

CONTACT GEOMETRY IN ELECTROMAGNETISM

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Abstract—In the first part of this work we show that, by working in Fourier space, the Bohren decomposition and the Helmholtz's decomposition can be combined into one decomposition. This yields a completely mathematical decomposition, which decomposes an arbitrary vector field on \mathbb{R}^3 into three components. A key property of the decomposition is that it commutes both with the curl operator and with the time derivative. We can therefore apply this decomposition to Maxwell's equations without assuming anything about the media. As a result, we show that Maxwell's equations split into three completely uncoupled sets of equations. Further, when a medium is introduced, these decomposed Maxwell's equations either remain uncoupled, or become coupled depending on the complexity of the medium.

In the second part of this work, we give a short introduction to contact geometry and then study its relation to electromagnetism. By studying examples, we show that the decomposed fields in the decomposed Maxwell's equations always seem to induce contact structures. For instance, for a plane wave, the decomposed fields are the right and left hand circularly polarized components, and each of these induce their own contact structure. Moreover, we show that each contact structure induces its own *Carnot-Carathéodory metric*, and the path traversed by the circularly polarized waves seem to coincide with the geodesics of these metrics.

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1. INTRODUCTION

In electromagnetics, *chiral media* is media where electromagnetic waves can propagate with different phase velocities depending on their handedness. In other words, a left hand circularly polarized wave can propagate with a different phase velocity than a right hand circularly polarized wave. In modern electromagnetism, there are many (more or less equivalent) macroscopic models for such media. One such mathematical model is given by the constitutive equations

$$\mathbf{D} = \epsilon \mathbf{E} + \xi \mathbf{H}, \quad (1)$$

$$\mathbf{B} = \mu \mathbf{H} + \zeta \mathbf{E}. \quad (2)$$

(In these equations we have used standard notation for the time harmonic electromagnetic fields, and $\epsilon, \mu, \xi, \zeta$ are complex scalars that describe the media [1, 2].) With the above constitutive equations

one can, for instance, mathematically show that right and left hand circularly polarized waves can propagate with a different phase velocities. However, the main disadvantage of the above model is that it is algebraic. That is, although equations (1)–(2) do model chiral media, the equations in themselves do not have a direct geometrical or physical interpretation related to chiral media. This means that when we translate our physical description for chiral media into a mathematical one, we lose the geometric insight that we might have about handed behavior, circularly polarized waves, and mirror asymmetry. Since handedness is a very important phenomena in nature [3–5], it is motivated to search for a formulation for electromagnetism which directly could describe these phenomena. In this work we shall present one such formulation for electromagnetism. As an example, we use this formulation to give a geometric model for chiral media.

This study is divided into two parts. In the first part (Sections 2–4) we derive a handed formulation for electromagnetism. This formulation is derived from the traditional Maxwell’s equations by a *helicity decomposition*, which decomposes Maxwell’s equations into three parts. This decomposition can be seen as a generalization of the *Bohren decomposition* and a refinement of the *Helmholtz’s decomposition*. This decomposition is well-known in fluid mechanics [6–10].

The aim of the second part of this work is to try to describe the *internal geometry* of electromagnetism. In other words, the aim is to find a geometric structure, which would describe the geometry of space as an electromagnetic wave would see space. What we here exactly mean by geometry is not clear since there does not seem to exist any such geometric structure for electromagnetism (see [11]). In this work we will neither present any such canonical geometric structure for electromagnetism. However, we will show ample evidence, which suggests two things. First, in order to study this geometry, one must take into account the handed behavior of electromagnetism. For instance, in the scattering of a plane wave in chiral media, one must take into account the wave’s handedness. For this, the decomposed Maxwell’s equations form an ideal framework. Second, we will show that the geometry of electromagnetism seems to be related to contact geometry. In Section 5 we give a short introduction to contact geometry, and in Section 6 we give examples of contact structures derived from the decomposed Maxwell’s equations. In the last section of this work (Section 7) we draw the conclusions and give some suggestions for further work.

2. HELICITY AND BELTRAMI FIELDS

In Sections 2–4 we will work with possibly complex valued vector fields. These are vector fields defined on an open simply connected set $\Omega \subset \mathbb{R}^3$ with possibly complex component functions. If Ω has a boundary, we also assume that the boundary is smooth. We further assume that the component functions of all vector fields are Lebesgue measurable functions $\Omega \rightarrow \mathbb{C}$. The *Lebesgue integral* of a measurable function $f : \Omega \rightarrow \mathbb{C}$ is denoted by $\int_{\Omega} f(\mathbf{x})d\mathbf{x}$. Similarly, the integral of a vector field \mathbf{F} is defined componentwise, and is denoted by $\int_{\Omega} \mathbf{F}(\mathbf{x})d\mathbf{x}$. In this work $i = \sqrt{-1}$ is the complex unit, and $\Re\{x\}$ and $\Im\{x\}$ are the real and imaginary parts of a complex number, vector, or matrix x . Similarly, the complex conjugate of x is written as x^* .

We next define helicity. It is a scalar associated with a vector field that measures the amount of handed twisting in the vector field. Depending on how the vector field twists, it's helicity can be positive, negative, or zero.

Definition 2.1 (Helicity) *Let \mathbf{F} and \mathbf{G} be real valued vector fields on a simply connected open set $\Omega \subset \mathbb{R}^3$. The helicity of \mathbf{F} is the real number defined as*

$$\mathcal{H}(\mathbf{F}) = \int_{\Omega} \mathbf{F} \cdot \nabla \times \mathbf{F} d\mathbf{x}. \quad (3)$$

We will also say that $\mathbf{F} \cdot \nabla \times \mathbf{F}$ is the helicity density of \mathbf{F} . The cross-helicity of \mathbf{F} and \mathbf{G} is defined as

$$\mathcal{H}(\mathbf{F}, \mathbf{G}) = \int_{\Omega} \mathbf{F} \cdot \nabla \times \mathbf{G} d\mathbf{x}. \quad (4)$$

(It should be pointed out that the above definition of helicity is slightly non-standard. See e.g., [12]. However, the present definition of helicity is motivated since it is related to contact geometry [11].) In the above definition we have not defined the precise function space for \mathbf{F} and \mathbf{G} . However, the aim of the present section is only to give a short heuristic introduction to helicity. We therefore post-poned the definition of this function space to Section 3. In the present section, we therefore tacitly assume that all objects are sufficiently smooth and well behaved so that all derivatives and integrals are well behaved.

We next show how helicity density is related to the polarization of *time harmonic plane waves*. These are real valued vector fields in \mathbb{R}^3 that can be written as

$$\mathbf{F}(z, t) = \Re \left\{ \mathbf{A} e^{i(kz - \omega t)} \right\} \quad (5)$$

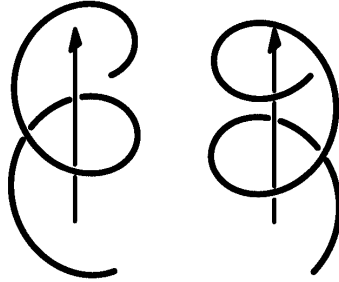


Figure 1. \mathbf{E}_+ and \mathbf{E}_- in Example 2.2.

for some Cartesian coordinates x, y, z , some positive real numbers k, ω , and a complex constant vector \mathbf{A} with no z -component. For \mathbf{F} , the z -axis is the *direction of propagation*, and t is the time parameter. The vector \mathbf{A} determines the *polarization* of the wave. We define the *handedness* of circularly polarized waves as follows. A circularly polarized plane-wave is *right-hand polarized*, if its helicity density is negative, and *left-hand polarized*, if its helicity density is positive. This definition is motivated by the next example.

