still spatially-dependent and therefore require further manipulation. This leads to series which allow us to exploit the orthogonality properties of the vector spherical harmonics, and thus facilitate the solution for the scattering coefficients involved in the boundary conditions relations.

At this stage we are ready to calculate the scattered field, which is given as a superposition (integral) of plane waves. Inasmuch as the frequencies of the scattered plane waves in the integrand are dependent on the directions of the waves, the complex path integrals are complicated and cannot be represented in terms of series of vector spherical waves, as was done for scattering in the absence of motion. For the near field in the vicinity of the scatterer a first order approximation is provided, but this is inadequate for larger distances. Mimicking the approach taken in [2] for scattering by a moving cylinder, series of inverse-distance powers differential operators are used, facilitating moderate and far field representations.

Because of the complicated calculations, we have to heap up new definitions and substitutions for expressions as we move along. While this has the potential of encumbering the presentation, it should be easily handled by mathematical numerical packages which allow continuous nesting of parameters. For the detailed organization of the subsequent material see also the summary below.

2. MATHEMATICAL OVERVIEW

The mathematical background for the present investigation is given in some detail. Basic relations are included here for completeness, because there are some subtle differences in definition and notation used by various authors. Sources for spherical vector waves and the related harmonics are Morse and Feshbach [3], and Stratton [4]. We draw heavily on Twersky [5, 6], and use some of his notation.

Essentially, the present investigation deals with the solution of the Helmholtz vector wave equation in simple media

$$\begin{aligned} \partial_{\boldsymbol{r}}\partial_{\boldsymbol{r}}\cdot\boldsymbol{u}-\partial_{\boldsymbol{r}}\times\partial_{\boldsymbol{r}}\times\boldsymbol{u}+k^2\boldsymbol{u}=0, \qquad \boldsymbol{u}=\boldsymbol{u}(\boldsymbol{r},t) \\ k^2=\omega^2\mu\varepsilon, \qquad \boldsymbol{r}=\boldsymbol{r}(r,\theta,\psi) \end{aligned} \tag{9}$$

in terms of spherical coordinates $\boldsymbol{r}=\boldsymbol{r}(r,\theta,\psi)$, where for plane waves the time-dependence of $\boldsymbol{u}(\boldsymbol{r},t)$ is harmonic according to $e^{-i\omega t}$. The solutions are represented in terms of solutions of the associated scalar wave equation

$$\left(\partial_{\boldsymbol{r}}^2+k^2\right)\varphi(\boldsymbol{r})=0, \quad \partial_{\boldsymbol{r}}^2=\partial_{\boldsymbol{r}}\cdot\partial_{\boldsymbol{r}}, \quad \boldsymbol{u}(\boldsymbol{r},t)=\varphi(\boldsymbol{r})e^{-i\omega t} \tag{10}$$

Three independent solutions of (9) are constructed

$$\boldsymbol{L} = \partial_{\boldsymbol{r}}\varphi, \quad \boldsymbol{M} = \partial_{\boldsymbol{r}} \times \hat{\boldsymbol{a}}\varphi, \quad \boldsymbol{N} = \partial_{\boldsymbol{r}} \times \boldsymbol{M}/k, \quad \boldsymbol{M} = \partial_{\boldsymbol{r}} \times \boldsymbol{N}/k \quad (11)$$

where in (11) $\hat{\boldsymbol{a}}$ denotes a unit vector satisfying $\partial_{\boldsymbol{r}} \times \hat{\boldsymbol{a}} = 0$. Note that $\partial_{\boldsymbol{r}} \cdot \boldsymbol{M} = 0$, $\partial_{\boldsymbol{r}} \cdot \boldsymbol{N} = 0$, therefore solutions of

$$\left(\partial_{\boldsymbol{r}} \times \partial_{\boldsymbol{r}} \times -k^2\right) \boldsymbol{u}(\boldsymbol{r}, t) = 0, \quad \partial_{\boldsymbol{r}} \cdot \boldsymbol{u}(\boldsymbol{r}, t) = 0 \quad (12)$$

are prescribed by Maxwell's equations for fields in sourceless simple uniform media. The solution $\boldsymbol{L} = \partial_{\boldsymbol{r}}\varphi$ nevertheless satisfies (9), (10), and is needed for completeness. It also features in the velocity-dependent problem defined below.

The solutions for (10) are given by

$$\begin{aligned} \varphi_{nm} &= z_n(kr) Y_n^m(\hat{\boldsymbol{r}}) e^{-i\omega t}, \quad Y_n^m(\hat{\boldsymbol{r}}) = P_n^m(C_\theta) e^{im\psi} \\ Y_n^{-m} &= P_n^{-m} e^{-im\psi} = \gamma_{nm} M P_n^m e^{-im\psi} \\ P_n^m &= 0 \text{ for } |m| > n, \ \gamma_{nm} = (n-m)!/(n+m)!, \ M = (-1)^m \end{aligned} \quad (13)$$

where $z_n(kr)$ are generically the spherical Bessel functions of order n, and Y_n^m are the scalar spherical harmonics, defined in terms of the associated Legendre functions P_n^m and the azimuthal dependence $e^{im\psi}$. In particular, we shall exploit z_n in terms of the nonsingular spherical Bessel functions j_n in bounded sourceless domains, and the spherical Hankel functions of the first kind $h_n^{(1)} = h_n$ for the external domain. Together with the time exponential $e^{-i\omega t}$, h_n are chosen for representation of outgoing scattered waves. The functions of the second kind $h_n^{(2)}$ usually do not feature in our formulas. For orthogonality properties of Y_n^m in (13), see below (15), which also provides the orthogonality properties for $\boldsymbol{P}_n^m$.

Corresponding to (11), (13), vector spherical waves follow

$$\begin{aligned} \boldsymbol{M}_{nm}(k_\alpha \boldsymbol{r}) &= J_{zn1}^{k_\alpha r} \boldsymbol{C}_n^m(\hat{\boldsymbol{r}}), \quad \boldsymbol{N}_{nm}(k_\alpha \boldsymbol{r}) = J_{zn2}^{k_\alpha r} \boldsymbol{P}_n^m(\hat{\boldsymbol{r}}) + J_{zn3}^{k_\alpha r} \boldsymbol{B}_n^m(\hat{\boldsymbol{r}}) \\ \boldsymbol{L}_{nm}(k_\alpha \boldsymbol{r}) &= J_{zn4}^{k_\alpha r} \boldsymbol{P}_n^m(\hat{\boldsymbol{r}}) + J_{zn5}^{k_\alpha r} \boldsymbol{B}_n^m(\hat{\boldsymbol{r}}) \\ J_{zn1}^{k_\alpha r_\beta} &= z_n(k_\alpha r_\beta), \quad J_{zn2}^{k_\alpha r_\beta} = z_n(k_\alpha r_\beta)/(\kappa_n k_\alpha r_\beta) \\ J_{zn3}^{k_\alpha r_\beta} &= \partial_{k_\alpha r_T}[k_\alpha r_\beta z_n(k_\alpha r_\beta)]/k_\alpha r_\beta, \quad J_{zn4}^{k_\alpha r_\beta} = \partial_{k_\alpha r_\beta}[z_n(k_\alpha r_\beta)] \\ J_{zn5}^{k_\alpha r_\beta} &= z_n(k_\alpha r_\beta)/(k_\alpha r_\beta), \quad \kappa_n = 1/n(n+1) \\ \boldsymbol{C}_n^m(\hat{\boldsymbol{r}}) &= -\boldsymbol{r} \times \partial_{\boldsymbol{r}} Y_n^m(\hat{\boldsymbol{r}}) = -\hat{\boldsymbol{r}} \times \boldsymbol{B}_n^m = \left(\hat{\boldsymbol{\theta}} \partial_\psi / S_\theta - \hat{\boldsymbol{\psi}} \partial_\theta\right) Y_n^m(\hat{\boldsymbol{r}}) \\ \boldsymbol{B}_n^m(\hat{\boldsymbol{r}}) &= r \partial_{\boldsymbol{r}} Y_n^m(\hat{\boldsymbol{r}}) = \hat{\boldsymbol{r}} \times \boldsymbol{C}_n^m = \left(\hat{\boldsymbol{\psi}} \partial_\psi / S_\theta + \hat{\boldsymbol{\theta}} \partial_\theta\right) Y_n^m(\hat{\boldsymbol{r}}) \\ \boldsymbol{P}_n^m(\hat{\boldsymbol{r}}) &= \hat{\boldsymbol{r}} Y_n^m(\hat{\boldsymbol{r}}), \ \partial_\psi Y_n^m(\hat{\boldsymbol{r}}) = im Y_n^m(\hat{\boldsymbol{r}}), \ C_\zeta = \cos\zeta, \ S_\zeta = \sin\zeta \end{aligned} \quad (14)$$

in terms of the vector spherical harmonics $\boldsymbol{P}_n^m$, $\boldsymbol{B}_n^m$, $\boldsymbol{C}_n^m$. It is noted that rather than following Stratton [4] who defines $\boldsymbol{L} = \partial_{\boldsymbol{r}}\varphi$ as in (3), in Twersky [6] $\boldsymbol{L}_{nm}$ follows the definition $\boldsymbol{L} = \partial_{k\boldsymbol{r}}\varphi = \partial_{\boldsymbol{r}}\varphi/k$, and therefore differs by this factor from Stratton (see [4], p. 414). Of course this has no effect on the solutions, except when comparing the two conventions. The definition of the spherical Bessel functions, e.g., $h_0(kr) = e^{ikr}/ikr$ is identical.

The vector spherical harmonics satisfy spatial and functional orthogonality relations as follows

$$\begin{aligned}
&\boldsymbol{P}_n^m \cdot \boldsymbol{B}_n^m = \boldsymbol{P}_n^m \cdot \boldsymbol{C}_n^m = \boldsymbol{B}_n^m \cdot \boldsymbol{C}_n^m = 0,\\
&\int \boldsymbol{C}_n^{-m} \cdot \boldsymbol{C}_\nu^\mu d\Omega = \int \boldsymbol{B}_n^{-m} \cdot \boldsymbol{B}_\nu^\mu d\Omega = \alpha_{nm}\delta_{n\nu}\delta_{m\mu}\\
&\int \boldsymbol{P}_n^{-m} \cdot \boldsymbol{P}_\nu^\mu d\Omega = \int Y_n^{-m} Y_\nu^\mu d\Omega = \beta_{nm}\delta_{n\nu}\delta_{m\mu}\\
&\gamma_{nm}\int (P_n^m)^2 d\Omega = 4\pi\lambda_n,\ \gamma_{nm}\int \left(P_n^m/S_\theta\right)^2 d\Omega = 2\pi/m \qquad (15)\\
&\int P_n^m P_n^{-m}/S_\theta^2 d\Omega = \mu_m\\
&\int d\Omega = \int_{-\pi}^{\pi} d\psi \int_0^\pi S_\theta d\theta,\ \delta_{a\alpha} = \left\{\begin{array}{ll} 0, & a \neq \alpha\\ 1, & a = \alpha\end{array}\right\}\\
&\alpha_{nm} = M4\pi/N_n,\ \beta_{nm} = \alpha_{nm}\kappa_n,\ \lambda_n = 1/(2n+1),\ \mu_m = 2\pi M/m
\end{aligned}$$

Plane wave representations for the spherical vector wave functions will be used below for the implementation of the new boundary conditions and evaluation of the scattered field. Thus we have, correspondingly,

$$\begin{aligned}
&\left\{i^n \boldsymbol{M}_{nm}(k\boldsymbol{r}),\ i^{n-1}\boldsymbol{N}_{nm}(k\boldsymbol{r}),\ i^{n-1}\boldsymbol{L}_{nm}(k\boldsymbol{r})\right\}\\
&= \frac{1}{2\pi}\int e^{ikr\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}} \left\{\boldsymbol{C}_n^m(\hat{\boldsymbol{p}}),\ \boldsymbol{B}_n^m(\hat{\boldsymbol{p}}),\ \boldsymbol{P}_n^m(\hat{\boldsymbol{p}})\right\} d\Omega_{\hat{\boldsymbol{p}}},\ \hat{\boldsymbol{p}} = \hat{\boldsymbol{p}}(\beta,\alpha) \qquad (16)\\
&\int d\Omega_{\hat{\boldsymbol{p}}} = \left\{\begin{array}{ll} \displaystyle\int_{-\pi}^{\pi} d\beta \int_0^{(\pi/2)-i\infty} S_\alpha d\alpha, & z_n = h_n\\ \displaystyle\int_{-\pi}^{\pi} d\beta \int_0^{\pi} S_\alpha d\alpha, & z_n = 2j_n \end{array}\right\}
\end{aligned}$$

for each of the waves and its corresponding harmonic in (16). The factor *2* for the nonsingular Bessel functions in (16) is due to the fact that $2j_n = h_n^{(1)} + h_n^{(2)}$. For discussion of the integration limits see [4, 5, 7].

In general, arbitrary wave functions can be represented by series and their corresponding plane wave integrals

$$\begin{aligned}
\boldsymbol{u}(\boldsymbol{r},t) &= e^{-i\omega t}\Sigma_{nm} i^n (\boldsymbol{M}_{nm}(k\boldsymbol{r})c_{nm} - i\boldsymbol{N}_{nm}(k\boldsymbol{r})b_{nm} - i\boldsymbol{L}_{nm}(k\boldsymbol{r})p_{nm}) \\
&= \frac{1}{2\pi}e^{-i\omega t}\int e^{ik\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}}\boldsymbol{G}(\hat{\boldsymbol{p}})d\Omega_{\hat{\boldsymbol{p}}} \qquad (17)\\
\boldsymbol{G}(\hat{\boldsymbol{r}}) &= \Sigma_{nm}\left(\boldsymbol{C}_n^m(\hat{\boldsymbol{r}})c_{nm} + \boldsymbol{B}_n^m(\hat{\boldsymbol{r}})b_{nm} + \boldsymbol{P}_n^m(\hat{\boldsymbol{r}})p_{nm}\right), \\
\Sigma_{nm} &= \sum_{n=0}^{\infty}\sum_{m=-n}^{m=n}
\end{aligned}$$

where in (17) c_{nm}, b_{nm}, p_{nm} are coefficients, usually constants, but in any case not involving the integration variables symbolized by $\hat{\boldsymbol{p}}$, and for transversal waves as specified by (9), the longitudinal waves $\boldsymbol{L}_{nm}(\boldsymbol{r})$ in the series and their corresponding vector harmonics $\boldsymbol{P}_n^m$ in the integral, are omitted.

For two-dimensional problems Twersky [8] and Twersky *et al.* [9] derived the scattered field in terms of a differential operator representation involving inverse powers of the distance. The two-dimensional representation has been used for the moving cylinder and moving cylindrical medium [2]. A corresponding representation for the scalar three-dimensional case was also given [5] by Twersky, who also provides [6] the solution to the three-dimensional vector problem discussed here, in terms of a dyadic differential operator $\tilde{\boldsymbol{O}}(kr, \tilde{\boldsymbol{D}})$ and inverse powers of distance

$$\begin{aligned}
\boldsymbol{u}(\boldsymbol{r},t) &= e^{-i\omega t}h_0(kr)\tilde{\boldsymbol{O}}(kr,\tilde{\boldsymbol{D}})\cdot\boldsymbol{G}(\hat{\boldsymbol{r}}) \\
\tilde{\boldsymbol{O}} &= \sum_{\nu=0}^{\infty}\rho_\nu\tilde{\boldsymbol{D}}\cdot(\tilde{\boldsymbol{D}}-1.2\tilde{\boldsymbol{I}})\cdot(\tilde{\boldsymbol{D}}-2.3\tilde{\boldsymbol{I}})\cdots(\tilde{\boldsymbol{D}}-(\nu-1)\nu\tilde{\boldsymbol{I}}) \\
&= \tilde{\boldsymbol{I}}+\rho_1\tilde{\boldsymbol{D}}+\rho_2\tilde{\boldsymbol{D}}\cdot(\tilde{\boldsymbol{D}}-2\tilde{\boldsymbol{I}})+\rho_n\tilde{\boldsymbol{D}}\cdot(\tilde{\boldsymbol{D}}-1.2\tilde{\boldsymbol{I}})\cdots(\tilde{\boldsymbol{D}}-(n-1)n\tilde{\boldsymbol{I}}) \\
\rho_\nu &= (i/(2kr))^\nu/\nu! \qquad (18)\\
\tilde{\boldsymbol{D}} &= \hat{\boldsymbol{r}}(D+2)\hat{\boldsymbol{r}}+\hat{\boldsymbol{r}}2S_\theta^{-1}\partial_\theta(S_\theta\hat{\boldsymbol{\theta}})+\hat{\boldsymbol{r}}\partial_\psi\hat{\boldsymbol{\psi}}+\hat{\boldsymbol{\theta}}(D+S_\theta^{-2})\hat{\boldsymbol{\theta}} \\
&\quad +\hat{\boldsymbol{\theta}}2S_\theta^{-2}C_\theta\partial_\psi\hat{\boldsymbol{\psi}}-\hat{\boldsymbol{\theta}}\partial_\theta\hat{\boldsymbol{r}}+\hat{\boldsymbol{\psi}}(D+S_\theta^{-2})\hat{\boldsymbol{\psi}}-\hat{\boldsymbol{\psi}}2S_\theta^{-2}C_\theta\partial_\psi\hat{\boldsymbol{\theta}} \\
&\quad -\hat{\boldsymbol{\psi}}2S_\theta^{-1}\partial_\psi\hat{\boldsymbol{r}}, \quad D = S_\theta^{-2}\left[\partial_\psi^2+S_\theta\partial_\theta(S_\theta\partial_\theta)\right]
\end{aligned}$$

In (18) the application of the differential operator dyadic $\tilde{\boldsymbol{D}}$ first acts on the vector function $\boldsymbol{G}(\hat{\boldsymbol{r}})$, thus separating it to its components, e.g., $\hat{\boldsymbol{\theta}}\cdot\boldsymbol{G} = G_\theta$, then operating on these component according to the differential operators as prescribed, and finally attaching the appropriate antecedent unit vector as prescribed (cf. [6]). This is the usual scheme used for dyadics.