Example 2.2 (Helicity density and polarization) Let us define

$$\mathbf{E}_\pm(z, t) = \Re \left\{ (\mathbf{u}_x \pm i\mathbf{u}_y) e^{i(kz - \omega t)} \right\}.$$

For these fields, $\nabla \times \mathbf{E}_\pm = \pm k \mathbf{E}_\pm$. In other words, the fields are parallel and anti-parallel to their own curl. Hence the helicity densities of \mathbf{E}_+ and \mathbf{E}_- are positive, respectively negative, so \mathbf{E}_+ is left-hand circularly polarized and \mathbf{E}_- is right-hand circularly polarized. Figure 1 shows these fields for $t = 0$: \mathbf{E}_- rotates around the positive z -axis using the “right-hand rule”, and \mathbf{E}_+ rotates around the positive z -axis using the “left-hand rule”. Adding \mathbf{E}_+ and \mathbf{E}_- yields a linearly polarized plane wave with zero helicity. It follows that a linearly polarized plane wave carries no helicity, but it can be decomposed into two plane-waves with positive/negative helicity densities. \square

Helicity is closely related to *Beltrami fields*. These are vector fields $\mathbf{F} : \Omega \rightarrow \mathbb{R}^3$ in a simply connected open set $\Omega \subset \mathbb{R}^3$ that satisfy $\nabla \times \mathbf{F} = f \mathbf{F}$ for some function $f : \Omega \rightarrow \mathbb{R}$. (Here, again, we assume that all objects are sufficiently smooth.) Geometrically, the above equation states that the rotation of \mathbf{F} is everywhere parallel to the field. A characteristic feature for such fields is a constant twisting of the field. If $f > 0$, the field has positive helicity, and if $f < 0$, the field has negative helicity. If f is constant (as in Example 2.2), the

field is said to be a *Trkalian field*. Trkalian fields on \mathbb{R}^3 are classified in [13].

Beltrami fields appear in surprisingly many areas of physics. In plasma physics Beltrami fields are also called *force free fields*. For instance, the magnetic field inside ball lightnings and fusion reactors have been modeled by Beltrami fields [14, 15]. In electromagnetics, Beltrami fields are also called *wave field* [1, 2, 16]. Also, in fluid mechanics, the motion of particles in tornadoes and waterspouts have been modeled by Beltrami fields [17]. Beltrami fields also appear in gravitation research, quark physics and thermoacoustics [17]. In [18] it is shown that there is a one-to-one correspondence (up to a scaling) between non-vanishing Beltrami fields and contact structures.

3. THE HELICITY DECOMPOSITION

We next define the helicity decomposition, which decomposes a vector field on \mathbb{R}^3 into three components: one with zero helicity, one with positive helicity, and one with negative helicity. To define this decomposition, we shall need the Fourier transform. We therefore assume that the underlying space is \mathbb{R}^3 with Cartesian coordinates. For L^1 vector fields (whose all component functions are L^1 functions on \mathbb{R}^3), we define the Fourier transform \mathcal{F} and its inverse \mathcal{F}^{-1} as follows:

$$\mathcal{F}\{\mathbf{F}\}(\mathbf{k}) = \int_{\mathbb{R}^3} \mathbf{F}(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x}, \quad (6)$$

$$\mathcal{F}^{-1}\{\hat{\mathbf{F}}\}(\mathbf{x}) = \int_{\mathbb{R}^3} \hat{\mathbf{F}}(\mathbf{k}) e^{2\pi i \mathbf{k} \cdot \mathbf{x}} d\mathbf{k}. \quad (7)$$

We shall also write $\mathcal{F}\{\mathbf{F}\} = \hat{\mathbf{F}}$. Since $L^1 \cap L^2$ is dense in L^2 , the above Fourier transform extends to L^2 vector fields (see [19]). By means of the L^2 Fourier transform, we define curl and div as $\nabla \times \mathbf{F} = \mathcal{F}^{-1}(2\pi i \mathbf{k} \times \hat{\mathbf{F}})$, and $\nabla \cdot \mathbf{F} = \mathcal{F}^{-1}(2\pi i \mathbf{k} \cdot \hat{\mathbf{F}})$. Since we shall work with electromagnetic fields, it is natural to restrict our study to vector fields in L^2_{curl} ; L^2 vector fields whose curl is also an L^2 vector field.

From the definitions of curl and div, it follows that Helmholtz's decomposition for a vector field on \mathbb{R}^3 has the following interpretation in Fourier space: the curl-free component is normal to the $|\mathbf{k}|$ -sphere and the divergence-free component is tangential to the $|\mathbf{k}|$ -sphere. (Of course, since $\hat{\mathbf{F}}$ is complex, tangential and normal should here be understood in a complex sense.) When $\mathbf{k} \neq 0$, we can further decompose the tangential component using the projection operators

induced by the involution dyadic $i\mathbf{u}_r \times \bar{\bar{\mathbf{I}}}$ [11]. Here $\mathbf{u}_r = \frac{\mathbf{k}}{|\mathbf{k}|}$. (For an introduction to dyadic algebra, see [1, 11].) Also, the point $\mathbf{k} = 0$ poses no problem, since it has zero measure.

Definition 3.1 (Helicity decomposition) *Let \mathbf{F} be a real valued vector field in L^2_{curl} . For $\lambda = 0, \pm 1$, let*

$$\pi_\lambda \mathbf{F} = \mathcal{F}^{-1} \left\{ \bar{\bar{\mathbf{P}}}_\lambda \cdot \{\mathbf{F}\} \right\}.$$

where

$$\bar{\bar{\mathbf{P}}}_\lambda(\mathbf{k}) = \begin{cases} \frac{1}{2} (\bar{\bar{\mathbf{I}}} + i\lambda \mathbf{u}_r \times \bar{\bar{\mathbf{I}}}) \cdot \bar{\bar{\mathbf{P}}}_t & \text{when } \lambda = \pm 1, \mathbf{k} \neq 0, \\ \mathbf{u}_r \mathbf{u}_r & \text{when } \lambda = 0, \mathbf{k} \neq 0, \\ 0 & \text{when } \mathbf{k} = 0, \end{cases}$$

and $\bar{\bar{\mathbf{P}}}_t = (\bar{\bar{\mathbf{I}}} - \mathbf{u}_r \mathbf{u}_r)$. We also write $\mathbf{F}_\lambda = \pi_\lambda \mathbf{F}$.

Since $\bar{\bar{\mathbf{P}}}_\lambda^*(\mathbf{k}) = \bar{\bar{\mathbf{P}}}_\lambda(-\mathbf{k})$, it follows that the decomposed fields are real valued. They are also vector fields in L^2_{curl} . We here note that since we have set $\bar{\bar{\mathbf{P}}}_\lambda$ to zero when $\mathbf{k} = 0$, $\bar{\bar{\mathbf{P}}}_\lambda$ are only dyadics almost everywhere. Despite this we call $\bar{\bar{\mathbf{P}}}_\lambda$ dyadics.

The $\bar{\bar{\mathbf{P}}}_\pm$ dyadics can also be derived from the *Bohren decomposition* known in electromagnetism. For the sourceless Helmholtz's equation $\nabla \times \nabla \times \mathbf{E} = k^2 \mathbf{E}$ ($k > 0$ real), the Bohren decomposition decomposes \mathbf{E} as $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$, where $\mathbf{E}_\pm = \frac{1}{2}(\mathbf{E} \pm \frac{1}{k} \nabla \times \mathbf{E})$. If we Fourier transform Helmholtz's equation, we can formally show that $\hat{\mathbf{E}}$ can be non-zero only on the shell $|\mathbf{k}| = k/(2\pi)$. Using this result, we can simplify the expressions for \mathbf{E}_\pm and obtain the same $\bar{\bar{\mathbf{P}}}_\pm$ dyadics as in Definition 3.1.

We next list some properties of the helicity decomposition. For a more detailed discussion, see [11]. First, from the definition of curl and div, it follows that the decomposed fields satisfy $\nabla \cdot \mathbf{F}_\pm = 0$ and $\nabla \times \mathbf{F}_0 = 0$. Hence, the helicity decomposition is a refinement of Helmholtz's decomposition. Also, using the identity $\nabla \times \mathbf{F}_\lambda = \mathcal{F}^{-1}\{2\pi\lambda|\mathbf{k}|\hat{\mathbf{F}}_\lambda\}$, it follows that $\mathcal{H}(\mathbf{F}_0) = 0$, $\mathcal{H}(\mathbf{F}_+) \geq 0$, and $\mathcal{H}(\mathbf{F}_-) \leq 0$ with equality only for $\mathbf{F}_\pm = 0$.

We also have that $\pi_\lambda \pi_\kappa \mathbf{F} = \delta_{\lambda\kappa} \pi_\lambda \mathbf{F}$ for all λ and κ . Since π_λ are self adjoint operators (in the real L^2 inner product), it follows that distinct components in the helicity decomposition are orthogonal and have zero cross-helicity. A key property of the helicity decomposition is that it commutes with the curl operator, i.e., for all λ

$$\pi_\lambda(\nabla \times \mathbf{F}) = \nabla \times (\pi_\lambda \mathbf{F}).$$

This last relation is essential for decomposing Maxwell's equations.

For a time dependent vector field, the helicity decomposition is defined pointwise. If we can assume that time derivative and spatial integration commutes, then the helicity decomposition commutes with the time derivative, i.e.,

$$\frac{\partial}{\partial t}\pi_\lambda\mathbf{F} = \pi_\lambda\frac{\partial}{\partial t}\mathbf{F}.$$

The helicity decomposition also commutes with Cartesian coordinate changes, spatial convolutions, and temporal convolutions [11].

3.1. The Moses Decomposition

Next we define the *Moses decomposition* [6] which provides a basis in Fourier space for the helicity decomposition. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be an orthonormal basis for \mathbb{R}^3 , and let $\mathbf{k} = \sum k_i\mathbf{u}_i$ and $k = |\mathbf{k}|$. Then the Moses decomposition introduces the complex basis $\{\mathbf{Q}_0(\mathbf{k}), \mathbf{Q}_+(\mathbf{k}), \mathbf{Q}_-(\mathbf{k})\}$ in Fourier space $\mathbb{R}^3 \setminus \{0\}$ by

$$\mathbf{Q}_0(\mathbf{k}) = -(k_1, k_2, k_3)/k,$$

and for $\lambda = \pm 1$,

$$\mathbf{Q}_\lambda(\mathbf{k}) = -\frac{\lambda}{\sqrt{2}} \left(\frac{k_1(k_1 + i\lambda k_2)}{k(k + k_3)} - 1, \frac{k_2(k_1 + i\lambda k_2)}{k(k + k_3)} - i\lambda, \frac{k_1 + i\lambda k_2}{k} \right).$$

The properties of the \mathbf{Q}_λ vectors are investigated in [6]. In the same reference the definition of the \mathbf{Q}_λ vectors is also motivated. Here we only mention that they are both orthonormal, i.e., $\mathbf{Q}_\lambda(\mathbf{k}) \cdot \mathbf{Q}_\kappa^*(\mathbf{k}) = \delta_{\lambda\kappa}$ and complete. Moreover, they satisfy $\mathbf{k} \times \mathbf{Q}_\lambda(\mathbf{k}) = -i\lambda|\mathbf{k}|\mathbf{Q}_\lambda(\mathbf{k})$ for $\lambda = \pm 1$, which is the key property, which makes the Moses decomposition well behaved under curl.

The *Moses decomposition* in physical space is defined as the projection onto the \mathbf{Q}_λ basis in Fourier space. If we denote the projection operators in physical space by $\tilde{\pi}_\lambda$ then

$$\tilde{\pi}_\lambda\mathbf{F} = \mathcal{F}^{-1}\{f_\lambda(\mathbf{k})\mathbf{Q}_\lambda(\mathbf{k})\},$$

where $f_\lambda(\mathbf{k}) = \mathcal{F}\{\mathbf{F}\}(\mathbf{k}) \cdot \mathbf{Q}_\lambda^*(\mathbf{k})$ and $\lambda = 0, \pm 1$. The decomposed field $\tilde{\pi}_\lambda\mathbf{F}$ are real valued if \mathbf{F} is real valued. The corresponding condition for f_λ is as follows: $\tilde{\pi}_\lambda\mathbf{F}$ is real valued if and only if $f_\lambda(-\mathbf{k}) = \phi_\lambda(\mathbf{k})f_\lambda^*(\mathbf{k})$, where $\phi_\lambda(\mathbf{k}) = -\frac{k_1 - i\lambda k_2}{k_1 + i\lambda k_2}$ [6]. Any such function f_λ can almost everywhere be written as

$$f_\lambda(\mathbf{k}) = \frac{1}{2}(\xi(\mathbf{k}) + \phi_\lambda(\mathbf{k})\xi(-\mathbf{k})) + i\frac{1}{2}(\xi(\mathbf{k}) - \phi_\lambda(\mathbf{k})\xi(-\mathbf{k}))$$

for some function $\xi : \mathbb{R}^3 \rightarrow \mathbb{R}$. It follows that each decomposed field in the Moses decomposition depends only on one scalar function from \mathbb{R}^3 to \mathbb{R} . Using computer algebra, we can show that the $\overline{\overline{\mathbf{P}}}_\lambda$ dyadics represent almost everywhere the same mapping as $\mathbf{Q}_\lambda \mathbf{Q}_\lambda^{*\text{T}}$ [11]. Hence the helicity decomposition is identical to the Moses decomposition for L^2 vector fields. In consequence each component in the helicity decomposition depend only on one real scalar function.

4. HELICITY DECOMPOSITION IN ELECTROMAGNETICS

In this section we apply the helicity decomposition to Maxwell's equations. Without any assumptions on the media we prove that Maxwell's equations for the fields $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ decompose into three uncoupled sets of equations; one set involving only the $+$ -components, one set involving only the $-$ -components, and one set involving only the 0 -components of the fields.

4.1. Decomposition of Maxwell's Equations

Maxwell's equations can be written down in a variety of different mathematical formalisms. However, to apply the helicity decomposition to Maxwell's equations, we formulate them using vector fields on \mathbb{R}^3 with Cartesian coordinates. Maxwell's equations then read

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{M}, \quad (8)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \quad (9)$$

$$\nabla \cdot \mathbf{D} = \rho, \quad (10)$$

$$\nabla \cdot \mathbf{B} = \rho_m. \quad (11)$$

In the above, \mathbf{E} and \mathbf{H} are the electric and magnetic field intensities, \mathbf{D} and \mathbf{B} are the electric and magnetic flux densities, and ρ and \mathbf{J} are the charge density and current. We have also included magnetic charge density ρ_m and magnetic current \mathbf{M} . To solve Maxwell's equations, these must be accompanied by a set of *constitutive equations* that relate the fields $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$. We shall assume that \mathbf{D} and \mathbf{B} can be solved as functionals of \mathbf{E} and \mathbf{H} , i.e.,

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H}), \quad (12)$$

$$\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H}). \quad (13)$$

We next apply the helicity decomposition to Maxwell's equations. For this purpose, we shall assume that all the vector fields in Maxwell's equations are time dependent vector fields in a function space where the helicity decomposition is defined and time derivatives commute with the decomposition.

The $+$ -component of the first two Maxwell's equations (8)–(9) are

$$\nabla \times \mathbf{E}_+ = -\frac{\partial \mathbf{B}_+}{\partial t} - \mathbf{M}_+, \quad (14)$$

$$\nabla \times \mathbf{H}_+ = \frac{\partial \mathbf{D}_+}{\partial t} + \mathbf{J}_+, \quad (15)$$

the $-$ -components are

$$\nabla \times \mathbf{E}_- = -\frac{\partial \mathbf{B}_-}{\partial t} - \mathbf{M}_-, \quad (16)$$

$$\nabla \times \mathbf{H}_- = \frac{\partial \mathbf{D}_-}{\partial t} + \mathbf{J}_-, \quad (17)$$

and the 0-components are

$$\frac{\partial \mathbf{B}_0}{\partial t} = -\mathbf{M}_0, \quad (18)$$

$$\frac{\partial \mathbf{D}_0}{\partial t} = -\mathbf{J}_0. \quad (19)$$

Further, inserting $\mathbf{D} = \mathbf{D}_0 + \mathbf{D}_+ + \mathbf{D}_-$ and $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_+ + \mathbf{B}_-$ into equations (10)–(11) yields

$$\nabla \cdot \mathbf{D}_0 = \rho, \quad (20)$$

$$\nabla \cdot \mathbf{B}_0 = \rho_m. \quad (21)$$

Equations (14)–(21) constitute the decomposed Maxwell's equations. These equations give an alternative, but completely equivalent formulation for nonrelativistic electromagnetism in \mathbb{R}^3 . Here, of course, when we say electromagnetism, we mean it in the broad sense, and not as the theory of the *electric* and the *magnetic* field. These fields are not present in the above equations. In fact, none of the original fields \mathbf{E} , \mathbf{D} , \mathbf{B} , \mathbf{H} , \mathbf{J} or \mathbf{M} are present in equations (14)–(21). Instead, each of these have split into three components, and each component is governed by its own set of equations: the $+$ -components are governed by equations (14)–(15), the $-$ -components are governed by equations (16)–(17), and the 0-components are governed by equations (18)–(21). Since the 0-components of the electromagnetic fields are curl-free, we can identify them with the non-radiating fields. Correspondingly, we can identify the \pm -components with the radiating fields.