Note that unlike Stratton [4], who expresses the vector spherical waves and harmonics (16) in terms of series involving even and odd harmonics, with C_ψ, S_ψ, and takes the appropriate limits for m, Twersky [5, 6] and others use complex Fourier series with $e^{im\psi}$, and the appropriate limits on the sum with respect to m. Let us consider the relation between the two representations. The surface harmonics are defined in (13), whence it follows that

$$\begin{aligned} Y^e_{nm} &= Y^m_{\boldsymbol{R}n} = P^m_n C_{m\psi} = \left(Y^m_n + Y^{-m}_n M/\gamma_{nm}\right)/2 \\ Y^o_{nm} &= Y^m_{\boldsymbol{I}n} = P^m_n S_{m\psi} = \left(Y^m_n - Y^{-m}_n M/\gamma_{nm}\right)/(2i) \\ Y^e_{nm} &\pm iY^o_{nm} = P^m_n e^{\pm im\psi}, \quad \partial_\psi Y^m_{\left\{\substack{\boldsymbol{R}\\ \boldsymbol{I}}\right\}n} = \{\mp\} m Y^m_{\left\{\substack{\boldsymbol{I}\\ \boldsymbol{R}}\right\}n} \end{aligned} \tag{19}$$

establishing the connection to the even and odd spherical harmonics used by Stratton [4] and elsewhere, e.g., [3]. The notation $Y^m_{\boldsymbol{R}n}$, $Y^m_{\boldsymbol{I}n}$ indicates that we take the real, imaginary, part of $e^{im\psi}$, respectively (assuming real arguments everywhere).

From (14) and (19) follow the relations of the vector spherical harmonics and waves written in terms of the exponential representation, to the even and odd ones

$$\begin{aligned} \boldsymbol{Q}^m_{\boldsymbol{R}n} + i\boldsymbol{Q}^m_{\boldsymbol{I}n} &= \boldsymbol{Q}^m_n, \quad \overline{\partial}_\psi \boldsymbol{Q}^m_n = im\boldsymbol{Q}^m_n \\ \overline{\partial}_\psi \boldsymbol{Q}^m_{\boldsymbol{R}n} &= -m\boldsymbol{Q}^m_{\boldsymbol{I}n}, \quad \overline{\partial}_\psi \boldsymbol{Q}^m_{\boldsymbol{I}n} = m\boldsymbol{Q}^m_{\boldsymbol{R}n} \\ \boldsymbol{Q} &= \boldsymbol{P},\ \boldsymbol{B},\ \boldsymbol{C},\ \boldsymbol{L},\ \boldsymbol{M},\ \boldsymbol{N} \end{aligned} \tag{20}$$

with the understanding that $\overline{\partial}_\psi$ acts only on $e^{im\psi}$ and its real and imaginary parts, and does not affect unit vectors and the spherical Bessel functions and associated derivatives as detailed in (14).

The present section recapitulated the various mathematical tools needed subsequently for discussing the velocity dependent scattering problems. Additional formulas will be added as needed.

3. THE BOUNDARY VALUE PROBLEM FOR A MOVING SPHERE

The center of a sphere of radius R moves according to $x = y = 0,\ z = \nu t,\ \boldsymbol{v} = \hat{\boldsymbol{z}}\nu$, along the direction of propagation of the excitation wave, $\hat{\boldsymbol{k}}_{ex}$. Accordingly we define a local coordinate system, denoted by index T, in which the location of the boundary is time-independent

$$\begin{aligned} x_T &= \boldsymbol{r}_T \cdot \hat{\boldsymbol{x}}_T = r_T S_{\theta_T} C_{\psi_T} = x = \boldsymbol{r} \cdot \hat{\boldsymbol{x}} = r S_\theta C_\psi \\ y_T &= \boldsymbol{r}_T \cdot \hat{\boldsymbol{y}}_T = r_T S_{\theta_T T} S_{\psi_T} = y = \boldsymbol{r} \cdot \hat{\boldsymbol{y}} = r S_\theta S_\psi \\ z_T &= \boldsymbol{r}_T \cdot \hat{\boldsymbol{z}}_T = r_T C_{\theta_T} = z - \nu t = \boldsymbol{r} \cdot \hat{\boldsymbol{z}} - \nu t = r C_\theta - \nu t \end{aligned} \tag{21}$$

This is not a new spatiotemporal frame of reference in the relativistic sense. In fact, we refer to one and only frame of reference $(\boldsymbol{r}, t)$, in which the local coordinate system $(\boldsymbol{r}_T, t)$ is *parametrized* by the time t.

The excitation plane wave in the external medium (indicated by index "*1*") is given by

$$\begin{aligned} \boldsymbol{E}_{ex} &= \hat{\boldsymbol{x}} E_{ex}, \quad \boldsymbol{H}_{ex} = \hat{\boldsymbol{y}} H_{ex} e^{i\varphi_{ex}} \\ E_{ex} &= E_{ex0} e^{i\varphi_{ex}}, \quad H_{ex} = H_{ex0} e^{i\varphi_{ex}} \\ \varphi_{ex} &= k_{ex} z - \omega_{ex} t \\ k_{ex}/\omega_{ex} &= (\mu^{(1)} \varepsilon^{(1)})^{1/2} = 1/\nu_{ph}^{(1)} \\ E_{ex}/H_{ex} &= (\mu^{(1)}/\varepsilon^{(1)})^{1/2} = \zeta^{(1)} \end{aligned} \tag{22}$$

propagating in the direction $\hat{\boldsymbol{k}}_{ex} = \hat{\boldsymbol{z}}$, with $\boldsymbol{E}_{ex}$, $\boldsymbol{H}_{ex}$ polarized along the x, y axes, respectively, thus conforming with the geometry used by Stratton [4] (see p. 564 there).

The new boundary conditions (7), (8) are used here for $\boldsymbol{v}^{(b\mapsto 2)} = 0$, yielding at some point referred to by $\boldsymbol{r}_T$, e.g., $\boldsymbol{r}_T = 0$ a field signal

$$\begin{aligned} \boldsymbol{E}_{exT} &= (\boldsymbol{E}_{ex} + \boldsymbol{v} \times \boldsymbol{B}_{ex})|_{r_T=0} = (\boldsymbol{E}_{ex} + \boldsymbol{v} \times \hat{\boldsymbol{y}} \mu^{(1)} H_{ex})|_{r_T=0} = \hat{\boldsymbol{x}} E_{exT} \\ E_{exT} &= E_{0T} e^{i\varphi_{0T}}, \quad E_{0T} = E_{ex0}(1 - \beta^{(1)}), \\ \beta^{(1)} &= \nu/\nu_{ph}^{(1)}, \quad \nu_{ph}^{(1)} = (\mu^{(1)} \varepsilon^{(1)})^{-1/2} \end{aligned} \tag{23}$$

and similarly

$$\begin{aligned} \boldsymbol{H}_{exT} &= (\boldsymbol{H}_{ex} - \boldsymbol{v} \times \boldsymbol{D}_{ex})|_{r_T=0} = (\boldsymbol{H}_{ex} - \boldsymbol{v} \times \varepsilon^{(1)} \boldsymbol{E}_{ex})|_{r_T=0} = \hat{\boldsymbol{y}} H_{exT} \\ H_{exT} &= H_{0T} e^{i\varphi_{0T}} = E_{0T} e^{i\varphi_{0T}}/\zeta^{(1)}, \quad H_{0T} = H_{ex0}(1 - \beta^{(1)}) \end{aligned} \tag{24}$$

In (23), (24) the amplitude is multiplied by a phase factor $e^{i\varphi_{0T}}$, providing a reference phase, as explained below. Note that by considering fields at some fixed point, e.g., at a boundary we are not dealing any more with a field *wave*, rather, we are considering a field *signal*, which does not satisfy the wave equations (9), (10).

Simple substitution of (21) into φ_{ex} in (22), would constitute a Galilean transformation of the plane wave into a new spatiotemporal frame of reference. We already know that this technique does not tally with relativistic results. This point has been discussed in [2] in some detail. Essentially, what the present model prescribes is to find the time-dependent field signal at some arbitrary reference point in the boundary's local coordinate system $(\boldsymbol{r}_T, t)$, and then consider phase shifts associated with other locations. To do so we have to include the

Fresnel drag effect [2, 10–12]. What this means in the present context is that we have to use

$$\boldsymbol{k}^{(b)} = \boldsymbol{k}^{(1)} - \boldsymbol{v}^{(b\mapsto 1)}\omega^{(1)}/c^2 \tag{25}$$

where instead of the propagation vector of a plane wave $\boldsymbol{k}^{(1)}$, given in the external medium "*1*" at rest, we consider $\boldsymbol{k}^{(b)}$ in the boundary local coordinate system, as prescribed by (25).

Accordingly, in (23), (24) we compute the signal at $\boldsymbol{r}_T = 0$ according to (21), (22). Although this point is not accessible to the external waves, it provides a convenient reference because of the symmetry with respect to the spherical boundary surface. Of course any other reference point can be used.

Thus we get

$$\varphi_{0T} = \varphi_{ex}|_{\boldsymbol{r}_T=0} = -\omega_T t, \quad \omega_T = \omega_{ex}(1-\beta^{(1)}) \tag{26}$$

It is noted that (26) prescribes a new frequency ω_T which agrees with the first order in ν/c frequency transformation, and this is the point where the Galilean and Lorentzian transformations are in agreement, to the first order in ν/c.

At any other location $\boldsymbol{r}_T$ we have to include an appropriate phase shift as prescribed according to (25), embodying the Fresnel drag effect. Specifically, at the sphere's surface $r_T = R$ we have a time-dependent signal $e^{i\varphi_T}$ with

$$\varphi_T = k_T R C_{\theta_T} - \omega_T t,\ k_T = k_{ex}(1-\beta^{(1)}A^{(1)}),\ A^{(1)} = \left(\nu_{ph}^{(1)}/c\right)^2 \tag{27}$$

In free space $A^{(1)} = 1$, and consequently in (27) $k_T = k_{ex}(1-\beta^{(1)})$ reduces to the first order in ν/c relativistic Doppler effect in free space as given in [2]. Observe that (22), (27) imply $\varphi_T \neq \varphi_{ex}$, unless we qualify t. This point does not invalidate the following analysis, and will be picked up again in Section 6.

From (14), (22)–(27) and Stratton [4, p. 564], the excitation field at the boundary is recast in terms of spherical vector functions

$$\begin{aligned}
\boldsymbol{E}_{exT} &= \overline{e}\Sigma_{n1} I_n \left(\boldsymbol{M}^{exT}_{\boldsymbol{I}n1} - i\boldsymbol{N}^{exT}_{\boldsymbol{R}n1}\right)\\
\boldsymbol{H}_{exT} &= -\overline{h}\Sigma_{n1} I_n \left(\boldsymbol{M}^{exT}_{\boldsymbol{R}n1} + i\boldsymbol{N}^{exT}_{\boldsymbol{I}n1}\right)\\
\boldsymbol{M}^{exT}_{n1}(k_T\boldsymbol{R}) &= J^{k_T R}_{jn1}\boldsymbol{C}^1_n(\hat{\boldsymbol{r}}_T)\\
\boldsymbol{N}^{exT}_{n1}(k_T\boldsymbol{R}) &= J^{k_T R}_{jn2}\boldsymbol{P}^1_n(\hat{\boldsymbol{r}}_T) + J^{k_T R}_{jn3}\boldsymbol{B}^1_n(\hat{\boldsymbol{r}}_T)\\
\overline{e} &= E_{0T}e^{-i\omega_T t},\ \overline{h} = H_{0T}e^{-i\omega_T t}\\
I_n &= i^n N_n, \quad N_n = \kappa_n/\lambda_n
\end{aligned} \tag{28}$$

In the absence of motion, corresponding to (28), the nonsingular internal fields for $r_T \leq R$ are represented at the boundary (cf. Stratton [4], p. 565) as

$$\begin{aligned} \boldsymbol{E}_{inT} &= \overline{e}\Sigma_{n1} I_n \left(c_{n1}^{in} \boldsymbol{M}_{\boldsymbol{I}n1}^{in} - i b_{n1}^{in} \boldsymbol{N}_{\boldsymbol{R}n1}^{in} \right) \\ \boldsymbol{H}_{inT} &= -\overline{h}\Sigma_{n1} I_n \left(b_{n1}^{in} \boldsymbol{M}_{\boldsymbol{R}n1}^{in} + i c_{n1}^{in} \boldsymbol{N}_{\boldsymbol{I}n1}^{in} \right) \\ \boldsymbol{M}_{n1}^{in}(k_{in}\boldsymbol{R}) &= J_{jn1}^{k_{in}R} \boldsymbol{C}_n^1(\hat{\boldsymbol{r}}_T) \\ \boldsymbol{N}_{n1}^{in}(k_{in}\boldsymbol{R}) &= J_{jn2}^{k_{in}R} \boldsymbol{P}_n^1(\hat{\boldsymbol{r}}_T) + J_{jn3}^{k_{in}R} \boldsymbol{B}_n^1(\hat{\boldsymbol{r}}_T) \\ k_{in}/\omega_T &= (\mu^{(2)}\varepsilon^{(2)})^{1/2} = 1/\nu_{ph}^{(2)} \end{aligned} \tag{29}$$

In (29) k_{in} characterizes the interior medium. Motion might change the field signal at the boundary and introduce additional terms to the internal field. Note that in (29) we have already normalized the coefficients with respect to E_{0T}, H_{0T} of the incident wave. For the field signal corresponding to $\boldsymbol{E}_{in}$, $\boldsymbol{H}_{in}$ inside the sphere, we substituted $R = r_T$ in (29).

The construction of the scattered wave in the exterior embedding domain must provide for a time dependence $e^{-i\omega_T t}$ at the boundary $r_T = R$, and for outgoing transversal electromagnetic waves in this medium, so that the scattered wave satisfies (17), (18), which satisfy (9) too.

Similarly to (22), consider a single plane wave propagating in an arbitrary direction indicated by $\hat{\boldsymbol{k}}_p$

$$\begin{aligned} \boldsymbol{E}_p &= \hat{\boldsymbol{e}}_p E_p, \ E_p = E_{p0} e^{i\varphi_p}, \ \boldsymbol{H}_p = \hat{\boldsymbol{k}}_p \times \hat{\boldsymbol{e}}_p H_p, \ H_p = H_{p0} e^{i\varphi_p} \\ \varphi_p &= \boldsymbol{k}_p \cdot \boldsymbol{r} - \omega_p t = k_{px}x + k_{py}y + k_{pz}z - \omega_p t \\ k_p/\omega_p &= 1/\nu_{ph}^{(1)}, \quad E_p/H_p = \zeta^{(1)} \end{aligned} \tag{30}$$

similarly to (26) we have

$$\begin{aligned} \varphi_{p0} &= \varphi_p|_{\boldsymbol{r}_T=0} = -\omega_{pT} t \\ \omega_{pT} &= \omega_p(1 - \beta^{(1)} C_\alpha) \end{aligned} \tag{31}$$

which for $\alpha = 0$ reduce to (26). Inasmuch as the boundary conditions must be satisfied at all times, the scattered wave must be constructed in such a way that on the sphere's surface the time dependence is identical to that of the incident wave, therefore

$$\begin{aligned} \omega_{pT} &= \omega_T \\ \omega_p &= \omega_{ex}(1 - \beta^{(1)})/(1 - \beta^{(1)} C_\alpha) \\ &= \omega_{ex}(1 + \beta^{(1)}(C_\alpha - 1)) + O(\beta^{(1)})^2 \end{aligned} \tag{32}$$

The last line (32) applies to first order in $\beta^{(1)}$, which is the first order approximation used throughout.