One interpretation of the above is that the fundamental quantities in electromagnetic field theory are not the 6 vector fields \mathbf{E} , \mathbf{D} , \mathbf{B} , \mathbf{H} , \mathbf{J} and \mathbf{M} , but the 18 decomposed fields \mathbf{E}_λ , \mathbf{D}_λ , \mathbf{B}_λ , \mathbf{H}_λ , \mathbf{J}_λ and \mathbf{M}_λ . By Section 3.1, we know that each decomposed field depends only on one real scalar function. Hence the decomposed fields have the same degrees of freedom as the original fields, which depend on 18 Cartesian coordinate functions in \mathbb{R}^3 . However, for the decomposed fields each of the 18 components has a clear physical interpretation. This is not true for the 18 Cartesian component functions for the original fields since the choice of coordinate axes is arbitrary, i.e., does not depend on physics.

We now see that the decomposed Maxwell's equations for the decomposed fields give a much more detailed view of electromagnetism than the traditional Maxwell's equations. For instance, we immediately see the handed nature of electromagnetism. The fields with positive helicity are governed by a different set of equations than the fields with negative helicity. Although these $+$ - and the $-$ -equations are structurally identical, they are formulated on different function spaces. From this observation, it follows that Maxwell's equations are not handed; electromagnetism does not prefer one handedness over the other. Another important observation is that there is no coupling between the different sets of equations. For instance, the equations for \mathbf{E}_+ , \mathbf{D}_+ , \mathbf{B}_+ , \mathbf{H}_+ do not depend on \mathbf{E}_- , \mathbf{D}_- , \mathbf{B}_- , \mathbf{H}_- and vice-versa. Physically this means that these fields propagate independently of each other; the fields with positive helicity do not "see" the fields with negative helicity. This is in sharp contrast to the traditional Maxwell's equations in Cartesian coordinates, where the curl operator couples the x , y and z components of the fields [6].

From the decomposed Maxwell's equations, it can also be seen that the decomposed components of the fields are completely determined by the corresponding components of the sources. This result can be interpreted through *Curie's principle*. It is a general principle in science, which states that a symmetry in the effect can be traced back to a symmetry in the cause [5].

The main disadvantages of the decomposition is that it does not preserve the support of the fields. For instance, even if \mathbf{J} is non-zero only in some small region of \mathbb{R}^3 (for instance inside an antenna), the decomposed fields \mathbf{J}_λ can be non-zero in all of \mathbb{R}^3 (see e.g., [20]). For sources this is problematic since the Green's dyad is singular in the origin [1]. We will not consider this problem.

4.2. Decomposition of the Constitutive Equations

In the previous section we saw that using the helicity decomposition, Maxwell's equations decompose into three uncoupled parts. This result was independent of any choice of media. We also saw that there were numerous advantages of treating these decomposed fields as fundamental quantities in electromagnetism. It is therefore also motivated to seek a formulation for the constitutive equations in terms of these fields. Ideally, such a formulation could give qualitative information about the coupling of say \mathbf{D}_- and \mathbf{E}_+ in different scattering problems. However, even for simple geometries such as a dielectric sphere, it seems to be very difficult to find such a formulation for the constitutive equations. For instance, if $\mathbf{D} = \epsilon(\mathbf{x})\mathbf{E}$, where $\epsilon(\mathbf{x})$ is a real function, then

$$\mathbf{D}_\lambda = \pi_\lambda(\epsilon(\mathbf{x})\mathbf{E}_+) + \pi_\lambda(\epsilon(\mathbf{x})\mathbf{E}_-) + \pi_\lambda(\epsilon(\mathbf{x})\mathbf{E}_0).$$

From this equation we can only deduce that depending on the properties of $\epsilon(\mathbf{x})$ there might be coupling between \mathbf{E}_+ , \mathbf{E}_- , \mathbf{E}_0 , and \mathbf{D}_λ . Unfortunately, this equation gives no deeper insight or qualitative information about the scattering process.

In vacuum, the constitutive equations for the decomposed fields take the form

$$\begin{aligned}\mathbf{D}_\lambda &= \epsilon\mathbf{E}_\lambda, \\ \mathbf{B}_\lambda &= \mu\mathbf{H}_\lambda.\end{aligned}$$

It follows that in this medium, the medium does not couple the decomposition. In other words, the response of this medium does not depend on the helicity of the fields.

For chiral media, let us introduce the constitutive equations

$$\begin{aligned}\mathbf{D}_\lambda &= \epsilon_\lambda\mathbf{E}_\lambda, \\ \mathbf{B}_\lambda &= \mu_\lambda\mathbf{H}_\lambda.\end{aligned}$$

These equations contain six real (constant) medium parameters; ϵ_0, μ_0 describe the response of fields with zero helicity, ϵ_+, μ_+ describe the response of the fields with positive helicity, and ϵ_-, μ_- describe the response of the fields with negative helicity. Let us now use the fact that the helicity decomposition (at least formally) contains the Bohren decomposition as a special case. It then follows that right hand circularly polarized waves see the medium as a different medium than a left hand circularly polarized wave. We can therefore say that the above constitutive equations have a direct geometrical interpretation. They

also imply the following constitutive equations for the undecomposed fields:

$$\mathbf{D} = \frac{\epsilon_+ \sqrt{\epsilon_- \mu_+} + \epsilon_- \sqrt{\epsilon_+ \mu_-}}{\sqrt{\epsilon_- \mu_+} + \sqrt{\epsilon_+ \mu_-}} \mathbf{E} + i(\epsilon_+ - \epsilon_-) \frac{\sqrt{\mu_- \mu_+}}{\sqrt{\epsilon_- \mu_+} + \sqrt{\epsilon_+ \mu_-}} \mathbf{H}, \quad (22)$$

$$\mathbf{B} = \frac{\mu_+ \sqrt{\epsilon_+ \mu_-} + \mu_- \sqrt{\epsilon_- \mu_+}}{\sqrt{\epsilon_- \mu_+} + \sqrt{\epsilon_+ \mu_-}} \mathbf{H} + i(\mu_- - \mu_+) \frac{\sqrt{\epsilon_+ \epsilon_-}}{\sqrt{\epsilon_- \mu_+} + \sqrt{\epsilon_+ \mu_-}} \mathbf{E}. \quad (23)$$

These equations are derived by a formal calculation similar to the derivation of the helicity decomposition from the Bohren decomposition. In this derivation, the sources are assumed to be zero [11].

4.3. Scalar Formulation for Electromagnetics

In this section we use the Moses representation for the decomposed fields to formulate the decomposed Maxwell's equations. This will yield a completely scalar formulation for electromagnetism. From this formulation we derive a new duality transformation [1] with relation to linear symplectic geometry [21].

We shall use the same notation as in Section 3.1. If \mathbf{E} is a vector field, then $\mathbf{E}_\lambda = \mathcal{F}^{-1}\{e_\lambda \mathbf{Q}_\lambda\}$, so, for λ in $\{0, \pm 1\}$, e_λ are functions $\mathbb{R}^3 \rightarrow \mathbb{C}$ that determines \mathbf{E}_λ . Similarly, we denote by $d_\lambda, b_\lambda, h_\lambda, j_\lambda, m_\lambda$ the functions that determine $\mathbf{D}_\lambda, \mathbf{B}_\lambda, \mathbf{H}_\lambda, \mathbf{J}_\lambda, \mathbf{M}_\lambda$. Then, defining

$$\mathbf{e} = \begin{pmatrix} e_- \\ h_+ \\ e_+ \\ h_- \end{pmatrix}, \quad \mathbf{J} = \left(\begin{array}{c|c} & 1 \\ \hline & 1 \\ -1 & \\ \hline & -1 \end{array} \right), \quad \mathbf{f} = \begin{pmatrix} b_+ \\ d_- \\ b_- \\ d_+ \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} m_+ \\ j_- \\ m_- \\ j_+ \end{pmatrix},$$

Maxwell's equations take the form

$$-2\pi |\mathbf{k}| \mathbf{J} \mathbf{e} = \partial_t \mathbf{f} + \mathbf{s}, \quad (24)$$

and

$$\partial_t b_0 = -m_0, \quad (25)$$

$$\partial_t d_0 = -j_0, \quad (26)$$

$$-2\pi i |\mathbf{k}| b_0 = \hat{\rho}_m, \quad (27)$$

$$-2\pi i |\mathbf{k}| d_0 = \hat{\rho}, \quad (28)$$

where $\hat{\rho}$ and $\hat{\rho}_m$ are the L^2 scalar Fourier transforms of ρ and ρ_m , and ∂_t is the time derivative.

We next derive a dual transformation for the linear equation (24). To do this, we first make the assumption that we can write $\mathbf{f} = \mathbf{M}\mathbf{e}$ for some, possibly complex, invertible 4×4 matrix \mathbf{M} , that is possibly a function of \mathbf{k} , but not a function of t . For \mathbf{f} to be real valued, \mathbf{M} should also satisfy $\mathbf{M}(-\mathbf{k}) = \mathbf{M}^*(\mathbf{k})$. A necessary condition for writing $\mathbf{f} = \mathbf{M}\mathbf{e}$ is that there is no coupling between the 0-components and the \pm -components of the fields. In scattering problems, this assumption should hold with good accuracy if the scatterer is much smaller than the wavelength of the wave. Under this assumption, equation (24) reads

$$-2\pi|\mathbf{k}|\mathbf{J}\mathbf{e} = \partial_t\mathbf{M}\mathbf{e} + \mathbf{s}. \quad (29)$$

This is a set of ordinary differential equations which can be solved using traditional methods [6]. If equation (29) holds, then we say that \mathbf{e} is the solution corresponding to the source \mathbf{s} in the medium \mathbf{M} . Next we introduce a transformation matrix \mathbf{T} , which should have the same properties as \mathbf{M} . By writing $\mathbf{e} = \mathbf{T}\mathbf{T}^{-1}\mathbf{e}$, equation (29) can be manipulated into the form

$$-2\pi|\mathbf{k}|\mathbf{J}(\mathbf{T}^{-1}\mathbf{e}) = \partial_t\mathbf{J}\mathbf{T}^{-1}\mathbf{J}^{-1}\mathbf{M}\mathbf{T}(\mathbf{T}^{-1}\mathbf{e}) + \mathbf{J}\mathbf{T}^{-1}\mathbf{J}^{-1}\mathbf{s}.$$

This gives the following duality transformation. If \mathbf{e} is the solution corresponding to the source \mathbf{s} in the medium \mathbf{M} , then the transformed field $\mathbf{e}' = \mathbf{T}^{-1}\mathbf{e}$ is a solution corresponding to the source $\mathbf{s}' = \mathbf{J}\mathbf{T}^{-1}\mathbf{J}^{-1}\mathbf{s}$ in the medium $\mathbf{M}' = \mathbf{J}\mathbf{T}^{-1}\mathbf{J}^{-1}\mathbf{M}\mathbf{T}$.

Let us assume that \mathbf{T} further satisfies the relation $\mathbf{T}\mathbf{J}\mathbf{T}^H = \mathbf{J}$, where $\mathbf{T}^H = \mathbf{T}^{T*}$. Then \mathbf{s}' and \mathbf{M}' take the simple forms $\mathbf{s}' = \mathbf{T}^H\mathbf{s}$ and $\mathbf{M}' = \mathbf{T}^H\mathbf{M}\mathbf{T}$. These forms are obtained using $\mathbf{J}^{-1} = \mathbf{J}^T = -\mathbf{J}$. Here, the condition for \mathbf{T} is a natural generalization of real *symplectic matrices* to complex matrices; a real 4×4 matrix \mathbf{T} is *symplectic* if $\mathbf{T}\mathbf{J}\mathbf{T}^T = \mathbf{J}$ [21].

5. CONTACT GEOMETRY

Contact geometry is the study of *contact structures*. These are certain topological structures that can exist on odd dimensional manifolds. Similarly, *symplectic geometry* is the study of *symplectic structures*. These are also certain topological structures, but these can only exist on even dimensional manifolds. These theories are dual in the sense that they are closely related and have many results in common. One can very roughly say that if the fundamental quantity in Riemannian geometry is length, then the fundamental quantity in symplectic

geometry is directed area, and the fundamental quantity in contact geometry is a certain twisting behavior. A characteristic feature for both contact and symplectic geometry is that they have both been found in numerous areas of physics and mathematics (see [11]).

Since both contact and symplectic structures are purely topological structures, they do not depend on any metric structure of the underlying space. Therefore it is not motivated to study these structures using standard vector analysis, where geometry and topology is intertwined. For these reasons, we will use the language of differential forms on manifolds to describe contact geometry. We will use the same definition of a manifold as in [22]. An n -dimensional manifold M^n is a topological Hausdorff space with countable base that is locally homeomorphic to \mathbb{R}^n [22]. In addition, we shall always assume that all transition functions are C^∞ -smooth. That is, we shall only consider C^∞ -smooth manifolds. The space of differential p -forms on M^n is denoted by $\Omega^p(M^n)$, and the tangent space of M^n is denoted by TM^n . The Einstein summing convention is used throughout. Hereafter, we shall assume that all mathematical objects (e.g., functions, p -forms and vector fields) are C^∞ -smooth. This is a standard assumption in differential geometry. However, since the natural function space for electromagnetism is L^2_{curl} , this assumption gives some mathematical problems when studying “contact and symplectic geometry in electromagnetism”. We shall not study this problem.

5.1. Contact Structures

Contact geometry is, in general, an odd dimensional theory. However, in view of our application to electromagnetism, we shall only study contact structures in three dimensions. On a 3-manifold, we define a contact structure as a plane field that is completely non-integrable:

Definition 5.1 (Planefield) A planefield ξ on a 3-manifold M^3 is a smooth mapping $p \mapsto \xi_p$ defined for all $p \in M^3$ such that ξ_p is a 2-dimensional vector subspace of the tangent space $T_p M^3$ (the tangent space of M^3 at p).

Definition 5.2 Let ξ be a planefield on a 3-manifold M . Then ξ is integrable at $p \in M$, if there exists a smooth surface S passing through p such that ξ is tangential to S in some neighborhood of p . Moreover ξ is integrable, if ξ is integrable at every point of M .

Definition 5.3 [18] A planefield ξ on a 3-manifold M^3 is a contact structure, if and only if ξ is everywhere non-integrable.

On a 3-manifold, a two dimensional plane field is locally determined as the kernel of a 1-form. A contact structure ξ that globally can be written as the kernel of a 1-form is said to be *transversally oriented*. Then $\xi = \ker \alpha$ for some $\alpha \in \Omega^1(M^3)$, and α is said to be a *contact form* for ξ . We will only consider such contact structures. Due to the next theorem, this is a standard assumption in contact geometry.

Theorem 5.4 (Frobenius theorem)[23, 24] *Let α be a 1-form on a 3-manifold. The plane field $\xi = \ker \alpha$ is a contact structure if and only if $\alpha \wedge d\alpha$ is nowhere zero.*

The above theorem shows that if M^3 has a contact structure, then the contact structure induces an orientation on M^3 given by the volume-form $\alpha \wedge d\alpha$. It is then possible to compare *orientations* of contact structures as follows. The contact structures $\ker \alpha$ and $\ker \alpha'$ have the *same (opposite) orientation*, if $\alpha \wedge d\alpha = f\alpha' \wedge d\alpha'$ for a positive (negative) function f . If we scale α by some positive or negative function f , then the plane field $\ker \alpha$ is clearly invariant, so $\ker \alpha$ and $\ker f\alpha$ have the same orientation. However, the induced volume-form $\alpha \wedge d\alpha$ depends on the choice of α .