At any other location $\boldsymbol{r}_T$ we have to include an appropriate phase shift as prescribed (25). Similarly to (27) we now have at the sphere's surface $e^{i\varphi_{pT}}$ with

$$\begin{aligned}
\varphi_{pT} &= \boldsymbol{k}_{pT}\cdot\hat{\boldsymbol{r}}_T R-\omega_T t\\
&= k_{pTx}RS_{\theta_T}C_{\psi_T}+k_{pTy}RS_{\theta_T T}S_{\psi_T}+k_{pTz}RC_{\theta_T}-\omega_T t\\
k_{pTx} &= k_{px},\quad k_{pTy}=k_{py}\\
k_{pTz} &= k_p(C_\alpha-\beta^{(1)}A^{(1)})=k_{pz}-k_p\beta^{(1)}A^{(1)}\\
\varphi_{pT} &= \boldsymbol{k}_p\cdot\hat{\boldsymbol{r}}_T R+\beta^{(1)}A-\omega_T t,\ K_{ex}=k_{ex}R,\ A=-K_{ex}A^{(1)}C_{\theta_T}
\end{aligned}\tag{33}$$

Once Again note that (30), (33) imply $\varphi_{pT}\neq\varphi_p$, unless we qualify t.

Incorporating (32) into (33) yields

$$\varphi_{pT}=K_{ex}\hat{\boldsymbol{k}}_p\cdot\hat{\boldsymbol{r}}_T-\omega_T t+\beta^{(1)}K_{ex}(C\alpha-1)\hat{\boldsymbol{k}}_p\cdot\hat{\boldsymbol{r}}_T-\beta^{(1)}K_{ex}A^{(1)}C_{\theta_T}\tag{34}$$

Based on (7), (8), (30) and similarly to (23), (24), at the boundary, we have to consider the signal

$$\begin{aligned}
\boldsymbol{E}_{pT} &= \boldsymbol{E}_p+\boldsymbol{v}\times\boldsymbol{B}_p=\boldsymbol{E}_p+\beta^{(1)}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{h}}_pE_p\\
&= E_p\left(\hat{\boldsymbol{e}}_p+\beta^{(1)}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_p\times\hat{\boldsymbol{e}}_p\right)\\
\boldsymbol{H}_{pT} &= \boldsymbol{H}_p-\boldsymbol{v}\times\boldsymbol{D}_p=\boldsymbol{H}_p-\beta^{(1)}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{e}}_pH_p\\
&= H_p\left(\hat{\boldsymbol{h}}_p+\beta^{(1)}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_p\times\hat{\boldsymbol{h}}_p\right)\\
\left\{\begin{array}{c}\boldsymbol{E}_{pT}\\ \boldsymbol{H}_{pT}\end{array}\right\} &= (1+\beta^{(1)}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_p\times)\left\{\begin{array}{c}\boldsymbol{E}_p\\ \boldsymbol{H}_p\end{array}\right\}
\end{aligned}\tag{35}$$

Clearly, $\boldsymbol{E}_p$, $\boldsymbol{H}_p$ constitute transversal plane electromagnetic waves, but the field signals $\boldsymbol{E}_{pT}$, $\boldsymbol{H}_{pT}$ at the boundary, due to the operator $(1+\beta^{(1)}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_p\times)$, generally contain longitudinal components. However, note that $\boldsymbol{E}_{pT}$, $\boldsymbol{H}_{pT}$, involved in the solution of the boundary condition problem, are signals, not waves satisfying the wave equation (9).

Corresponding to (28), (29) Stratton [4] (see p. 564) represents the scattered field signal in the absence of motion as

$$\begin{aligned}
\boldsymbol{E}_{scT} &= E_{ex0}e^{-i\omega_{ex}t}\Sigma_{n1}I_n\left(c_{n1}^{sc}\boldsymbol{M}_{\boldsymbol{I}n1}^{sc}-ib_{n1}^{sc}\boldsymbol{N}_{\boldsymbol{R}n1}^{sc}\right)\\
\boldsymbol{H}_{scT} &= -H_{ex0}e^{-i\omega_{ex}t}\Sigma_{n1}I_n\left(b_{n1}^{sc}\boldsymbol{M}_{\boldsymbol{R}n1}^{sc}+ic_{n1}^{sc}\boldsymbol{N}_{\boldsymbol{I}n1}^{sc}\right)\\
\boldsymbol{M}_{n1}^{sc}(k_{sc}\boldsymbol{R}) &= J_{hn1}^{k_{sc}R}\boldsymbol{C}_n^1(\hat{\boldsymbol{r}}_T)\\
\boldsymbol{N}_{n1}^{sc}(k_{sc}\boldsymbol{R}) &= J_{hn2}^{k_{sc}R}\boldsymbol{P}_n^1(\hat{\boldsymbol{r}}_T)+J_{hn3}^{k_{sc}R}\boldsymbol{B}_n^1(\hat{\boldsymbol{r}}_T),\ k_{sc}=k_{ex}
\end{aligned}\tag{36}$$

In (36) the coefficients c_n^{sc}, b_n^{sc} (cf. [4]) are derivable from the boundary conditions in the absence of motion. Using the relations in (13), the plane wave representation (16), and (36), the scattered wave for a sphere at rest is represented as a superposition (integral) of transversal plane waves

$$\begin{aligned}
\left\{ \begin{array}{c} \boldsymbol{E}_{scT} \\ \boldsymbol{H}_{scT} \end{array} \right\} &= \left\{ \begin{array}{c} +E_{ex0} \\ -H_{ex0} \end{array} \right\} e^{-i\omega_{ex}t} \frac{1}{2\pi} \int e^{iK_{ex}\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T} \boldsymbol{g}_{\left\{ \boldsymbol{E} \atop \boldsymbol{H} \right\}}(\hat{\boldsymbol{p}}) d\Omega_{\hat{\boldsymbol{p}}} \\
\boldsymbol{g}_{\boldsymbol{E}} &= \Sigma_{n1} \left(d_{1n1}^{sc} \boldsymbol{C}_{\boldsymbol{I}n}^1 + d_{2n1}^{sc} \boldsymbol{B}_{\boldsymbol{R}n}^1 \right) \qquad (37) \\
\boldsymbol{g}_{\boldsymbol{H}} &= \Sigma_{n1} \left(d_{2n1}^{sc} \boldsymbol{C}_{\boldsymbol{R}n}^1 - d_{1n1}^{sc} \boldsymbol{B}_{\boldsymbol{I}n}^1 \right) \\
d_{1n1}^{sc} &= N_n c_{n1}^{sc}, \quad d_{2n1}^{sc} = N_n b_{n1}^{sc}, \quad \boldsymbol{r} \times \boldsymbol{g}_{\left\{ \boldsymbol{E} \atop \boldsymbol{H} \right\}} = \{\mp\} \boldsymbol{g}_{\left\{ \boldsymbol{H} \atop \boldsymbol{E} \right\}}
\end{aligned}$$

where for the object at rest $\hat{\boldsymbol{r}}_T$ reduces to $\hat{\boldsymbol{r}}$.

Corresponding to (37), the scattered wave in the presence of the moving sphere will be constructed as a superposition (integral) of waves

$$\left\{ \begin{array}{c} \boldsymbol{E}_{sc} \\ \boldsymbol{H}_{sc} \end{array} \right\} = \left\{ \begin{array}{c} +E_{0T} \\ -H_{0T} \end{array} \right\} \frac{1}{2\pi} \int_c e^{i\varphi_p(\hat{\boldsymbol{p}})} \boldsymbol{G}_{\left\{ \boldsymbol{E} \atop \boldsymbol{H} \right\}}(\hat{\boldsymbol{p}}) d\Omega_{\hat{\boldsymbol{p}}} \qquad (38)$$

arbitrarily normalized to E_{0T}, H_{0T}, $E_{0T}/H_{0T} = \zeta^{(1)}$ in order to simplify subsequent expressions. The contour c in (38) needs to be determined such that the scattered fields constitute outgoing waves, and the new velocity dependent scattering amplitudes $\boldsymbol{G}_{\boldsymbol{E}}$, $\boldsymbol{G}_{\boldsymbol{H}}$, must be computed. Finally the scattered wave outlined in (37) will have to be converted to forms that can be readily computed. For vanishing velocity (38) reduces to (37).

4. CALCULATION OF THE VELOCITY-DEPENDENT TERMS

The statement of the scattering problem for the moving sphere culminated in having the integral expression (38). We now embark on the tedious and complicated odyssey of recasting the first order in $\beta^{(1)}$ velocity terms in series of vector spherical harmonics. Only in this form orthogonality considerations can be exploited in the computation of the scattering coefficients included in $\boldsymbol{G}_{\left\{ \boldsymbol{E} \atop \boldsymbol{H} \right\}}$, and $\left\{ \begin{array}{c} \boldsymbol{E}_{sc} \\ \boldsymbol{H}_{sc} \end{array} \right\}$, (38).

Assuming R to be sufficiently small to justify the approximation of an exponential by its leading Taylor expansion terms, i.e., $e^a \approx 1+a$, in view of (30), (33)–(35), at the boundary (38) becomes, to the first order in $\beta^{(1)}$

$$
\begin{aligned}
\left\{\begin{matrix}\boldsymbol{E}_{scT}\\ \boldsymbol{H}_{scT}\end{matrix}\right\} &= \left\{\begin{matrix}+\overline{e}\\ -\overline{h}\end{matrix}\right\}\frac{1}{2\pi}\int_c e^{iK_{ex}\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T}(\boldsymbol{G}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}(\hat{\boldsymbol{p}})+\beta^{(1)}\boldsymbol{f}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}(\hat{\boldsymbol{p}}))d\Omega_{\hat{\boldsymbol{p}}}\\
\boldsymbol{f}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}(\hat{\boldsymbol{p}}) &= iK_{ex}(C_\alpha-1)(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T)\boldsymbol{g}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}+(\hat{\boldsymbol{z}}\cdot\boldsymbol{g}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}})\hat{\boldsymbol{p}}\\
&\quad -(\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{p}})\boldsymbol{g}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}+A\boldsymbol{g}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}\\
&= iK_{ex}(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T)\boldsymbol{h}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}+(\hat{\boldsymbol{z}}\cdot\boldsymbol{g}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}})\hat{\boldsymbol{p}}\\
&\quad -\boldsymbol{g}^c_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}+A\boldsymbol{g}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}},\quad K_{ex}=k_{ex}R \qquad (39)\\
\boldsymbol{h}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}} &= \boldsymbol{g}^c_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}-\boldsymbol{g}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}},\ \boldsymbol{g}^c_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}(\hat{\boldsymbol{p}})=C_\alpha\boldsymbol{g}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}=(\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{p}})\boldsymbol{g}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}\\
\boldsymbol{h}_{\boldsymbol{H}} &= -\hat{\boldsymbol{p}}\times\boldsymbol{h}_{\boldsymbol{E}},\quad \boldsymbol{f}_{\boldsymbol{E}}=\hat{\boldsymbol{p}}\times\boldsymbol{h}_{\boldsymbol{H}}+(\hat{\boldsymbol{z}}\cdot\boldsymbol{g}_{\boldsymbol{E}})\hat{\boldsymbol{p}}\\
\boldsymbol{f}_{\boldsymbol{H}} &= -\hat{\boldsymbol{p}}\times\boldsymbol{h}_{\boldsymbol{E}}+(\hat{\boldsymbol{z}}\cdot\boldsymbol{g}_{\boldsymbol{H}})\hat{\boldsymbol{p}}
\end{aligned}
$$

In (39), because of the factor $\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T$ in the exponential, and the time factor involving ω_T, the spherical vector waves (14) are now prescribed in terms of these coordinates, and in (39) we choose for c the integral prescribed by (16). The leading term is $\boldsymbol{G}$, which will be calculated subsequently according to the boundary conditions equations, and the next term involving $\beta^{(1)}$ is multiplied by the zero order approximation of $\boldsymbol{G}$, namely the term $\boldsymbol{g}$. The zero velocity scattering amplitudes $\boldsymbol{g}_{\boldsymbol{E}}$, $\boldsymbol{g}_{\boldsymbol{H}}$ are considered to be known (e.g., see Stratton [4], p. 565). It therefore remains to show how $\boldsymbol{G}$ is computed in terms of vector spherical harmonics. Note that unlike (38), in (39) we have longitudinal terms in direction $\hat{\boldsymbol{p}}$, but we are not dealing here with a wave proper, and in (39) $\boldsymbol{E}_{scT}$, $\boldsymbol{H}_{scT}$ are not solutions of the electromagnetic wave equations. Some of the following mathematical results were given previously [13].

In (39) $A\boldsymbol{g}_{\left\{\begin{smallmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{smallmatrix}\right\}}$ simply means (37) is modified by this factor

$$
\begin{aligned}
A\boldsymbol{g}_{\boldsymbol{E}} &= \Sigma_{n1}\left(d^{sc}_{3n1}\boldsymbol{C}^1_{\boldsymbol{I}n}+d^{sc}_{4n1}\boldsymbol{B}^1_{\boldsymbol{R}n}\right)\\
A\boldsymbol{g}_{\boldsymbol{H}} &= \Sigma_{n1}\left(d^{sc}_{4n1}\boldsymbol{C}^1_{\boldsymbol{R}n}-d^{sc}_{3n1}\boldsymbol{B}^1_{\boldsymbol{I}n}\right) \qquad (40)\\
d^{sc}_{3n1} &= Ad^{sc}_{1n1},\quad d^{sc}_{4n1}=Ad^{sc}_{2n1}
\end{aligned}
$$

Note that according to (33) A involves R, C_{θ_T} but for the time being we treat them as constants because they are not involved in the integration in (39).

Consider the term $\boldsymbol{g}^c = C_\alpha \boldsymbol{g}$ in (39). For convenience the calculations are expressed in terms of the coordinates $\hat{\boldsymbol{r}}$ instead of $\hat{\boldsymbol{p}}$, which makes it easier to look up pertinent formulas in the literature. Later on the expressions we derive will be recast in terms of the pertinent coordinates $\hat{\boldsymbol{p}}$ or $\hat{\boldsymbol{r}}_T$, as needed. Accordingly we have to evaluate $C_\theta \boldsymbol{C}_n^m(\hat{\boldsymbol{r}})$, $C_\theta \boldsymbol{B}_n^m(\hat{\boldsymbol{r}})$ which can be recast as

$$\begin{aligned} C_\theta \boldsymbol{C}_n^m(\hat{\boldsymbol{r}}) &= -C_\theta \boldsymbol{r} \times \partial_{\boldsymbol{r}} Y_n^m \\ &= -\boldsymbol{r} \times \partial_{\boldsymbol{r}}(C_\theta Y_n^m) + Y_n^m \boldsymbol{r} \times \partial_{\boldsymbol{r}} C_\theta \\ &= -\boldsymbol{r} \times \partial_{\boldsymbol{r}}(C_\theta Y_n^m) - \hat{\boldsymbol{\psi}} S_\theta Y_n^m \end{aligned} \tag{41}$$

Exploiting identities given by [3] (see p. 1326) and [4] (see p. 401)

$$\begin{aligned} C_\theta P_n^m &= a_{1nm} P_{n+1}^m + a_{2nm} P_{n-1}^m \\ a_{1nm} &= \lambda_n (n-m+1), \; a_{2nm} = \lambda_n (n+m) \end{aligned} \tag{42}$$

Thus we get for the first term in (41)

$$\begin{aligned} -\boldsymbol{r} \times \partial_{\boldsymbol{r}}(C_\theta Y_n^m) &= -\boldsymbol{r} \times \partial_{\boldsymbol{r}} \left(a_{1nm} Y_{n+1}^m + a_{2nm} Y_{n-1}^m \right) \\ &= a_{1nm} \boldsymbol{C}_{n+1}^m + a_{2nm} \boldsymbol{C}_{n-1}^m \end{aligned} \tag{43}$$

The second term (41) is recast in a series of vector spherical harmonics

$$-\hat{\boldsymbol{\psi}} S_\theta Y_n^m = \Sigma_{qm} \left(\overline{A}_q^m \boldsymbol{C}_q^m + \overline{B}_q^m \boldsymbol{B}_q^m \right) \tag{44}$$

where in (44) the bar on $\overline{A}_q^m$, $\overline{B}_q^m$ is used in order not to confuse these coefficients from A_q^m, B_q^m introduced below.