Example 5.5 (The standard structures on \mathbb{R}^3) On \mathbb{R}^3 with coordinates x, y , and z , let $\alpha_{\pm} = xdy \pm dz$. Then $\alpha_{\pm} \wedge d\alpha_{\pm} = \pm dx \wedge dy \wedge dz$, so $\ker \alpha_{\pm}$ are contact structures with opposite orientations. Usually, either of these are called the *standard contact structure* on \mathbb{R}^3 . Since we have no reason to prefer one orientation over the other, we here call $\ker \alpha_+$ and $\ker \alpha_-$ the *standard contact structures* on \mathbb{R}^3 .

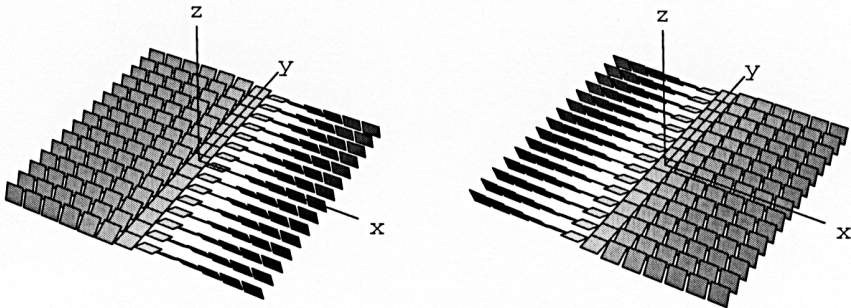


Figure 2. The standard structures on \mathbb{R}^3 .

In Figure 2 the plane fields $\ker \alpha_+$ and $\ker \alpha_-$ are plotted in the xy -plane. Since α_{\pm} do not depend on z , the plane fields are only plotted for $z = 0$. The plots show the vector spaces that α_{\pm} map to zero as small tiles. For instance, when $x = 0$, $\alpha_{\pm} = \pm dz$. That means that

(at $x = 0$) all vectors in the xy -plane are mapped to zero. At $x = 0$, the tiles are thus oriented perpendicular to the z -direction. \square

5.2. Contact Structures, Beltrami Fields, and Helicity

By definition, a contact structure can not be a tangential to any smooth surface. In other words, contact structures must be constantly twisting so that the planes, i.e., vector sub-spaces, can not be “stitched” together into a smooth surface. This characteristic twisting can, for instance, be seen in Figure 2. In Section 2, we noted that Beltrami fields also possess a characteristic twisting. It is therefore not surprising that there is a connection between Beltrami fields and contact structures. This correspondence is established in [18]. It is shown that every Beltrami field induces a contact structure and a converse: If ξ is a contact structure, then there exist a Riemannian metric and a vector field X (determined up to a scaling) such that $\nabla \times X = X$. In this section we prove Proposition 5.6, which shows how contact structures can be generated from certain Beltrami fields on 3-manifolds. For this reason, we must first generalize the definition of the curl operator and Beltrami fields to 3-manifolds.

To transform vectors into 1-forms and vice-versa, we use the standard isomorphisms induced by the Riemannian metric $g = g_{ij}dx^i \otimes dx^j$ [24]. By contracting the metric with the vector field $X = X^i \frac{\partial}{\partial x^i}$, we obtain the 1-form $X^\flat = g_{ij}X^i dx^j$. This \flat -mapping transforms vector fields into 1-forms. Since g_{ij} is positive definite, the \flat -mapping also has an inverse, a \sharp -mapping. If $\alpha = \alpha_i dx^i$ is a 1-form, then $\alpha^\sharp = g^{ij}\alpha_i \frac{\partial}{\partial x^j}$ where g^{ij} are the elements of the matrix $(g_{ij})^{-1}$ [24]. On an oriented Riemannian 3-manifold, we define the *curl* of a vector field X as the vector field $\nabla \times X$ for which $(\nabla \times X)^\flat = *dX^\flat$. Here $*$ is the *Hodge star operator* [24, 25]. On an oriented 3-manifold M^3 , we then say that an everywhere non-vanishing vector field X is a *Beltrami vector field*, if $*dX^\flat = fX^\flat$ for some smooth function $f : M^3 \rightarrow \mathbb{R}$. Reading this as an equation for the 1-form X^\flat , it is motivated to call X^\flat a *Beltrami 1-form*. Further, if f does not vanish at any point of M^3 , then α is a *rotational Beltrami 1-form*.

Theorem 5.6 (Etnyre, Ghrist) [18] *Let M^3 be a Riemannian 3-manifold, and let α be a non-vanishing rotational Beltrami 1-form on M^3 . Then α is a contact form on M^3 .*

Proof. We have $*d\alpha = f\alpha$ for some non-vanishing function $f : M^3 \rightarrow \mathbb{R}$. Then $\alpha \wedge d\alpha = f\alpha \wedge *\alpha$. The claim follows since $\alpha \wedge *\alpha$ only vanishes where α vanishes. \square

Example 5.7 (The standard overtwisted contact structures)

On \mathbb{R}^3 with coordinates x, y , and z , let us define $\alpha_{\pm} = \cos(kx)dz \pm \sin(kx)dy$. For these, we have that $*d\alpha_{\pm} = \pm k\alpha_{\pm}$, so α_{\pm} are rotational Beltrami 1-forms. Hence, by Theorem 5.6, it follows that $\ker \alpha_{\pm}$ are contact structures (unless $k = 0$). We also have that $\alpha_{\pm} \wedge d\alpha_{\pm} = \pm kdx \wedge dy \wedge dz$. Thus $\ker \alpha_{\pm}$ are contact structures with opposite orientations. The structures $\ker \alpha_{\pm}$ are called the *standard overtwisted contact structures on \mathbb{R}^3* . In Figure 3 these are plotted when $k = 1$ and x range from $-\pi$ to π . \square

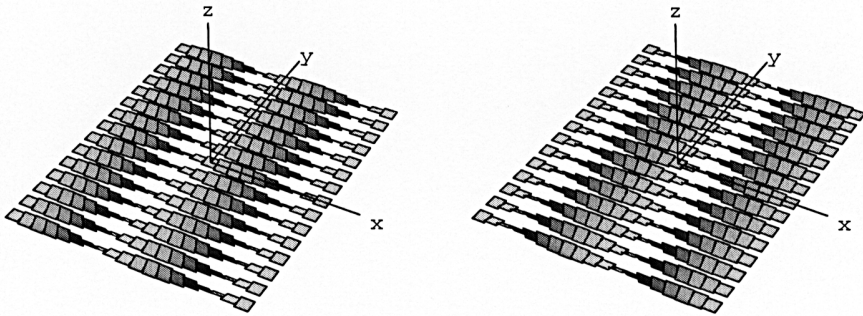


Figure 3. The standard overtwisted contact structures on \mathbb{R}^3 .

The previous example suggests that the orientation of a contact structure is related to the handedness of the twisting in the contact structure. This is indeed the case as we next show. More precisely, we show that $\alpha \wedge d\alpha$ is the equivalent to the helicity density $\mathbf{F} \cdot \nabla \times \mathbf{F}$ of a vector field. In Section 2 we showed that the sign of $\mathbf{F} \cdot \nabla \times \mathbf{F}$ is a measure of the handed twisting of a vector field. Therefore, since $\alpha \wedge d\alpha$ can never vanish (i.e., change sign) in a transversally oriented contact structure, we can interpret contact structures as everywhere twisting structures with a constant handedness.

On a Riemannian manifold, the *dot product* of two vector fields X, Y is defined as $X \cdot Y = g(X, Y)$. Also, for 1-forms α and β , we have that $g(\alpha^\sharp, \beta^\sharp)dV = \alpha \wedge * \beta$. Then, from $\mathbf{F} \cdot \nabla \times \mathbf{F}dV = F^\flat \wedge (\nabla \times F)^\flat = F^\flat \wedge d(F^\flat)$ it follows that it is not natural to define helicity for the vector field F . Instead, helicity should be defined for the 1-form F^\flat . Then helicity does not depend on the metric.