Multiplying (44) by $\cdot \boldsymbol{C}_q^{-m}$ and integrating according to (15) yields on the right of (44) $\overline{A}_q^m \alpha_{qm}$, while on the left we obtain

$$-\int \hat{\boldsymbol{\psi}} S_\theta Y_n^m \cdot \boldsymbol{C}_q^{-m} d\Omega = \int Y_n^m S_\theta \partial_\theta Y_q^{-m} d\Omega \tag{45}$$

The differential relations for P_n^m functions given by Stratton [4] (see p. 402) can be combined into a new identity

$$\begin{aligned} S_\theta \partial_\theta Y_n^m &= a_{3nm} Y_{n-1}^m + a_{4nm} Y_{n+1}^m \\ a_{3nm} &= -\lambda_n (n+m)(n+1), \; a_{4nm} = \lambda_n n (n+1-m) \end{aligned} \tag{46}$$

Substituting (46) into (45), and using (15), yields

$$\begin{aligned} \overline{A}_q^m \alpha_{qm} &= a_{3q;-m} \int Y_n^m Y_{q-1}^{-m} d\Omega + a_{4q;-m} \int Y_n^m Y_{q+1}^{-m} d\Omega \\ &= (a_{3q;-m} \beta_{nm} \delta_{n;q-1} + a_{4q;-m} \beta_{nm} \delta_{n;q+1}) \end{aligned} \tag{47}$$

indicating that only the terms for which $q = n+1,\ q = n-1$ are non-vanishing. We finally find the terms

$$\begin{aligned} \overline{A}_{n+1}^m &= a_{5nm} = -\lambda_n(n+1-m)/(n+1) \\ \overline{A}_{n-1}^m &= a_{6nm} = \lambda_n(n+m)/n \end{aligned} \tag{48}$$

Similarly, by multiplying (44) by $\cdot\boldsymbol{B}_q^{-m}$ on both sides and integrating, using the definition of $\boldsymbol{B}_q^{-m}$ given in (13), (14) and the orthogonality relations (15) we find for (44)

$$\overline{B}_n^m = im\kappa_n = ia_{7nm} \tag{49}$$

Thus we derive an expression for $C_\theta \boldsymbol{C}_n^m$, and from (14) follows a similar expression for $C_\theta \boldsymbol{B}_n^m$

$$\begin{aligned} C_\theta \boldsymbol{C}_n^m &= a_{8nm}\boldsymbol{C}_{n+1}^m + a_{9nm}\boldsymbol{C}_{n-1}^m + ia_{7nm}\boldsymbol{B}_n^m \\ C_\theta \boldsymbol{B}_n^m &= a_{8nm}\boldsymbol{B}_{n+1}^m + a_{9nm}\boldsymbol{B}_{n-1}^m - ia_{7nm}\boldsymbol{C}_n^m \\ C_\theta \boldsymbol{C}_{\left\{\boldsymbol{R} \atop \boldsymbol{I}\right\}n}^m &= a_{8nm}\boldsymbol{C}_{\left\{\boldsymbol{R} \atop \boldsymbol{I}\right\}n+1}^m + a_{9nm}\boldsymbol{C}_{\left\{\boldsymbol{R} \atop \boldsymbol{I}\right\}n-1}^m \{\mp\} a_{7nm}\boldsymbol{B}_{\left\{\boldsymbol{I} \atop \boldsymbol{R}\right\}n}^m \\ C_\theta \boldsymbol{B}_{\left\{\boldsymbol{R} \atop \boldsymbol{I}\right\}n}^m &= a_{8nm}\boldsymbol{B}_{\left\{\boldsymbol{R} \atop \boldsymbol{I}\right\}n+1}^m + a_{9nm}\boldsymbol{B}_{\left\{\boldsymbol{R} \atop \boldsymbol{I}\right\}n-1}^m \{\pm\} a_{7nm}\boldsymbol{C}_{\left\{\boldsymbol{I} \atop \boldsymbol{R}\right\}n}^m \\ a_{8nm} &= a_{1nm} + a_{5nm}, \quad a_{9nm} = a_{2nm} + a_{6nm} \end{aligned} \tag{50}$$

where the relation between $C_\theta \boldsymbol{C}_n^m$ and $C_\theta \boldsymbol{B}_n^m$ follows from the definitions in (14). The rules (20) prescribe $C_\theta \boldsymbol{C}_{\left\{\boldsymbol{R} \atop \boldsymbol{I}\right\}n}^m$, $C_\theta \boldsymbol{B}_{\left\{\boldsymbol{R} \atop \boldsymbol{I}\right\}n}^m$.

Applying (50) to (39), yields

$$\begin{aligned} (\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}})\boldsymbol{g}_{\boldsymbol{E}} &= \boldsymbol{g}_{\boldsymbol{E}}^c = \Sigma_{n1}\Big(d_{1n1}^{sc}\Big(a_{8n1}\boldsymbol{C}_{\boldsymbol{I}n+1}^1 + a_{9n1}\boldsymbol{C}_{\boldsymbol{I}n-1}^1 + a_{7n1}\boldsymbol{B}_{\boldsymbol{R}n}^1\Big) \\ &\quad + d_{2n1}^{sc}\Big(a_{8n1}\boldsymbol{B}_{\boldsymbol{R}n+1}^1 + a_{9n1}\boldsymbol{B}_{\boldsymbol{R}n-1}^1 + a_{7n1}\boldsymbol{C}_{\boldsymbol{I}n}^1\Big)\Big) \\ &= \Sigma_{n1}\Big(d_{5n1}^{sc}\boldsymbol{C}_{\boldsymbol{I}n}^1 + d_{6n1}^{sc}\boldsymbol{B}_{\boldsymbol{R}n}^1\Big) \\ (\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}})\boldsymbol{g}_{\boldsymbol{H}} &= \boldsymbol{g}_{\boldsymbol{H}}^c = -\boldsymbol{r}\times\boldsymbol{g}_{\boldsymbol{E}}^c = \Sigma_{n1}\Big(d_{6n1}^{sc}\boldsymbol{C}_{\boldsymbol{R}n}^1 - d_{5n1}^{sc}\boldsymbol{B}_{\boldsymbol{I}n}^1\Big) \\ d_{5nm}^{sc} &= d_{2nm}^{sc}a_{7nm} + d_{1;n-1;m}^{sc}a_{8;n-1;m} + d_{1;n+1;m}^{sc}a_{9;n+1;m} \\ d_{6nm}^{sc} &= d_{1nm}^{sc}a_{7nm} + d_{2;n-1;m}^{sc}a_{8;n-1;m} + d_{2;n+1;m}^{sc}a_{9;n+1;m} \end{aligned} \tag{51}$$

Now the term $\boldsymbol{h}$ in (39) is recast as

$$\begin{aligned} \boldsymbol{h}_{\boldsymbol{E}} &= \boldsymbol{g}_{\boldsymbol{E}}^c - \boldsymbol{g}_{\boldsymbol{E}} = \Sigma_{n1}\Big(d_{7n1}^{sc}\boldsymbol{C}_{\boldsymbol{I}n}^1 + d_{8n1}^{sc}\boldsymbol{B}_{\boldsymbol{R}n}^1\Big) \\ \boldsymbol{h}_{\boldsymbol{H}} &= -\boldsymbol{r}\times\boldsymbol{h}_{\boldsymbol{E}} = \Sigma_{n1}\Big(d_{8n1}^{sc}\boldsymbol{C}_{\boldsymbol{R}n}^1 - d_{7n1}^{sc}\boldsymbol{B}_{\boldsymbol{I}n}^1\Big) \\ d_{7nm}^{sc} &= d_{5nm}^{sc} - d_{1nm}^{sc}, \quad d_{8nm}^{sc} = d_{6nm}^{sc} - d_{2nm}^{sc} \end{aligned} \tag{52}$$

The term corresponding to $(\hat{\boldsymbol{z}}\cdot\boldsymbol{g})\hat{\boldsymbol{p}}$ in (39) leads to longitudinal vector harmonics of type $\boldsymbol{P}_n^m$. At a first glance one is tempted to discard these terms, because we are interested only in transversal waves. But on inspection of (14), (17) it is evident that the associated $\boldsymbol{L}_{nm}$ functions still involve the transversal $\boldsymbol{B}_n^m$. From (14), (19), (20), (46) we find

$$
\begin{aligned}
(\hat{\boldsymbol{z}}\cdot\boldsymbol{C}_n^m)\hat{\boldsymbol{r}} &= -\hat{\boldsymbol{r}}\partial_\psi Y_n^m = -\overline{\partial}_\psi\boldsymbol{P}_n^m = -im\boldsymbol{P}_n^m \\
(\hat{\boldsymbol{z}}\cdot\boldsymbol{B}_n^m)\hat{\boldsymbol{r}} &= -\hat{\boldsymbol{r}}S_\theta\partial_\theta Y_n^m = -a_{3nm}\boldsymbol{P}_{n-1}^m - a_{4nm}\boldsymbol{P}_{n+1}^m \\
(\hat{\boldsymbol{z}}\cdot\boldsymbol{C}^m_{\left\{\boldsymbol{R}\atop\boldsymbol{I}\right\}n})\hat{\boldsymbol{r}} &= -\hat{\boldsymbol{r}}\partial_\psi Y^m_{\left\{\boldsymbol{R}\atop\boldsymbol{I}\right\}n} = \{\pm\}m\boldsymbol{P}^m_{\left\{\boldsymbol{I}\atop\boldsymbol{R}\right\}n} \qquad (53)\\
(\hat{\boldsymbol{z}}\cdot\boldsymbol{B}^m_{\left\{\boldsymbol{R}\atop\boldsymbol{I}\right\}n})\hat{\boldsymbol{r}} &= -\hat{\boldsymbol{r}}S_\theta\partial_\theta Y^m_{\left\{\boldsymbol{R}\atop\boldsymbol{I}\right\}n} = -a_{3nm}\boldsymbol{P}^m_{\left\{\boldsymbol{R}\atop\boldsymbol{I}\right\}n-1} - a_{4nm}\boldsymbol{P}^m_{\left\{\boldsymbol{R}\atop\boldsymbol{I}\right\}n+1} \\
(\hat{\boldsymbol{z}}\cdot\boldsymbol{g}_{\boldsymbol{E}})\hat{\boldsymbol{r}} &= \Sigma_{n1}\left(d^{sc}_{1n1}\hat{\boldsymbol{z}}\cdot\boldsymbol{C}^1_{\boldsymbol{I}n} + d^{sc}_{2n1}\hat{\boldsymbol{z}}\cdot\boldsymbol{B}^1_{\boldsymbol{R}n}\right) = \Sigma_{n1}d^{sc}_{9n1}\boldsymbol{P}^1_{\boldsymbol{R}n} \\
(\hat{\boldsymbol{z}}\cdot\boldsymbol{g}_{\boldsymbol{H}})\hat{\boldsymbol{r}} &= \Sigma_{n1}\left(d^{sc}_{2n1}\hat{\boldsymbol{z}}\cdot\boldsymbol{C}^1_{\boldsymbol{R}n} - d^{sc}_{1n1}\hat{\boldsymbol{z}}\cdot\boldsymbol{B}^1_{\boldsymbol{I}n}\right) = \Sigma_{n1}d^{sc}_{10n1}\boldsymbol{P}^1_{\boldsymbol{I}n} \\
d^{sc}_{9nm} &= -\left(md^{sc}_{1nm} + d^{sc}_{2;n+1;m}a_{3;n+1;m} + d^{sc}_{2;n-1;m}a_{4;n-1;m}\right) \\
d^{sc}_{10nm} &= md^{sc}_{2nm} + d^{sc}_{1;n+1;m}a_{3;n+1;m} + d^{sc}_{1;n-1;m}a_{4;n-1;m}
\end{aligned}
$$

The term $(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T)\boldsymbol{h}(\hat{\boldsymbol{p}})$ in (39) requires a clear distinction of the vectors $\hat{\boldsymbol{p}}$, $\hat{\boldsymbol{r}}_T$. Here we have

$$
(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T)\boldsymbol{h}(\hat{\boldsymbol{p}}) = S_{\theta_T}S_\alpha(C_{\psi_T}C_\beta + S_{\psi_T}S_\beta)\boldsymbol{h}(\hat{\boldsymbol{p}}) + C_{\theta_T}C_\alpha\boldsymbol{h}(\hat{\boldsymbol{p}}) \qquad (54)
$$

The last term involves $C_\alpha\boldsymbol{h}(\hat{\boldsymbol{p}})$, which temporarily will be expressed in terms of $\boldsymbol{r}$ coordinates as $C_\theta\boldsymbol{h}(\hat{\boldsymbol{r}})$. Corresponding to (50)–(52) we have for this term

$$
\begin{aligned}
C_\theta\boldsymbol{h}_{\boldsymbol{E}} &= \Sigma_{n1}\left(d^{sc}_{7n1}C_\theta C^1_{\boldsymbol{I}n} + d^{sc}_{8n1}C_\theta\boldsymbol{B}^1_{\boldsymbol{R}n}\right) \\
&= \Sigma_{n1}\left(d^{sc}_{11n1}\boldsymbol{C}^1_{\boldsymbol{I}n} + d^{sc}_{12n1}\boldsymbol{B}^1_{\boldsymbol{R}n}\right) \\
C_\theta\boldsymbol{h}_{\boldsymbol{H}} &= -C_\theta\boldsymbol{r}\times\boldsymbol{h}_{\boldsymbol{E}} = \Sigma_{n1}\left(d^{sc}_{12n1}\boldsymbol{C}^1_{\boldsymbol{R}n} - d^{sc}_{11n1}\boldsymbol{B}^1_{\boldsymbol{I}n}\right) \qquad (55)\\
d^{sc}_{11nm} &= d^{sc}_{7nm}\left(a_{8;n-1;m} + a_{9;n+1;m}\right) + d^{sc}_{8nm}a_{7nm} \\
d^{sc}_{12nm} &= d^{sc}_{8nm}\left(a_{8;n-1;m} + a_{9;n+1;m}\right) + d^{sc}_{7nm}a_{7nm}
\end{aligned}
$$

In (54), (55) appears also the factor C_{θ_T} which is not affected by the integration in (39), and will be further discussed below.

To evaluate the remaining terms in (54), it is noticed that they involve $e^{\pm i\beta}S_\alpha$, which in terms of $\boldsymbol{r}$ coordinates is expressed as $e^{\pm i\psi}S_\theta$. A little reflection on

$$
e^{\pm i\psi}S_\theta\boldsymbol{C}_n^m, \quad e^{\pm i\psi}S_\theta\boldsymbol{B}_n^m \qquad (56)
$$

reveals that here the azimuthal dependence must involve $m \pm 1$. Exploiting in (14) $\boldsymbol{B} = \hat{\boldsymbol{r}} \times \boldsymbol{C}$, $\boldsymbol{C} = -\hat{\boldsymbol{r}} \times \boldsymbol{B}$, in general we have

$$\begin{aligned} e^{\{\pm\}i\psi} S_\theta \boldsymbol{C}_n^m &= \Sigma_q (A_q^{\left\{ m+1 \atop m-1 \right\}} \boldsymbol{C}_q^{\left\{ m+1 \atop m-1 \right\}} + iB_q^{\left\{ m+1 \atop m-1 \right\}} \boldsymbol{B}_q^{\left\{ m+1 \atop m-1 \right\}}) \\ e^{\{\pm\}i\psi} S_\theta \boldsymbol{B}_n^m &= \Sigma_q (A_q^{\left\{ m+1 \atop m-1 \right\}} \boldsymbol{B}_q^{\left\{ m+1 \atop m-1 \right\}} - iB_q^{\left\{ m+1 \atop m-1 \right\}} \boldsymbol{C}_q^{\left\{ m+1 \atop m-1 \right\}}) \end{aligned} \tag{57}$$

where in (57) A_q^m, B_q^m are different from $\overline{A}_q^m$, $\overline{B}_q^m$ in (44), and the factor i is introduced to simplify other expressions below.