Definition 5.8 (Helicity) Let α be a 1-form on a 3-manifold M . The helicity of α is defined as

$$\mathcal{H}(\alpha) = \int_M \alpha \wedge d\alpha.$$

This definition of helicity generalizes Definition 2.1: under the assumptions in Definition 2.1, $\mathcal{H}(\alpha^\sharp) = \mathcal{H}(\alpha)$.

5.3. Darboux's Theorem for Contact Structures

A surprising property of contact structures (of same dimension) is that they all locally look the same. This result is known as Darboux's theorem, and its interpretation is that all interesting information about contact structures is of global nature. The study of these global properties is called *contact topology* [26]. For the contact structures in Examples 5.5 and 5.7, this local invariance can be seen by letting $x \rightarrow 0$. Then $\cos(x)dz \pm \sin(x)dy$ approaches $dz \pm xdy$; the standard structures on \mathbb{R}^3 . Thus, up to a rotation in the yz -plane, these overtwisted structures locally look like the standard structures on \mathbb{R}^3 .

Definition 5.9 [21] *Let ξ be a contact structure on a 3-manifold M , and let η be a contact structure on a 3-manifold N . The structures ξ and η are contactomorphic if there exists a diffeomorphism $f : M \rightarrow N$ such that $f_*\xi = \eta$. Then f is a contactomorphism.*

In the above definition f_* is the *push-forward* of the map $f : M \rightarrow N$ [24]. It maps vector fields on M to vector fields on N . The push-forward map naturally extends to plane fields on M . If $\xi = \text{span}\{X, Y\}$, then $f_*\xi = \text{span}\{f_*X, f_*Y\}$.

Theorem 5.10 (Darboux's theorem) [21] *Let ξ and ξ' be contact structures on two 3-manifolds M and N . Then ξ and ξ' are locally contactomorphic.*

The above result states the following. If $x \in M$ and $y \in N$, then there exist some neighborhoods $U \subset M$ ($x \in U$) and $V \subset N$ ($y \in V$) and a diffeomorphism $f : U \rightarrow V$, such that $\xi|_U$ is contactomorphic to $\eta|_V$. Here, $\xi|_U$ is the *restriction* of ξ to U . Darboux's theorem, for instance, states that any contact structure on a 3-manifold is locally contactomorphic in an orientation preserving way to one of the standard contact structures on \mathbb{R}^3 .

5.4. The Carnot-Carathéodory Metric

Theorem 5.11 [27] *Let ξ be a contact structure on a connected 3-manifold M^3 . Then any two points in M^3 can be connected by a piecewise smooth curve such that each component is tangential to the contact structure.*

Suppose ξ is a contact structure on a Riemannian 3-manifold M^3 . Then we can use Theorem 5.11 to define a new metric on M^3 . If we are given two points p and q on M^3 , then they can be connected by

some curve tangential to ξ . Since M^3 has a Riemannian metric, we can measure the length of this curve. Further, if there are many ways to connect the two points, we can take the infimum of the lengths of all such curves. The *Carnot-Carathéodory distance* between p and q is defined as this infimum. The Carnot-Carathéodory metric satisfies the axioms for a metric. However, it is not a Riemannian metric. Also, due to the infimum in the definition, it is usually only possible to calculate an upper bound for the Carnot-Carathéodory metric.

6. CONTACT GEOMETRY FROM HELMHOLTZ'S EQUATION

Next we study contact structures derived from solutions to Helmholtz's equation. More precisely, we start with a solution to Helmholtz's equations, decompose it, and show that the decomposed fields always seem to induce contact structures. To perform the decomposition in this section we shall use the Bohren decomposition. In Section 3 we saw that for solutions to the sourceless Helmholtz's equation, the helicity decomposition (at least formally) is equal to the Bohren decomposition. Thus, if \mathbf{E} is a solution to the sourceless Helmholtz's equations $\nabla \times (\nabla \times \mathbf{E}) = k^2 \mathbf{E}$, the decomposed fields in the time domain are

$$\mathbf{E}_{\pm} = \frac{1}{2} \Re \left\{ \left(\mathbf{E} \pm \frac{1}{k} \nabla \times \mathbf{E} \right) e^{-i\omega t} \right\}. \quad (30)$$

The advantage of using this formula is that it is local. We can therefore apply it to solutions which are not necessarily in L^2_{curl} . (We will, for instance, study contact structures for plane waves.) If \mathbf{E} is a solution to Helmholtz's equation, then \mathbf{E}_{\pm} are Beltrami fields. If they, in addition, do not vanish at any point, then they induce two contact structures, $\ker(\mathbf{E}_+)^{\flat}$ and $\ker(\mathbf{E}_-)^{\flat}$. In this section, we will always use the Cartesian metric. We shall therefore make no distinction between vector fields and 1-forms.

6.1. Contact Structures from Planewaves

From equation (30), it follows that the decomposed components for the plane wave 5 are

$$\mathbf{E}_{\pm}(z, t) = \frac{1}{2} \Re \left\{ (\mathbf{A} \pm i\mathbf{u}_z \times \mathbf{A}) e^{i(kz - \omega t)} \right\}. \quad (31)$$

These are circularly polarized plane waves with opposite orientations. More precisely, the helicity densities for the decomposed fields are constant and proportional to the energy densities of the decomposed

fields [11]. Thus, in general, a plane wave induces two contact structures; one for the RCP component and one for the LCP component.

If we let $\mathbf{A} = \mathbf{u}_x$, then the contact structures induced by the fields \mathbf{E}_\pm , look like the standard overtwisted contact structures in Figure 3, i.e., the contact planes constantly rotate around the direction of propagation. Since the value of t does not modify this behavior, we set $t = 0$ to simplify the analysis. Then we see that if an RCP (or LCP) plane wave passes through two points, then the path given by the Carnot-Carathéodory metric between these points is the straight line connecting the points. Thus, for a plane wave in isotropic homogeneous space, the Carnot-Carathéodory metric describes the path traversed by the wave.

Let us next consider an RCP wave which changes direction due to a plane boundary. From this wave, we then get one contact structure. (Here we do not take into account the (non-smooth) sudden change in direction due to the boundary.) For this contact structure, it would seem very plausible that the Carnot-Carathéodory metric describes the propagation of the RCP wave. Indeed, suppose we take one point above the boundary and one point below the boundary such that the wave passes through both points. Then the broken line that describes how the wave connects these points is tangential to the contact structure, i.e., an admissible path for the Carnot-Carathéodory metric. Unfortunately, it seems to be quite difficult to show that no shorter path tangential to the contact structure exists. It would, however, seem very natural that the minimizing path would be the piecewise straight line. If that is indeed the case, then at least for plane waves and plane boundaries, the Carnot-Carathéodory metric would correctly describe the path of RCP/LCP waves. Since the RCP and LCP components induce two different contact structures and thus two different Carnot-Carathéodory metrics, these induced metrics would take into account the different scattering behaviors for different polarizations in chiral media.

6.2. Contact Structures in a Rectangular Waveguide

In this section we consider solutions to Helmholtz's equation in a rectangular waveguide. Since explicit expressions for the solutions are known, it is straightforward to decompose these using equation 30 (see [11]). In Figures 4–6 the plane fields induced by the $+$ -components of the electric field for the TE_{01} , TE_{11} , TM_{11} , TE_{21} are shown. All solutions are 2π periodic, but they are only plotted for $z = 0, \frac{1}{3}\frac{\pi}{2}, \frac{2}{3}\frac{\pi}{2}$, and $\frac{\pi}{2}$ as these plots show the basic twisting behavior for the plane field. All the plots are plotted for $t = 0$. Also, we only plot the $+$ -