As prescribed by (15), orthogonality requires that the two sides of the equations (57) contain the same factor $e^{i\{m\pm 1\}\psi}$, therefore exponentials with index $\{\pm\}$ must be associated with coefficients $\left\{ m+1 \atop m-1 \right\}$, correspondingly. Consider

$$e^{i\psi} S_\theta \boldsymbol{C}_n^m = \Sigma_q \left(A_q^{m+1} \boldsymbol{C}_q^{m+1} + iB_q^{m+1} \boldsymbol{B}_q^{m+1} \right) \tag{58}$$

Multiplying (58) by $\cdot \boldsymbol{C}_q^{-(m+1)}$, and integrating and exploiting the orthogonality properties in (15), yields

$$\begin{aligned} &\int e^{i\psi} S_\theta \boldsymbol{C}_n^m \cdot \boldsymbol{C}_q^{-(m+1)} d\Omega = \alpha_{q;m+1} A_q^{m+1} \\ &= -\int \left(\hat{\boldsymbol{\theta}} im - \hat{\boldsymbol{\psi}} S_\theta \partial_\theta \right) P_n^m \cdot \left(\hat{\boldsymbol{\theta}} i(m+1) S_\theta^{-1} + \hat{\boldsymbol{\psi}} \partial_\theta \right) P_q^{-(m+1)} d\Omega \\ &= \int \Big(m(m+1) S_\theta P_n^m P_q^{-(m+1)} + S_\theta (S_\theta \partial_\theta P_n^m) S_\theta \partial_\theta P_q^{-(m+1)} \Big) S_\theta^{-2} d\Omega \end{aligned} \tag{59}$$

We exploit formulas from [4] (see p. 401), and [3] (see p. 1325)

$$\begin{aligned} S_\theta P_n^m &= a_{10nm}^+ P_{n-1}^{m+1} + a_{11nm}^+ P_{n+1}^{m+1} = a_{10nm}^- P_{n-1}^{m-1} + a_{11nm}^- P_{n+1}^{m-1} \\ a_{10nm}^+ &= -\lambda_n, \ a_{11nm}^+ = \lambda_n, \ a_{10nm}^- = \lambda_n (n+m)(n+m-1) \\ a_{11nm}^- &= -\lambda_n (n-m+1)(n-m+2) \end{aligned} \tag{60}$$

We choose for (59) the formula in (60) with the upper index needed in the orthogonality relations (15). Also using (46) we find

$$\begin{aligned} \alpha_{q;m+1} A_q^{m+1} &= \int \Big(a_{12nm}^+ Q_{n-1;q} + a_{13nm}^+ Q_{n+1;q} + a_{14nm}^+ Q_{n-2;q-1} \\ &\quad + a_{15nm}^+ Q_{n;q-1} + a_{16mn}^+ Q_{n+2;q-1} + a_{17nm}^+ Q_{n-2;q+1} \\ &\quad + a_{18nm}^+ Q_{n;q+1} + a_{19nm}^+ Q_{n+2;q+1} \Big) S_\theta^{-2} d\Omega \\ a_{12nm}^+ &= m(m+1) a_{10nm}^+, \ a_{13nm}^+ = m(m+1) a_{11nm}^+ \end{aligned}$$

$$
\begin{aligned}
a^+_{14nm} &= a_{3nm}a^+_{10;n-1;m}a_{3;q;-(m+1)} \\
a^+_{15nm} &= \left(a_{3nm}a^+_{11;n-1;m} + a_{4nm}a^+_{10;n+1;m}\right) a_{3q;-(m+1)} \\
a^+_{16nm} &= a_{4nm}a^+_{11;n+1;m}a_{3q;-(m+1)} \\
a^+_{17nm} &= a_{3nm}a^+_{10;n-1;m}a_{4q;-(m+1)} \\
a^+_{18nm} &= a^+_{15nm}a_{4q;-(m+1)}/a_{3q;-(m+1)} \\
a^+_{19nm} &= a_{4nm}a^+_{11;n+1;m}a_{4q;-(m+1)},\ Q_{ab} = P^{m+1}_a P^{-(m+1)}_b
\end{aligned} \tag{61}
$$

Using the last integral in (15) we finally derive

$$
\begin{aligned}
A^{m+1}_{n-1} &= \left(a^+_{12nm} + a^+_{14nm} + a^+_{18nm}\right)\mu_{m+1}/\alpha_{n-1;m+1} \\
A^{m+1}_{n+1} &= \left(a^+_{13nm} + a^+_{15nm} + a^+_{19nm}\right)\mu_{m+1}/\alpha_{n+1;m+1} \\
A^{m+1}_{n+3} &= a^+_{16nm}\mu_{m+1}/\alpha_{n+3;m+1} \\
A^{m+1}_{n-3} &= a^+_{17nm}\mu_{m+1}/\alpha_{n-3;m+1},\ A^{m+1}_n = A^{m+1}_{n\pm2} = 0
\end{aligned} \tag{62}
$$

Similarly, multiplying both sides of (57) by $\boldsymbol{B}^{-(m+1)}_q$ and exploiting the orthogonality properties in (15), yields similarly to (59)

$$
\begin{aligned}
&\int e^{i\psi}S_\theta \boldsymbol{C}^m_n \cdot \boldsymbol{B}^{-(m+1)}_q d\Omega = \alpha_{q;m+1} i B^{m+1}_q \\
=&\int \left(\hat{\boldsymbol{\theta}} im - \hat{\boldsymbol{\psi}} S_\theta\partial_\theta\right) P^m_n \cdot \left(\hat{\boldsymbol{\theta}}\partial_\theta - i\hat{\boldsymbol{\psi}}(m+1)S^{-1}_\theta\right) P^{-(m+1)}_q d\Omega \\
=&\ i\int\left(mS_\theta P^m_n S_\theta\partial_\theta P^{-(m+1)}_q + (m+1)S_\theta P^{-(m+1)}_q S_\theta\partial_\theta P^m_n\right) S^{-2}_\theta d\Omega
\end{aligned} \tag{63}
$$

Similarly to (61)

$$
\begin{aligned}
\alpha_{q;m+1}B^{m+1}_q =& \int \Big(a^+_{20nm}Q_{n-1;q-1} + a^+_{21nm}Q_{n-1;q+1} + a^+_{22nm}Q_{n+1;q-1} \\
&+ a^+_{23nm}Q_{n+1;q+1} + a^+_{24nm}U_{n-1;q-1} + a^+_{25nm}U_{n-1;q+1} \\
&+ a^+_{26nm}U_{n+1;q-1} + a^+_{27nm}U_{n+1;q+1}\Big) S^{-2}_\theta d\Omega
\end{aligned}
$$

$$
\begin{aligned}
a^+_{20nm} &= ma^+_{10nm}a_{3;q;-(m+1)},\ a^+_{21nm} = ma^+_{10nm}a_{4q;-(m+1)} \\
a^+_{22nm} &= ma^+_{11nm}a_{3q;-(m+1)},\ a^+_{23nm} = ma^+_{11nm}a_{4q;-(m+1)} \\
a^+_{24nm} &= (m+1)a_{3nm}a^+_{10q;-(m+1)} \\
a^+_{25nm} &= (m+1)a_{3nm}a^+_{11q;-(m+1)} \\
a^+_{26nm} &= (m+1)a_{4nm}a^+_{10q;-(m+1)} \\
a^+_{27nm} &= (m+1)a_{4nm}a^+_{11q;-(m+1)},\ U_{ab} = P^m_a P^{-m}_b
\end{aligned} \tag{64}
$$

Similarly to (62) we finally find

$$\begin{aligned}
B_n^{m+1} &= \Big((a^+_{20nm} + a^+_{23nm})\mu_{m+1} + (a^+_{24nm} + a^+_{27nm})\mu_m\Big)\,/\alpha_{n;m+1} \\
B_{n-2}^{m+1} &= \Big(a^+_{21nm}\mu_{m+1} + a^+_{25nm}\mu_m\Big)\,/\alpha_{n-2;m+1} \\
B_{n+2}^{m+1} &= \Big(a^+_{22nm}\mu_{m+1} + a^+_{26nm}\mu_m\Big)\,/\alpha_{n+2;m+1} \\
B_{n\pm1}^{m+1} &= B_{n\pm3}^{m+1} = 0
\end{aligned} \tag{65}$$

Consider now (57) with index $\{-\}$, associated with $\{m-1\}$, and modify (58), (59) accordingly. Use (60) with the formula for $m-1$, and consistently replace superscript "+" by "−". Consistently modify (61)–(65), including the exchange of the exponent $i\psi$ by $-i\psi$. This yields the analogs of (62), (65). Finally substituting (62), (65) in (57) we get

$$\begin{aligned}
e^{\{\pm\}i\psi} S_\theta \boldsymbol{C}_n^m &= \Sigma_{\nu \triangleright n\pm3}\big(A_\nu^{\left\{ {m+1 \atop m-1} \right\}} \boldsymbol{C}_\nu^{\left\{ {m+1 \atop m-1} \right\}} + iB_\nu^{\left\{ {m+1 \atop m-1} \right\}} \boldsymbol{B}_\nu^{\left\{ {m+1 \atop m-1} \right\}}\big) \\
e^{\{\pm\}i\psi} S_\theta \boldsymbol{B}_n^m &= \Sigma_{\nu \triangleright n\pm3}\big(A_\nu^{\left\{ {m+1 \atop m-1} \right\}} \boldsymbol{B}_\nu^{\left\{ {m+1 \atop m-1} \right\}} - iB_\nu^{\left\{ {m+1 \atop m-1} \right\}} \boldsymbol{C}_\nu^{\left\{ {m+1 \atop m-1} \right\}}\big)
\end{aligned} \tag{66}$$

where in (66) $\Sigma_{\nu \triangleright n\pm3}$ denotes summing over ν from $\nu = n-3$ to $\nu = n+3$.

By combining the two equations (66) for $e^{i\psi}$, $e^{-i\psi}$ (again representing results in terms coordinate system $\boldsymbol{r}$), the terms involving C_β, S_β in (54) are recast in vector spherical harmonics

$$\begin{aligned}
\left\{ {C_\psi \atop iS_\psi} \right\} S_\theta \boldsymbol{C}_n^m &= \frac{1}{2}\Sigma_{\nu \triangleright n\pm3}\Big(A_\nu^{m+1}\boldsymbol{C}_\nu^{m+1}\{\pm\}A_\nu^{m-1}\boldsymbol{C}_\nu^{m-1} \\
&\qquad + iB_\nu^{m+1}\boldsymbol{B}_\nu^{m+1}\{\pm\}iB_\nu^{m-1}\boldsymbol{B}_\nu^{m-1}\Big) \\
&= \frac{1}{2}\Sigma_{\nu \triangleright n\pm3}^{\{\pm\};\mu \triangleright m\pm1}\left(A_\nu^\mu \boldsymbol{C}_\nu^\mu + iB_\nu^\mu \boldsymbol{B}_\nu^\mu\right) \\
\left\{ {C_\psi \atop iS_\psi} \right\} S_\theta \boldsymbol{B}_n^m &= \frac{1}{2}\Sigma_{\nu \triangleright n\pm3}^{\{\pm\};\mu \triangleright m\pm1}\left(A_\nu^\mu \boldsymbol{B}_\nu^\mu - iB_\nu^\mu \boldsymbol{C}_\nu^\mu\right), \ \Sigma_{\nu \triangleright n\pm3} = \sum_{\nu=n-3}^{n+3}
\end{aligned} \tag{67}$$

In (67) the compaction effected in the notation $\Sigma_{\nu \triangleright n\pm3}^{\{\pm\};\mu \triangleright m\pm1}$ still contains all the ingredients: it prescribes that we sum over ν according to $\Sigma_{\nu \triangleright n\pm3}$, and sum over μ for the values $m=-1,\ m=1$, and assign the sign "+" or "−" between the summands according to the sign in the superscript.

According to (20), (67) yields

$$C_\psi S_\theta \boldsymbol{C}^m_{\left\{ {\boldsymbol{R} \atop \boldsymbol{I}} \right\} n} = \frac{1}{2}\Sigma_{\nu \triangleright n\pm3}^{+;\mu \triangleright m\pm1}\big(A_\nu^\mu \boldsymbol{C}^\mu_{\left\{ {\boldsymbol{R} \atop \boldsymbol{I}} \right\} \nu}\{\mp\}B_\nu^\mu \boldsymbol{B}^\mu_{\left\{ {\boldsymbol{I} \atop \boldsymbol{R}} \right\} \nu}\big)$$

$$
\begin{aligned}
S_\psi S_\theta \boldsymbol{C}^m_{\left\{\begin{smallmatrix}\boldsymbol{R}\\ \boldsymbol{I}\end{smallmatrix}\right\}n} &= \frac{1}{2}\Sigma^{-;\mu\triangleright m\pm 1}_{\nu\triangleright n\pm 3}(\{\pm\}A^\mu_\nu \boldsymbol{C}^\mu_{\left\{\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{R}\end{smallmatrix}\right\}\nu} + B^\mu_\nu \boldsymbol{B}^\mu_{\left\{\begin{smallmatrix}\boldsymbol{R}\\ \boldsymbol{I}\end{smallmatrix}\right\}\nu})\\
C_\psi S_\theta \boldsymbol{B}^m_{\left\{\begin{smallmatrix}\boldsymbol{R}\\ \boldsymbol{I}\end{smallmatrix}\right\}n} &= \frac{1}{2}\Sigma^{+;\mu\triangleright m\pm 1}_{\nu\triangleright n\pm 3}(A^\mu_\nu \boldsymbol{B}^\mu_{\left\{\begin{smallmatrix}\boldsymbol{R}\\ \boldsymbol{I}\end{smallmatrix}\right\}\nu} \{\pm\} B^\mu_\nu \boldsymbol{C}^\mu_{\left\{\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{R}\end{smallmatrix}\right\}\nu}) \qquad (68)\\
S_\psi S_\theta \boldsymbol{B}^m_{\left\{\begin{smallmatrix}\boldsymbol{R}\\ \boldsymbol{I}\end{smallmatrix}\right\}n} &= \frac{1}{2}\Sigma^{-;\mu\triangleright m\pm 1}_{\nu\triangleright n\pm 3}(\{\pm\}A^\mu_\nu \boldsymbol{B}^\mu_{\left\{\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{R}\end{smallmatrix}\right\}\nu} - B^\mu_\nu \boldsymbol{C}^\mu_{\left\{\begin{smallmatrix}\boldsymbol{R}\\ \boldsymbol{I}\end{smallmatrix}\right\}\nu})
\end{aligned}
$$

Apply now (68) to $\boldsymbol{h}$ defined in (39), (52), this yields

$$
\begin{aligned}
C_\psi S_\theta \boldsymbol{h}_{\boldsymbol{E}} &= \Sigma\left(d^{+;sc}_{13\nu\mu}\boldsymbol{C}^\mu_{\boldsymbol{I}\nu} + d^{+;sc}_{14\nu\mu}\boldsymbol{B}^\mu_{\boldsymbol{R}\nu}\right)\\
C_\psi S_\theta \boldsymbol{h}_{\boldsymbol{H}} &= \Sigma\left(d^{+;sc}_{14\nu\mu}\boldsymbol{C}^\mu_{\boldsymbol{R}\nu} - d^{+;sc}_{13\nu\mu}\boldsymbol{B}^\mu_{\boldsymbol{I}\nu}\right)\\
S_\psi S_\theta \boldsymbol{h}_{\boldsymbol{E}} &= \Sigma\left(-d^{-;sc}_{13\nu\mu}\boldsymbol{C}^\mu_{\boldsymbol{R}\nu} + d^{-;sc}_{14\nu\mu}\boldsymbol{B}^\mu_{\boldsymbol{I}\nu}\right)\\
S_\psi S_\theta \boldsymbol{h}_{\boldsymbol{H}} &= \Sigma\left(d^{-;sc}_{14\nu\mu}\boldsymbol{C}^\mu_{\boldsymbol{I}\nu} - d^{-;sc}_{13\nu\mu}\boldsymbol{B}^\mu_{\boldsymbol{R}\nu}\right) \qquad (69)\\
d^{+;sc}_{13\nu\mu} &= \frac{1}{2}\left(A^\mu_\nu \Lambda d^{sc}_{7\nu\mu} + B^\mu_\nu \Lambda d^{sc}_{8\nu\mu}\right)\\
d^{+;sc}_{14\nu\mu} &= \frac{1}{2}\left(A^\mu_\nu \Lambda d^{sc}_{8\nu\mu} + B^\mu_\nu \Lambda d^{sc}_{7\nu\mu}\right)
\end{aligned}
$$

In (69) and subsequently, the sign notation in the superscript, appearing in (67), (68), is delegated to the coefficients so that $\Sigma = \Sigma_{n1}\Sigma^{\mu\triangleright m\pm 1}_{\nu\triangleright n\pm 3}$ can be used for adding or subtracting summands. Also note that Λ is an *operator* changing the indices of the *immediately following* coefficient, e.g., $\Lambda\{\nu,\mu\} = \{n,1\}$. Thus the notation Λ acts similarly to $\delta_{\nu n}\delta_{\mu 1}$, but its position antecedent to the immediately following operand is crucial. Using Λ means that the relevant term $f(\nu,\mu)$ does not participate in the summation indicated by $\mu \triangleright m \pm 1,\ \nu \triangleright n \pm 3$, although the term appears under the summation sign.

By now we have derived explicit expressions for all the terms multiplying $\beta^{(1)}$ in (39), in terms of spherical vector harmonics.

These will now be combined, once again reverting to the appropriate original coordinates $\hat{\boldsymbol{p}},\ \hat{\boldsymbol{r}}_T$

$$
\begin{aligned}
\boldsymbol{f}_{\boldsymbol{E}} &= \Sigma\Big(d^{sc}_{15\nu\mu}\boldsymbol{C}^\mu_{\boldsymbol{I}\nu} + d^{sc}_{16\nu\mu}\boldsymbol{B}^\mu_{\boldsymbol{R}\nu} + d^{sc}_{17\nu\mu}\boldsymbol{C}^\mu_{\boldsymbol{R}\nu} + d^{sc}_{18\nu\mu}\boldsymbol{B}^\mu_{\boldsymbol{I}\nu} + d^{sc}_{19\nu\mu}\boldsymbol{P}^\mu_{\boldsymbol{R}\nu}\Big)\\
\boldsymbol{f}_{\boldsymbol{H}} &= \Sigma\Big(-d^{sc}_{15\nu\mu}\boldsymbol{B}^\mu_{\boldsymbol{I}\nu} + d^{sc}_{16\nu\mu}\boldsymbol{C}^\mu_{\boldsymbol{R}\nu} - d^{sc}_{17\nu\mu}\boldsymbol{B}^\mu_{\boldsymbol{R}\nu} + d^{sc}_{18\nu\mu}\boldsymbol{C}^\mu_{\boldsymbol{I}\nu} + d^{sc}_{20\nu\mu}\boldsymbol{P}^\mu_{\boldsymbol{I}\nu}\Big)\\
\Sigma &= \Sigma_{n1}\Sigma^{\mu\triangleright m\pm 1}_{\nu\triangleright n\pm 3},\ d^{sc}_{15\nu\mu} = \Lambda\left(d^{sc}_{3\nu\mu} - d^{sc}_{5\nu\mu} + Dd^{sc}_{11\nu\mu}\right)\Lambda + Ed^{+;sc}_{13\nu\mu}\\
d^{sc}_{16\nu\mu} &= \Lambda\left(d^{sc}_{4\nu\mu} - d^{sc}_{6\nu\mu} + Dd^{sc}_{12\nu\mu}\right)\Lambda + Ed^{+;sc}_{14\nu\mu} \qquad (70)
\end{aligned}
$$

$$d^{sc}_{17\nu\mu} = -Fd^{-;sc}_{13\nu\mu},\ d^{sc}_{18\nu\mu}=Fd^{-;sc}_{14\nu\mu},\ d^{sc}_{19\nu\mu}=\Lambda d^{sc}_{9\nu\mu}\Lambda,\ d^{sc}_{20\nu\mu}=\Lambda d^{sc}_{10\nu\mu}\Lambda$$
$$D = iK_{ex}C_{\theta_T},\ E = iK_{ex}S_{\theta_T}C_{\psi_T},\ F = iK_{ex}S_{\theta_T}S_{\psi_T}$$

In (70) the operator Λ appears preceding and following terms as in $\Lambda d^{sc}_{9\nu\mu}\Lambda$, indicating that we have both $\Lambda d^{sc}_{9\nu\mu}$ and $\Lambda \boldsymbol{P}^{\mu}_{\boldsymbol{R}\nu}$to consider.

Inspecting (16), (39), and noting that in (70) K_{ex}, S_{θ_T}, C_{θ_T}, S_{ψ_T}, C_{ψ_T}, embodied in $D = iK_{ex}C_{\theta_T},\ E = iK_{ex}S_{\theta_T}S_{\psi_T},\ F = iK_{ex}S_{\theta_T}S_{\psi_T}$, are not involved in the integration, we get

$$\begin{aligned}
\left\{\begin{matrix}\boldsymbol{E_f}(K_{ex}\hat{\boldsymbol{r}}_T\\ \boldsymbol{H_f}(K_{ex}\hat{\boldsymbol{r}}_T\end{matrix}\right\} &= \left\{\begin{matrix}+\overline{e}\\ -\overline{h}\end{matrix}\right\}\frac{1}{2\pi}\int e^{iK_{ex}\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T}\beta^{(1)}\boldsymbol{f}_{\left\{\begin{matrix}\boldsymbol{E}\\ \boldsymbol{H}\end{matrix}\right\}}(\hat{\boldsymbol{p}})d\Omega_{\hat{\boldsymbol{p}}}\\
\boldsymbol{E_f} &= \overline{e}\beta^{(1)}\Sigma i^{\nu}\left(d^{sc}_{15\nu\mu}\boldsymbol{M}^{scT}_{\boldsymbol{I}\nu\mu} - id^{sc}_{16\nu\mu}\boldsymbol{N}^{scT}_{\boldsymbol{R}\nu\mu} + d^{sc}_{17\nu\mu}\boldsymbol{M}^{scT}_{\boldsymbol{R}\nu\mu}\right.\\
&\quad\left. - id^{sc}_{18\nu\mu}\boldsymbol{N}^{scT}_{\boldsymbol{I}\nu\mu} - id^{sc}_{19\nu\mu}\boldsymbol{L}^{scT}_{\boldsymbol{R}\nu\mu}\right)\\
&= \overline{e}\beta^{(1)}\Sigma i^{\nu}\left(d^{sc}_{21\nu\mu}\boldsymbol{C}^{\mu}_{\boldsymbol{R}\nu} + d^{sc}_{22\nu\mu}\boldsymbol{C}^{\mu}_{\boldsymbol{I}\nu} + d^{sc}_{23\nu\mu}\boldsymbol{B}^{\mu}_{\boldsymbol{R}\nu}\right.\\
&\quad\left. + d^{sc}_{24\nu\mu}\boldsymbol{B}^{\mu}_{\boldsymbol{I}\nu} + d^{sc}_{25\nu\mu}\boldsymbol{P}^{\mu}_{\boldsymbol{R}\nu} + d^{sc}_{26\nu\mu}\boldsymbol{P}^{\mu}_{\boldsymbol{I}\nu}\right) \qquad (71)\\
\boldsymbol{H_f} &= -\overline{h}\beta^{(1)}\Sigma i^{\nu}\left(id^{sc}_{15\nu\mu}\boldsymbol{N}^{scT}_{\boldsymbol{I}\nu\mu} + d^{sc}_{16\nu\mu}\boldsymbol{M}^{scT}_{\boldsymbol{R}\nu\mu}\right.\\
&\quad\left. + id^{sc}_{17\nu\mu}\boldsymbol{N}^{scT}_{\boldsymbol{R}\nu\mu} + d^{sc}_{18\nu\mu}\boldsymbol{M}^{scT}_{\boldsymbol{I}\nu\mu} - id^{sc}_{20\nu\mu}\boldsymbol{L}^{scT}_{\boldsymbol{I}\nu\mu}\right)\\
&= -\overline{h}\beta^{(1)}\Sigma i^{\nu}\left(d^{sc}_{27\nu\mu}\boldsymbol{C}^{\mu}_{\boldsymbol{R}\nu} + d^{sc}_{28\nu\mu}\boldsymbol{C}^{\mu}_{\boldsymbol{I}\nu} + d^{sc}_{29\nu\mu}\boldsymbol{B}^{\mu}_{\boldsymbol{R}\nu}\right.\\
&\quad\left. + d^{sc}_{30\nu\mu}\boldsymbol{B}^{\mu}_{\boldsymbol{I}\nu} + d^{sc}_{31\nu\mu}\boldsymbol{P}^{\mu}_{\boldsymbol{R}\nu} + d^{sc}_{32\nu\mu}\boldsymbol{P}^{\mu}_{\boldsymbol{I}\nu}\right)
\end{aligned}$$

where in (71) the coefficients are related to previous ones by

$$\begin{aligned}
d^{sc}_{21\nu\mu} &= d^{sc}_{17\nu\mu}J^{K_{ex}}_{h\nu1}, \qquad d^{sc}_{22\nu\mu} = d^{sc}_{15\nu\mu}J^{K_{ex}}_{h\nu1}\\
d^{sc}_{23\nu\mu} &= -i\left(d^{sc}_{16\nu\mu}J^{K_{ex}}_{h\nu3} + d^{sc}_{19\nu\mu}J^{K_{ex}}_{h\nu5}\right)\\
d^{sc}_{24\nu\mu} &= -id^{sc}_{18\nu\mu}J^{K_{ex}}_{h\nu3},\ d^{sc}_{25\nu\mu} = -i\left(d^{sc}_{16\nu\mu}J^{K_{ex}}_{h\nu2} + d^{sc}_{19\nu\mu}J^{K_{ex}}_{h\nu4}\right) \qquad (72)\\
d^{sc}_{26\nu\mu} &= -id^{sc}_{18\nu\mu}J^{K_{ex}}_{h\nu2},\ d^{sc}_{27\nu\mu} = d^{sc}_{16\nu\mu}J^{K_{ex}}_{h\nu1},\ d^{sc}_{28\nu\mu} = d^{sc}_{18\nu\mu}J^{K_{ex}}_{h\nu1}\\
d^{sc}_{29\nu\mu} &= id^{sc}_{17\nu\mu}J^{K_{ex}}_{h\nu3}, \quad d^{sc}_{30\nu\mu} = i\left(d^{sc}_{15\nu\mu}J^{K_{ex}}_{h\nu3} - d^{sc}_{20\nu\mu}J^{K_{ex}}_{h\nu5}\right)\\
d^{sc}_{31\nu\mu} &= id^{sc}_{17\nu\mu}J^{K_{ex}}_{h\nu2}, \quad d^{sc}_{32\nu\mu} = i\left(d^{sc}_{15\nu\mu}J^{K_{ex}}_{h\nu2} - d^{sc}_{20\nu\mu}J^{K_{ex}}_{h\nu4}\right)
\end{aligned}$$

Note that in (71) $\boldsymbol{E_f}$, $\boldsymbol{H_f}$ are functions of $\boldsymbol{r}_T$, evaluated at the boundary $r_T = R$, therefore the vector spherical harmonics are functions of $\hat{\boldsymbol{r}}_T$.

With this we have finished the long and arduous task of calculating the velocity-dependent terms appearing in (39), but the coefficients in

(72) are still dependent on S_{θ_T}, C_{θ_T}, S_{ψ_T}, C_{ψ_T}, and not ready yet for the calculation of the scattering coefficients. This will be discussed in the next section.

5. CALCULATION OF THE SCATTERING COEFFICIENTS

We got now all the ingredients necessary for evaluating the scattering coefficients. According to (5)–(8), at the boundary only the tangential components $\boldsymbol{E_{f\|}}$, $\boldsymbol{H_{f\|}}$ in (71) are considered. Due to the spherical geometry of our present problem, we have to consider

$$\begin{aligned}
\boldsymbol{E_{f\|}} &= \overline{e}\beta^{(1)}\Sigma i^{\nu}\left(d^{sc}_{21\nu\mu}\boldsymbol{B}^{\mu}_{\boldsymbol{R}\nu} + d^{sc}_{22\nu\mu}\boldsymbol{B}^{\mu}_{\boldsymbol{I}\nu} - d^{sc}_{23\nu\mu}\boldsymbol{C}^{\mu}_{\boldsymbol{R}\nu} - d^{sc}_{24\nu\mu}\boldsymbol{C}^{\mu}_{\boldsymbol{I}\nu}\right)\\
\boldsymbol{H_{f\|}} &= -\overline{h}\beta^{(1)}\Sigma i^{\nu}\left(d^{sc}_{27\nu\mu}\boldsymbol{B}^{\mu}_{\boldsymbol{R}\nu} + d^{sc}_{28\nu\mu}\boldsymbol{B}^{\mu}_{\boldsymbol{I}\nu} - d^{sc}_{29\nu\mu}\boldsymbol{C}^{\mu}_{\boldsymbol{R}\nu} - d^{sc}_{30\nu\mu}\boldsymbol{C}^{\mu}_{\boldsymbol{I}\nu}\right)\\
\boldsymbol{E_{f\|}} &= \hat{\boldsymbol{r}}_T \times \boldsymbol{E_f}, \quad \boldsymbol{H_{f\|}} = \hat{\boldsymbol{r}}_T \times \boldsymbol{H_f} \qquad (73)
\end{aligned}$$

The longitudinal terms $\boldsymbol{P}^{\mu}_{\nu}$ of (71) do not feature.

In order to solve the boundary value problem, orthogonality properties must used. Hence in (73) the coefficients involving S_{θ_T}, C_{θ_T}, S_{ψ_T}, C_{ψ_T} times the involved vector spherical harmonics, must be recast in terms of vector spherical harmonics once again. All the necessary formulas have already been developed above, and we need to tend to the details. This operation will once again branch out the summation range.

Exploiting results from (50), (68), (70)–(73), $\boldsymbol{E_{f\|}}$ in (73) is recast as

$$\begin{aligned}
\boldsymbol{E_{f\|}} &= \overline{e}\beta^{(1)}\Sigma'\left(d^{sc}_{33\nu'\mu'}\boldsymbol{B}^{\mu'}_{\boldsymbol{I}\nu'} - d^{sc}_{34\nu'\mu'}\boldsymbol{C}^{\mu'}_{\boldsymbol{R},\nu'}\right)\\
d^{sc}_{33\nu'\mu'} &= A^{\mu'}_{\nu'}\left(d^{+;sc}_{35\nu'\mu'} - d^{-;sc}_{35\nu'\mu'}\right) + \Lambda'\left[i^{\nu'}J^{K_{ex}}_{h\nu'1}\Lambda\left(d^{sc}_{3\nu'\mu'} - d^{sc}_{5\nu'\mu'}\right)\Lambda\right]\Lambda'\\
&\quad + d^{sc}_{36;\nu'-1;\mu'}\Lambda' a_{8;\nu'-1;\mu'}\Lambda' + d^{sc}_{36;\nu'+1;\mu'}\Lambda' a_{9;\nu'+1;\mu'}\Lambda'\\
&\quad - d^{sc}_{37\nu'\mu'}\Lambda' a_{7\nu'\mu'}\Lambda' + B^{\mu'}_{\nu'}\left(d^{+;sc}_{38\nu'\mu'} - d^{-;sc}_{38\nu'\mu'}\right)\\
d^{sc}_{34\nu'\mu'} &= B^{\mu'}_{\nu'}\left(d^{+;sc}_{35\nu'\mu'} - d^{-;sc}_{35\nu'\mu'}\right) + d^{sc}_{36\nu'\mu'}\Lambda' a_{7\nu'\mu'}\Lambda'\\
&\quad + i\Lambda'\left[i^{\nu'}J^{K_{ex}}_{h\nu'3}\Lambda\left(d^{sc}_{6\nu'\mu'} - d^{sc}_{4\nu'\mu'}\right)\Lambda - i^{\nu'}J^{K_{ex}}_{h\nu'5}\Lambda d^{sc}_{9\nu'\mu'}\Lambda\right]\Lambda'\\
&\quad - d^{sc}_{37;\nu'-1;\mu'}\Lambda' a_{8;\nu'-;\mu'}\Lambda' - d^{sc}_{37;\nu'+1;\mu'}\Lambda' a_{9;\nu'+1;\mu'}\Lambda' \qquad (74)\\
&\quad + A^{\mu'}_{\nu'}\left(d^{+;sc}_{38\nu'\mu'} - d^{-;sc}_{38\nu'\mu'}\right)\\
d^{\{\pm\};sc}_{35\nu'\mu'} &= i\frac{1}{2}K_{ex}\Lambda'\left(i^{\nu'}d^{\{\pm\};sc}_{13\nu'\mu'}J^{K_{ex}}_{h\nu'1}\right)
\end{aligned}$$

$$
\begin{aligned}
d^{sc}_{36\nu'\mu'} &= iK_{ex}\Lambda'\left[i^{\nu'}J^{K_{ex}}_{h\nu'1}(\Lambda d^{sc}_{11\nu'\mu'}\Lambda)\right]\Lambda' \\
d^{sc}_{37\nu'\mu'} &= -K_{ex}\Lambda'\left[i^{\nu'}J^{K_{ex}}_{h\nu'3}(\Lambda d^{sc}_{12\nu'\mu'}\Lambda)\right]\Lambda' \\
d^{\{\pm\};sc}_{38\nu'\mu'} &= \frac{1}{2}K_{ex}\Lambda'\left(i^{\nu'}d^{\{\pm\};sc}_{14\nu'\mu'}J^{K_{ex}}_{h\nu'3}\right)
\end{aligned}
$$

In (74) the notation $\Sigma' = \Sigma_{\nu\mu}\Sigma^{\mu'\triangleright\mu\pm1}_{\nu'\triangleright\nu\pm3}$ denotes that for each value ν, μ we apply once more the scheme prescribed by Σ. Similarly to (69), in (74) we have $\Lambda'\{\nu',\mu'\} = \{\nu,\mu\}$. When we have an expression containing two operators, we start from the outermost operators, and continue to the inner operators, e.g., $\Lambda'[q_{1\nu'\mu'}(\Lambda q_{2\nu'\mu'}\Lambda)]\Lambda'\boldsymbol{q}_{3\nu'\mu'} = q_{1\nu\mu}q_{2n1}\boldsymbol{q}_{3n1}$. Also note that if the operator can only be applied once, hence for example $\Lambda'\Lambda' q_{1\nu'\mu'} = q_{1\nu\mu}$.

Similarly to (74), $\boldsymbol{H}_{\boldsymbol{f}\|}$ in (73) is recast as

$$
\begin{aligned}
\boldsymbol{H}_{\boldsymbol{f}\|} &= -\overline{h}\beta^{(1)}\Sigma'\left(d^{sc}_{39\nu'\mu'}\boldsymbol{B}^{\mu'}_{\boldsymbol{R}\nu'} - d^{sc}_{40\nu'\mu'}\boldsymbol{C}^{\mu'}_{\boldsymbol{I}\nu'}\right) \\
d^{sc}_{39\nu'\mu'} &= \Lambda'\left[i^{\nu'}J^{K_{ex}}_{h\nu'1}\Lambda\left(d^{sc}_{4\nu'\mu'} - d^{sc}_{6\nu'\mu'}\right)\Lambda - d^{sc}_{41;\nu'-1;\mu'}\Lambda' a_{8;\nu'-1;\mu'}\right]\Lambda' \\
&\quad - d^{sc}_{41;\nu'+1;\mu'}\Lambda' a_{9;\nu'+1;\mu'}\Lambda' + A^{\mu'}_{\nu'}\left(d^{-;sc}_{42\nu'\mu'} - d^{+;sc}_{42\nu'\mu'}\right) \\
&\quad + B^{\mu'}_{\nu'}\left(d^{-;sc}_{43\nu'\mu'} - d^{+;sc}_{43\nu'\mu'}\right) - d^{sc}_{44\nu'\mu'}\Lambda' a_{7\nu'\mu'}\Lambda' \\
d^{sc}_{40\nu'\mu'} &= d^{sc}_{41\nu'\mu'}\Lambda' a_{7\nu'\mu'}\Lambda' + B^{\mu'}_{\nu'}\left(d^{+;sc}_{42\nu'\mu'} - d^{-;sc}_{42\nu'\mu'}\right) \\
&\quad + A^{\mu'}_{\nu'}\left(d^{+;sc}_{43\nu'\mu'} - d^{-;sc}_{43\nu'\mu'}\right) + i\Lambda'\left[i^{\nu'}J^{K_{ex}}_{h\nu'3}\Lambda\left(d^{sc}_{3\nu'\mu'} - d^{sc}_{5\nu'\mu'}\right)\Lambda\right. \\
&\quad \left. - i^{\nu'}J^{K_{ex}}_{h\nu'5}\Lambda d^{sc}_{10\nu'\mu'}\Lambda\right]\Lambda' \qquad (75) \\
&\quad + d^{sc}_{44;\nu'-1;\mu'}\Lambda' a_{8;\nu'-1;\mu'}\Lambda' + d^{sc}_{44;\nu'+1;\mu'}\Lambda' a_{9;\nu'+1;\mu'}\Lambda' \\
d^{sc}_{41\nu'\mu'} &= iK_{ex}\Lambda'\left[i^{\nu'}J^{K_{ex}}_{h\nu'1}(\Lambda d^{sc}_{12\nu'\mu'}\Lambda)\right]\Lambda' \\
d^{\{\pm\};sc}_{42\nu'\mu'} &= i\frac{1}{2}K_{ex}\Lambda'\left(i^{\nu'}d^{\{\pm\};sc}_{14\nu'\mu'}J^{K_{ex}}_{h\nu'1}\right) \\
d^{\{\pm\};sc}_{43\nu'\mu'} &= \frac{1}{2}K_{ex}\Lambda'\left(i^{\nu'}d^{\{\pm\};sc}_{13\nu'\mu'}J^{K_{ex}}_{h\nu'3}\right) \\
d^{sc}_{44\nu'\mu'} &= K_{ex}\Lambda'\left[i^{\nu'}J^{K_{ex}}_{h\nu'3}(\Lambda d^{sc}_{11\nu'\mu'}\Lambda)\right]\Lambda'
\end{aligned}
$$

To accommodate to the summation conventions above, we recast (28) as

$$
\begin{aligned}
\boldsymbol{E}_{exT} &= \overline{e}\Sigma'\left(d^{ex}_{45\nu'\mu'}\boldsymbol{C}^{\mu'}_{\boldsymbol{I}\nu'} + d^{ex}_{46\nu'\mu'}\boldsymbol{B}^{\mu'}_{\boldsymbol{R}\nu'} + d^{ex}_{47\nu'\mu'}\boldsymbol{P}^{\mu'}_{\boldsymbol{R}\nu'}\right) \\
\boldsymbol{H}_{exT} &= -\overline{h}\Sigma'\left(d^{ex}_{45\nu'\mu'}\boldsymbol{C}^{\mu'}_{\boldsymbol{R}\nu'} + d^{ex}_{46\nu'\mu'}\boldsymbol{B}^{\mu'}_{\boldsymbol{I}\nu'} + d^{ex}_{47\nu'\mu'}\boldsymbol{P}^{\mu'}_{\boldsymbol{I}\nu'}\right)
\end{aligned}
$$

$$\begin{aligned} d^{ex}_{45\nu'\mu'} &= \Lambda'\Lambda I_{\nu'} J^{k_T R}_{j\nu' 1}\Lambda\Lambda', \; d^{ex}_{46\nu'\mu'} = i\Lambda'\Lambda I_{\nu'} J^{k_T R}_{j\nu' 3}\Lambda\Lambda' \\ d^{ex}_{47\nu'\mu'} &= i\Lambda'\Lambda I_{\nu'} J^{k_T R}_{j\nu' 2}\Lambda\Lambda' \end{aligned} \tag{76}$$

Similarly to (73)–(75), the field components tangential to the surface are given by

$$\begin{aligned} \boldsymbol{E}_{exT\|} &= \hat{\boldsymbol{r}}_T \times \boldsymbol{E}_{exT} = \overline{e}\Sigma' \left(d^{ex}_{45\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{I}\nu'} - d^{ex}_{46\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{R}\nu'} \right) \\ \boldsymbol{H}_{exT\|} &= \hat{\boldsymbol{r}}_T \times \boldsymbol{H}_{exT} = -\overline{h}\Sigma' \left(d^{ex}_{45\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{R}\nu'} - d^{ex}_{46\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{I}\nu'} \right) \end{aligned} \tag{77}$$

Similarly to the excitation fields (76), we recast the internal fields (29) as

$$\begin{aligned} \boldsymbol{E}_{inT} &= \overline{e}\Sigma' \left(d^{in}_{48\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{I}\nu'} + d^{in}_{49\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{R}\nu'} + d^{in}_{50\nu'\mu'} \boldsymbol{P}^{\mu'}_{\boldsymbol{R}\nu'} \right) \\ \boldsymbol{H}_{inT} &= -\overline{h}\Sigma' \left(d^{in}_{51\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{R}\nu'} + d^{in}_{52\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{I}\nu'} + d^{in}_{53\nu'\mu'} \boldsymbol{P}^{\mu'}_{\boldsymbol{I}\nu'} \right) \\ d^{ex}_{48\nu'\mu'} &= \Lambda'\Lambda I_{\nu'} c^{in}_{\nu'\mu'} J^{k_{in} R}_{j\nu' 1}\Lambda\Lambda', \; d^{ex}_{49\nu'\mu'} = -i\Lambda'\Lambda I_{\nu'} b^{in}_{\nu'\mu'} J^{k_{in} R}_{j\nu' 3}\Lambda\Lambda' \\ d^{ex}_{50\nu'\mu'} &= -i\Lambda'\Lambda I_{\nu'} b^{in}_{\nu'\mu'} J^{k_{in} R}_{j\nu' 2}\Lambda\Lambda' \\ d^{ex}_{51\nu'\mu'} &= \Lambda'\Lambda I_{\nu'} b^{in}_{\nu'\mu'} J^{k_{in} R}_{j\nu' 1}\Lambda\Lambda' = -i d^{ex}_{49\nu'\mu'} \Lambda'\Lambda J^{k_{in} R}_{j\nu' 1} / \Lambda'\Lambda J^{k_{in} R}_{j\nu' 3} \\ d^{ex}_{52\nu'\mu'} &= i\Lambda'\Lambda I_{\nu'} c^{in}_{\nu'\mu'} J^{k_{in} R}_{j\nu' 3}\Lambda\Lambda' = i d^{ex}_{48\nu'\mu'} \Lambda'\Lambda J^{k_{in} R}_{j\nu' 3} / \Lambda'\Lambda J^{k_{in} R}_{j\nu' 1} \\ d^{ex}_{53\nu'\mu'} &= i\Lambda'\Lambda I_{\nu'} c^{in}_{\nu'\mu'} J^{k_{in} R}_{j\nu' 2}\Lambda\Lambda' = d^{ex}_{52\nu'\mu'} \Lambda'\Lambda J^{k_{in} R}_{j\nu' 2} / \Lambda'\Lambda J^{k_{in} R}_{j\nu' 3} \end{aligned} \tag{78}$$

In (78) it is shown that the coefficients $d^{ex}_{48\nu'\mu'}$, $d^{ex}_{49\nu'\mu'}$, $d^{ex}_{50\nu'\mu'}$, associated with $\boldsymbol{E}_{inT}$ are related to $d^{ex}_{51\nu'\mu'}$, $d^{ex}_{52\nu'\mu'}$, $d^{ex}_{53\nu'\mu'}$, associated with $\boldsymbol{H}_{inT}$, respectively. This is a direct result of the fact that the interior of the sphere is a homogeneous medium at rest.

Similarly to (77), the fields corresponding to (78) that are tangential to the boundary are given by

$$\begin{aligned} \boldsymbol{E}_{inT\|} &= \hat{\boldsymbol{r}}_T \times \boldsymbol{E}_{inT} = \overline{e}\Sigma' \left(d^{in}_{48\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{I}\nu'} - d^{in}_{49\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{R}\nu'} \right) \\ \boldsymbol{H}_{inT\|} &= \hat{\boldsymbol{r}}_T \times \boldsymbol{H}_{inT} = -\overline{h}\Sigma' \left(d^{in}_{51\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{R}\nu'} - d^{in}_{52\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{I}\nu'} \right) \end{aligned} \tag{79}$$

and in (79) $d^{ex}_{48\nu'\mu'}$, $d^{ex}_{49\nu'\mu'}$ are related to $d^{ex}_{51\nu'\mu'}$, $d^{ex}_{52\nu'\mu'}$, respectively.

Turning once more to (39), consider $\left\{ \begin{smallmatrix} \boldsymbol{E}_G \\ \boldsymbol{H}_G \end{smallmatrix} \right\}$, the part of $\left\{ \begin{smallmatrix} \boldsymbol{E}_{scT} \\ \boldsymbol{H}_{scT} \end{smallmatrix} \right\}$ dependent on $\boldsymbol{G}$. Similarly to (36) $\left\{ \begin{smallmatrix} \boldsymbol{E}_G \\ \boldsymbol{H}_G \end{smallmatrix} \right\}$ is recast in terms of spherical vector wave functions of the argument $K_{ex} = k_{ex}R$, yielding

$$\left\{ \begin{array}{c} \boldsymbol{E}_G(K_{ex}\hat{\boldsymbol{r}}_T) \\ \boldsymbol{H}_G(K_{ex}\hat{\boldsymbol{r}}_T) \end{array} \right\} = \left\{ \begin{array}{c} +\overline{e} \\ -\overline{h} \end{array} \right\} \frac{1}{2\pi} \int e^{iK_{ex}\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T} \boldsymbol{G}(\hat{\boldsymbol{p}})_{\left\{ \begin{smallmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{smallmatrix} \right\}}) d\Omega_{\hat{\boldsymbol{p}}}$$

$$\begin{aligned} \boldsymbol{G_E} &= \Sigma' \left(d^G_{54\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{I}\nu'} + d^G_{55\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{R}\nu'} \right) \\ \boldsymbol{G_H} &= \Sigma' \left(d^G_{55\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{R}\nu'} - d^G_{54\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{I}\nu'} \right) \\ \boldsymbol{r} \times \boldsymbol{G}_{\left\{ \begin{smallmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{smallmatrix} \right\}} &= \{\mp\} \boldsymbol{G}_{\left\{ \begin{smallmatrix} \boldsymbol{H} \\ \boldsymbol{E} \end{smallmatrix} \right\}} \end{aligned} \tag{80}$$

Except for the time-dependence, the integrals in (37), (80) are identical in form, although they involve different functions, $\boldsymbol{g}_{\left\{ \begin{smallmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{smallmatrix} \right\}}$, $\boldsymbol{G}_{\left\{ \begin{smallmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{smallmatrix} \right\}}$, respectively. In the limit of vanishing velocity the two equations coalesce. Therefore we conclude that (80) formally corresponds to a homogeneous medium, hence the coefficients are related as indicated in (80). It is noted that $\left\{ \begin{smallmatrix} \boldsymbol{E}_G \\ \boldsymbol{H}_G \end{smallmatrix} \right\}$ in (80) is not one of the fields that can be measured separately. The analogy of (37) , (80) prescribes

$$\begin{aligned} \boldsymbol{E_G} &= \bar{e} \Sigma' i^{\nu'} \left(d^G_{54\nu'\mu'} \boldsymbol{M}^G_{\boldsymbol{I}\nu'\mu'} - i d^G_{55\nu'\mu'} \boldsymbol{N}^G_{\boldsymbol{R}\nu'\mu'} \right) \\ &= \bar{e} \Sigma' \left(d^G_{56\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{I}\nu'} + d^G_{57\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{R}\nu'} + d^G_{58\nu'\mu'} \boldsymbol{P}^{\mu'}_{\boldsymbol{R}\nu'} \right) \\ \boldsymbol{H_G} &= -\bar{h} \Sigma' i^{\nu'} \left(d^G_{55\nu'\mu'} \boldsymbol{M}^G_{\boldsymbol{R}\nu'\mu'} - i d^G_{54\nu'\mu'} \boldsymbol{N}^G_{\boldsymbol{I}\nu'\mu'} \right) \\ &= -\bar{h} \Sigma' \left(d^G_{59\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{R}\nu'} + d^G_{60\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{I}\nu'} + d^G_{61\nu'\mu'} \boldsymbol{P}^{\mu'}_{\boldsymbol{I}\nu'} \right) \\ d^G_{56\nu'\mu'} &= i^{\nu'} d^G_{54\nu'\mu'} J^{K_{ex}}_{h\nu'1}, \quad d^G_{57\nu'\mu'} = -i i^{\nu'} d^G_{55\nu'\mu'} J^{K_{ex}}_{h\nu'3} \\ d^G_{58\nu'\mu'} &= -i i^{\nu'} d^G_{55\nu'\mu'} J^{K_{ex}}_{h\nu'2}, \quad d^G_{59\nu'\mu'} = i^{\nu'} d^G_{55\nu'\mu'} J^{K_{ex}}_{h\nu'1} \\ d^G_{60\nu'\mu'} &= -i i^{\nu'} d^G_{54\nu'\mu'} J^{K_{ex}}_{h\nu'3}, \quad d^G_{61\nu'\mu'} = -i i^{\nu'} d^G_{54\nu'\mu'} J^{K_{ex}}_{h\nu'2} \end{aligned} \tag{81}$$

From (81), at the boundary the tangential fields are given by

$$\begin{aligned} \boldsymbol{E}_{\boldsymbol{G}\|} &= \hat{\boldsymbol{r}}_T \times \boldsymbol{E}_G = \bar{e} \Sigma' \left(d^G_{56\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{I}\nu'} - d^G_{57\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{R}\nu'} \right) \\ \boldsymbol{H}_{\boldsymbol{G}\|} &= \hat{\boldsymbol{r}}_T \times \boldsymbol{H}_G = -\bar{h} \Sigma' \left(d^G_{59\nu'\mu'} \boldsymbol{B}^{\mu'}_{\boldsymbol{R}\nu'} - d^G_{60\nu'\mu'} \boldsymbol{C}^{\mu'}_{\boldsymbol{I}\nu'} \right) \end{aligned} \tag{82}$$

Finally, from (74), (75), (77), (79), (82), the equations for the continuity of the tangential components of the $\boldsymbol{E}$, $\boldsymbol{H}$, fields, at the boundary, is obtained. Inasmuch as all the final coefficients are independent of θ, ψ, the series are functionally and spatially orthogonal, i.e., the terms in the series are independent, and the first line of (15) applies. Note that $\boldsymbol{E}_{\boldsymbol{f}\|}$, $\boldsymbol{H}_{\boldsymbol{f}\|}$, $\boldsymbol{E}_{exT\|}$, $\boldsymbol{H}_{exT\|}$ are known functions, derived from the velocity-independent Mie scattering problem. For a perfectly conducting sphere, or when the dielectric susceptibility in the exterior domain is negligible in comparison to that of the sphere, we have $\boldsymbol{E}_{inT\|} = 0$ at the surface, hence

$$\begin{aligned} \boldsymbol{E}_{exT\|} + \boldsymbol{E}_{\boldsymbol{G}\|} + \boldsymbol{E}_{\boldsymbol{f}\|} &= 0 \\ d^{ex}_{45\nu'\mu'} + d^{G}_{56\nu'\mu'} + \beta^{(1)} d^{sc}_{33\nu'\mu'} &= 0 \\ d^{ex}_{46\nu'\mu'} + d^{G}_{57\nu'\mu'} + \beta^{(1)} d^{sc}_{34\nu'\mu'} &= 0 \end{aligned} \tag{83}$$

prescribing two equations for the unknowns $d^{G}_{56\nu'\mu'}$, $d^{G}_{57\nu'\mu'}$. Similarly for the perfectly magnetic sphere, where the external magnetic permeability is negligible in comparison with that of the sphere, the boundary condition is $\boldsymbol{H}_{inT\|} = 0$ at the boundary, we find

$$\begin{aligned} \boldsymbol{H}_{exT\|} + \boldsymbol{H}_{\boldsymbol{G}\|} + \boldsymbol{H}_{\boldsymbol{f}\|} &= 0 \\ d^{ex}_{45\nu'\mu'} + d^{G}_{59\nu'\mu'} + \beta^{(1)} d^{sc}_{39\nu'\mu'} &= 0 \\ d^{ex}_{46\nu'\mu'} + d^{G}_{60\nu'\mu'} + \beta^{(1)} d^{sc}_{40\nu'\mu'} &= 0 \end{aligned} \tag{84}$$

prescribing two families of equations for the unknowns $d^{G}_{59\nu'\mu'}$, $d^{G}_{60\nu'\mu'}$.

The general case is prescribed by

$$\begin{aligned} \boldsymbol{E}_{exT\|} + \boldsymbol{E}_{\boldsymbol{G}\|} + \boldsymbol{E}_{\boldsymbol{f}\|} &= \boldsymbol{E}_{inT\|} \\ d^{ex}_{45\nu'\mu'} + d^{G}_{56\nu'\mu'} + \beta^{(1)} d^{sc}_{33\nu'\mu'} &= d^{in}_{48\nu'\mu'} \\ d^{ex}_{46\nu'\mu'} + d^{G}_{57\nu'\mu'} + \beta^{(1)} d^{sc}_{34\nu'\mu'} &= d^{in}_{49\nu'\mu'} \\ \boldsymbol{H}_{exT\|} + \boldsymbol{H}_{\boldsymbol{G}\|} + \boldsymbol{H}_{\boldsymbol{f}\|} &= \boldsymbol{H}_{inT\|} \\ d^{ex}_{45\nu'\mu'} + d^{G}_{59\nu'\mu'} + \beta^{(1)} d^{sc}_{39\nu'\mu'} &= d^{in}_{51\nu'\mu'} \\ d^{ex}_{46\nu'\mu'} + d^{G}_{60\nu'\mu'} + \beta^{(1)} d^{sc}_{40\nu'\mu'} &= d^{in}_{52\nu'\mu'} \end{aligned} \tag{85}$$

providing four families of equations for the eight families of unknowns $d^{G}_{56\nu'\mu'}$, $d^{G}_{57\nu'\mu'}$, $d^{G}_{59\nu'\mu'}$, $d^{G}_{60\nu'\mu'}$, $d^{in}_{48\nu'\mu'}$, $d^{in}_{49\nu'\mu'}$, $d^{in}_{51\nu'\mu'}$, $d^{in}_{52\nu'\mu'}$. From (81) we have additional two equations relating $d^{G}_{56\nu'\mu'}$, $d^{G}_{60\nu'\mu'}$ and $d^{G}_{57\nu'\mu'}$, $d^{G}_{59\nu'\mu'}$. From (78) we get additional two equations relating $d^{in}_{48\nu'\mu'}$, $d^{in}_{52\nu'\mu'}$, and $d^{in}_{49\nu'\mu'}$, $d^{in}_{51\nu'\mu'}$. Altogether we have eight families of equations for the eight families of unknowns, hence all coefficients are accounted for and can be calculated.

The representation of fields in terms of series of spherical harmonics has been achieved, and with that the problem of evaluating the scattering coefficients is considered as solved.

6. THE SCATTERED FIELD OF A MOVING SPHERE

With the coefficients of $\boldsymbol{G}_{\boldsymbol{E}}$, $\boldsymbol{G}_{\boldsymbol{H}}$ already provided according to (80), we now turn back to (38) to discuss the scattered field. The integral must be represented in a way which allows the calculation of the scattered field for arbitrary locations and times $(\boldsymbol{r}, t)$.

The phase $\varphi_p(\hat{\boldsymbol{p}})$ in the integrand (38) corresponds to (30). The phase used for the computation of the velocity-dependent field in (39) was defined for a single plane wave in (33). This is used throughout, culminating in (83)–(85). Both (30) and (33), (34) involve the same unit vector $\hat{\boldsymbol{k}}_p$, hence both (39) and (38) refer to the same unit vector $\hat{\boldsymbol{p}}$. To express (38) in the coordinates used for (39) we first need to remove from (39) the parts of $\boldsymbol{f}_{\left\{\substack{\boldsymbol{E}\\\boldsymbol{H}}\right\}}(\hat{\boldsymbol{p}})$ associated with the application of the boundary condition, as shown in (35). In addition we have to equate $\varphi_p = \varphi_{pT}$ to the first order in v/c. To achieve this goal, we need to replace the time t by a new definition t' as shown in (86). This has no effect on the solution of the boundary value problem. Specifically, (39) refers to the value at the boundary $r_T = R$, but the analysis equally applies to any other location $\boldsymbol{r}_T$.

Implementing these steps yields changes in eq. (86): add primes and last line as indicated below

$$\left\{\begin{array}{c}\boldsymbol{E}_{sc}\\\boldsymbol{H}_{sc}\end{array}\right\} = \left\{\begin{array}{c}+\overline{e}'\\-\overline{h}'\end{array}\right\}\frac{1}{2\pi}\int e^{iK'_{ex}\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T}(\boldsymbol{G}_{\left\{\substack{\boldsymbol{E}\\\boldsymbol{H}}\right\}}(\hat{\boldsymbol{p}}) + \beta^{(1)}\boldsymbol{f}'_{\left\{\substack{\boldsymbol{E}\\\boldsymbol{H}}\right\}}(\hat{\boldsymbol{p}}))d\Omega_{\hat{\boldsymbol{p}}}$$

$$\boldsymbol{f}'_{\left\{\substack{\boldsymbol{E}\\\boldsymbol{H}}\right\}}(\hat{\boldsymbol{p}}) = iK'_{ex}(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T)\boldsymbol{h}_{\left\{\substack{\boldsymbol{E}\\\boldsymbol{H}}\right\}} - iK'_{ex}A^{(1)}C_{\theta T}\boldsymbol{g}_{\left\{\substack{\boldsymbol{E}\\\boldsymbol{H}}\right\}},\ K'_{ex} = k_{ex}r_T \quad (86)$$

$$\left\{\begin{array}{c}+\overline{e}'\\-\overline{h}'\end{array}\right\} = \left\{\begin{array}{c}+E_{0T}\\-H_{0T}\end{array}\right\}e^{-i\omega_T t'},\quad t' = t - \boldsymbol{v}\cdot\boldsymbol{r}/c^2$$

We end up with a first order velocity-dependent approximation of the scattered field, expressed in terms of $\boldsymbol{r}_T$, which according to (21) is, a time-dependent quantity $\boldsymbol{r}_T = \boldsymbol{r} - \boldsymbol{v}t$. Similarly to (39) for points in the vicinity of the scatterer the approximation (86) is valid.

However, for large distances r_T from the scatterer, the exponential approximation $e^a \approx 1 + a$ fails and (86) is rendered inapplicable. Most cases of interest involve scattered fields at moderate and large distances. Therefore, for arbitrary distances r_T we must find a different representation. Twersky's differential operator representation (18), involving inverse powers of the distance, can be exploited. To that end, in (86) we bring back the velocity-dependent terms into the exponential, resulting in

$$\left\{\begin{array}{c}\boldsymbol{E}_{sc}\\\boldsymbol{H}_{sc}\end{array}\right\} = \left\{\begin{array}{c}+\overline{e}'\\-\overline{h}'\end{array}\right\}\frac{1}{2\pi}\int e^{iK'_{ex}\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T}\boldsymbol{G}'_{\left\{\substack{\boldsymbol{E}\\\boldsymbol{H}}\right\}}(\hat{\boldsymbol{p}})d\Omega_{\hat{\boldsymbol{p}}}$$

$$\boldsymbol{G}'_{\left\{\substack{\boldsymbol{E}\\\boldsymbol{H}}\right\}}(\hat{\boldsymbol{p}}) = \boldsymbol{G}_{\left\{\substack{\boldsymbol{E}\\\boldsymbol{H}}\right\}}(\hat{\boldsymbol{p}})e^{i\beta^{(1)}K'_{ex}[(C_\alpha - 1)(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}_T) - A^{(1)}C_{\theta_T}]} \quad (87)$$

$$\boldsymbol{r} \times \boldsymbol{G}'_{\left\{ \boldsymbol{E} \atop \boldsymbol{H} \right\}} = \{\mp\} \boldsymbol{G}'_{\left\{ \boldsymbol{H} \atop \boldsymbol{E} \right\}}$$

The new function $\boldsymbol{G}'$ in (87) involves $\boldsymbol{r}_T$ which does not participate in the integration with respect to $\hat{\boldsymbol{p}}$, hence the differential operator representation can be applied in the form

$$\begin{aligned}
\left\{ \begin{array}{c} \boldsymbol{E}_{sc} \\ \boldsymbol{H}_{sc} \end{array} \right\} &= \left\{ \begin{array}{c} +\overline{e}' \\ -\overline{h}' \end{array} \right\} h_0(K'_{ex}) \tilde{\boldsymbol{O}}(K'_{ex}, \tilde{\boldsymbol{D}}) \cdot \boldsymbol{G}'(\hat{\boldsymbol{p}}) \Big|_{\hat{\boldsymbol{p}} = \hat{\boldsymbol{r}}_T} \\
\tilde{\boldsymbol{O}} &= \sum_{\nu=0}^{\infty} \rho_\nu \tilde{\boldsymbol{D}} \cdot (\tilde{\boldsymbol{D}} - 1.2\tilde{\boldsymbol{I}}) \cdot (\tilde{\boldsymbol{D}} - 2.3\tilde{\boldsymbol{I}}) \cdots (\tilde{\boldsymbol{D}} - (\nu-1)\nu\tilde{\boldsymbol{I}}) \\
&= \tilde{\boldsymbol{I}} + \rho_1 \tilde{\boldsymbol{D}} + \rho_2 \tilde{\boldsymbol{D}} \cdot (\tilde{\boldsymbol{D}} - 2\tilde{\boldsymbol{I}}) + \rho_n \tilde{\boldsymbol{D}} \cdot (\tilde{\boldsymbol{D}} - 1.2\tilde{\boldsymbol{I}}) \\
&\quad \cdots (\tilde{\boldsymbol{D}} - (n-1)n\tilde{\boldsymbol{I}}) \qquad (88) \\
\tilde{\boldsymbol{D}} &= \hat{\boldsymbol{r}}_T (D+2) \hat{\boldsymbol{r}}_T + \hat{\boldsymbol{r}}_T 2 S_\alpha^{-1} \partial_\alpha (S_\alpha \hat{\boldsymbol{\alpha}}) + \hat{\boldsymbol{r}}_T \partial_\beta \hat{\boldsymbol{\beta}} \\
&\quad + \hat{\boldsymbol{\alpha}} (D + S_\alpha^{-2}) \hat{\boldsymbol{\alpha}} + \hat{\boldsymbol{\alpha}} 2 S_\alpha^{-2} C_\alpha \partial_\beta \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\alpha}} \partial_\alpha \hat{\boldsymbol{r}}_T \\
&\quad + \hat{\boldsymbol{\beta}} (D + S_\alpha^{-2}) \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}} 2 S_\alpha^{-2} C_\alpha \partial_\beta \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}} 2 S_\alpha^{-1} \partial_\beta \hat{\boldsymbol{r}}_T, \\
\rho_\nu &= (i/(2K'_{ex}))^\nu / \nu!, \; D = S_\alpha^{-2} \left[\partial_\beta^2 + S_\alpha \partial_\alpha (S_\alpha \partial_\alpha) \right]
\end{aligned}$$

with $|_{\hat{\boldsymbol{p}} = \hat{\boldsymbol{r}}_T}$ in (88) indicating that after operating on the terms involving coordinates $\hat{\boldsymbol{p}}$, coordinates $\hat{\boldsymbol{p}}$ are to be replaced by $\hat{\boldsymbol{r}}_T$. In the original expressions (18) this distinction was irrelevant, because the operand $\boldsymbol{G}$ depended on coordinates $\boldsymbol{r}$ only.

Once the scattered fields are found, one can substitute from (21) and (86) to finally obtain the results in terms of the initial $(\boldsymbol{r}, t)$ coordinates. Depending on the distance from the scatterer at a given time, one can make a decision as to where to truncate the differential operator series representation (88), in order to achieve the desired approximation.

With that the scattering problem in its entirety is considered to be solved.

7. SUMMARY AND CONCLUDING REMARKS

The Mie scattering problem for a sphere at rest [4] , as shown in (28), (29), (36), is a quite complicated problem, requiring prior knowledge of the vector spherical wave theory and the special functions involved. Therefore a recapitulation of some mathematical results and the introduction of notation is required.

Presently the first order in ν/c velocity-dependent problem of scattering by a moving sphere is investigated. The velocity-dependent boundary conditions based on the Lorentz force formulas (2)–(6) are incorporated . To the first order in the velocity, these formulas conform to the corresponding special-relativistic expressions [1, 2], and lead to simple expressions (7), (8).

The formalism is therefore straight-forward, but it turns out that in comparison to the Mie problem in the absence of motion, the implementation of the present scheme is much more complicated: Firstly, one has to investigate the behavior of plane waves under the present circumstances (22)–(35). Then, exploiting Sommerfeld-type integrals for the vector spherical waves (16), (17), (37), the scattered wave is represented as a superposition of plane waves (38), and the first order approximation for the field signals at the surface are derived (39).

In order to be able to calculate the scattering coefficients, a tedious process of expressing the fields in series of vector spherical harmonics is launched. There are two similar but distinct steps: One has to work under the integral sign in (39) with the first order terms $\beta^{(1)}\boldsymbol{f}_{\left\{\substack{\boldsymbol{E}\\ \boldsymbol{H}}\right\}}(\hat{\boldsymbol{p}})$, which are recast in series of spherical harmonics of the argument $\hat{\boldsymbol{p}}$.

Thus the field signals $\left\{\substack{\boldsymbol{E}_{scT}\\ \boldsymbol{H}_{scT}}\right\}$ are derived in terms of series of vector spherical harmonics as a function of $\boldsymbol{r}_T$, but again, the relevant coefficients are still dependent on $\hat{\boldsymbol{r}}_T$, and this dependence must be eliminated by recasting the series in terms of vector spherical harmonics with coefficients dependent on R only. When this task is finished, culminating in the equations (83)–(85), the calculation of the scattering coefficients is finished.

Finally, expressions for the scattered wave fields must be derived. A near zone approximation is given by (86), but this is of very limited usefulness. For moderate to large distances from the scatterer, we incorporate Twersky's [6] differential operator formula (18), involving inverse powers of the distances. The implementation of this formula for our present case and the necessary modifications are described in (88).

The present investigation is theoretical, and in the future some numerical simulations are planned, in order to bring out salient features of the solution. One important result which could be anticipated from the cylindrical case and the free-space spherical case [1, 2, 13] is the interaction of coefficients and the appearance of new, velocity-induced multipoles. Essentially, this is the result of the branching out of sums, both in the polar (n) and the azimuthal (m) indices. According to (62), for example, the n index spreads to $n \pm 3$ while the m index spreads to $m \pm 1$. Inasmuch as the process of getting coefficients free of functions is performed twice, in the final summation Σ' the range of this spreading is doubled.

8. LIST OF FREQUENTLY USED ABBREVIATIONS

$\gamma_{nm} = (n-m)!/(n+m)!$

$M = (-1)^m$

$J_{zn1}^{k_\alpha r_\beta} = z_n(k_\alpha r_\beta)$

$J_{zn2}^{k_\alpha r_\beta} = z_n(k_\alpha r_\beta)/(\kappa_n k_\alpha r_\beta)$

$\kappa_n = 1/n(n+1)$

$J_{zn3}^{k_\alpha r_\beta} = \partial_{k_\alpha r_T}[k_\alpha r_\beta z_n(k_\alpha r_\beta)]/k_\alpha r_\beta$

$J_{zn4}^{k_\alpha r_\beta} = \partial_{k_\alpha r_\beta}[z_n(k_\alpha r_\beta)]$

$J_{zn5}^{k_\alpha r_\beta} = z_n(k_\alpha r_\beta)/(k_\alpha r_\beta)$

$C_\zeta = \cos\zeta,\ S_\zeta = \sin\zeta$

$\delta_{a\alpha} = \begin{Bmatrix} 0 \text{ for } a\neq\alpha \\ 1 \text{ for } a=\alpha \end{Bmatrix}$

$\alpha_{nm} = M4\pi/N_n$

$\beta_{nm} = \alpha_{nm}\kappa_n$

$\lambda_n = 1/(2n+1)$

$\mu_m = 2\pi M/m$

$\beta^{(\alpha)} = \nu/\nu_{ph}^{(\alpha)}$

$\nu_{ph}^{(\alpha)} = (\mu^{(\alpha)}\varepsilon^{(\alpha)})^{-1/2}$

$\zeta(\alpha) = (\mu^{(\alpha)}/\varepsilon^{(\alpha)})^{1/2}$

$A^{(1)} = (\nu_{ph}^{(1)}/c)^2$

$\Sigma_{nm} = \sum_{n=0}^{\infty}\sum_{m=-n}^{m=n}$

$\overline{e} = E_{0T}e^{-i\omega_T t}$

$\overline{h} = H_{0T}e^{-i\omega_T t}$

$I_n = i^n N_n$

$N_n = \kappa_n/\lambda_n$

$A = -iK_{ex}A^{(1)}C_{\theta_T}$

$K_{ex} = k_{ex}R$

$\Sigma_{\nu\triangleright n\pm3} = \sum_{\nu=n-3}^{n+3}$

$\Sigma_{\nu\triangleright n\pm3}^{\{\pm\};\mu\triangleright m\pm1}$ see eq. (67)

$\Lambda\{\nu,\mu\} = \{n,1\}$ see eq. (69)

$D = iK_{ex}C_{\theta_T}$

$E = iK_{ex}S_{\theta_T}C_{\psi_T}$

$F = iK_{ex}S_{\theta_T}S_{\psi_T}$

$\Sigma = \Sigma_{n1}\Sigma_{\nu\triangleright n\pm3}^{\mu\triangleright n\pm3}$

$\boldsymbol{E}_\parallel = \hat{\boldsymbol{r}}_T\times\boldsymbol{E},\ \boldsymbol{H}_\parallel = \hat{\boldsymbol{r}}_T\times\boldsymbol{H}$

$\Sigma' = \Sigma_{\nu\mu}\Sigma_{\nu'\triangleright\nu\pm3}^{\mu'\triangleright\mu\pm1}$

$\Lambda'\{\nu',\mu'\} = \{\nu,\mu\}$ see eq. (74)

$\overline{e}' = E_{0T}e^{-i\omega_T t'}$

$\overline{h}' = H_{0T}e^{-i\omega_T t'}$

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