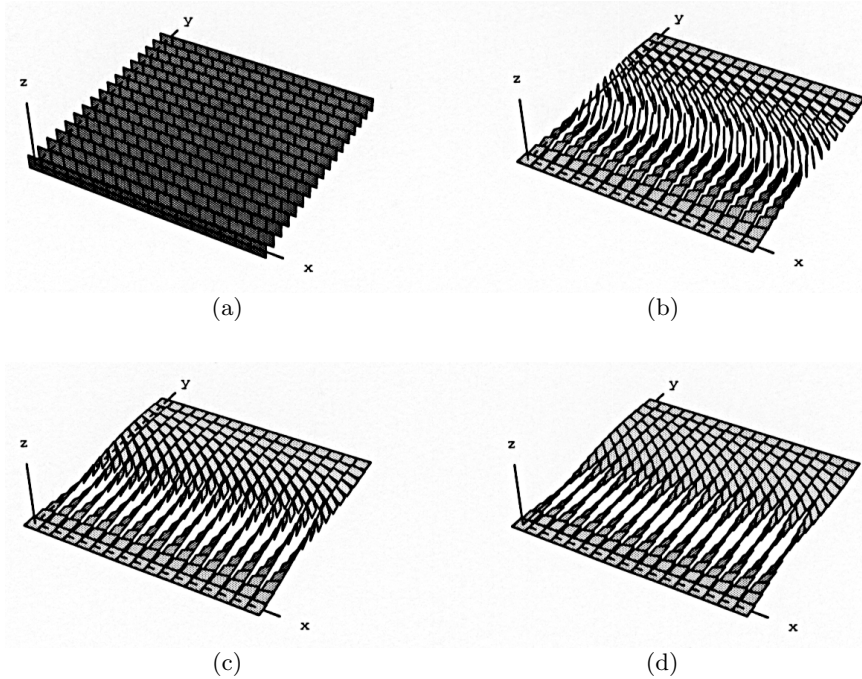


Figure 4. TE_{01} $+$ -field at $z = \frac{0}{3}\frac{\pi}{2}, \dots, \frac{3}{3}\frac{\pi}{2}$.

component since the $-$ -component is symmetrical; it simply twists with opposite helicity. Using computer algebra one can show that these are contact structures. This shows that contact structures is not something peculiar to only plane waves (i.e., linear optics), but contact structures also exist in more complicated solutions to Helmholtz's equation.

From these figures we can make an interesting observation. Namely, the TE and TM solutions are somehow symmetrical. The TE_{21} solution is obtained from the TM_{21} solution by shifting the solution in the xy -plane. This would suggest that (at least from a theoretical point of view) it is more natural to divide the fields inside a waveguide into $+$ -solutions and $-$ -solutions. The advantage of such a division would be that it would not be based on Cartesian coordinates (see [11]). Instead, a $+/-$ division would divide solutions in a waveguide into two sets of solutions which propagate independently of each other. This division thus has a physical interpretation. We could also say that the $+/-$ division represents the internal division of the fields in a waveguide whereas the TE/TM division is based on our Cartesian view of electromagnetism.

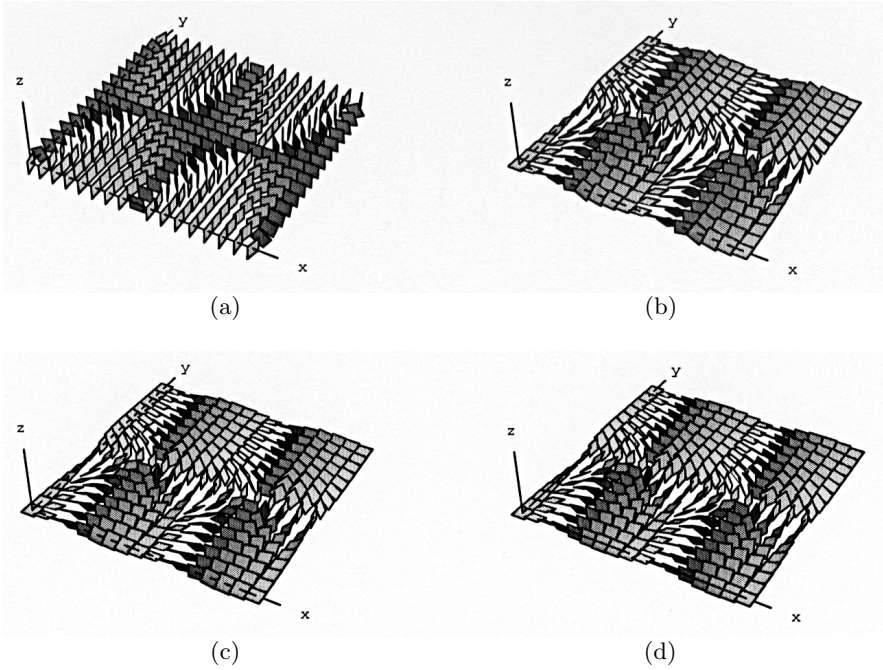


Figure 5. TE_{21} +-field at $z = \frac{0}{3}\frac{\pi}{2}, \dots, \frac{3}{3}\frac{\pi}{2}$.

6.3. Local Invariance of Helmholtz's Equations

Suppose we have two solutions \mathbf{E} and \mathbf{E}' to the Helmholtz's equation. By equation 30, these induce four Beltrami fields \mathbf{E}_{\pm} and \mathbf{E}'_{\pm} . For this section, let us assume that none of these fields vanish at any point. Then, by Theorem 5.6, they induce four contact structures. By Darboux's theorem, we know that any two contact structures are *locally* contactomorphic. Thus, the contact structures induced by \mathbf{E}_{+} and \mathbf{E}_{-} are locally contactomorphic to the contact structures induced by \mathbf{E}'_{+} and \mathbf{E}'_{-} . In addition, since the volume forms $(\mathbf{E}_{\pm})^b \wedge d(\mathbf{E}_{\pm})^b$ and $(\mathbf{E}'_{\pm})^b \wedge d(\mathbf{E}'_{\pm})^b$ have the same orientation, these contactomorphisms are both orientation preserving. By adding a possible scaling to these contactomorphisms, we can construct mappings f_{\pm} as in the diagram below.

$$\begin{array}{ccc}
 \mathbf{E} = \mathbf{E}_{+} + \mathbf{E}_{-} & & \\
 \downarrow f_{+} & & \downarrow f_{-} \\
 \mathbf{E}' = \mathbf{E}'_{+} + \mathbf{E}'_{-} & &
 \end{array}$$

This means that if we have two solutions \mathbf{E} and \mathbf{E}' to Helmholtz's equation, whose decomposed fields do not vanish, then locally \mathbf{E} can be

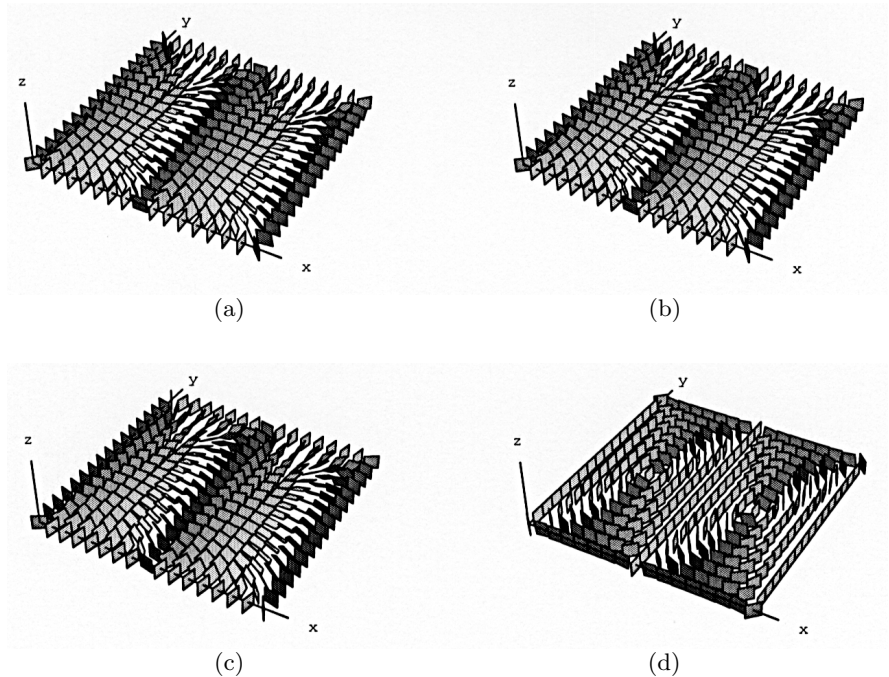


Figure 6. TM_{21} +-field at $z = \frac{0}{3}\frac{\pi}{2}, \dots, \frac{3}{3}\frac{\pi}{2}$.

transformed into \mathbf{E}' . From this result we can make two observations. First, to transform a solution to Helmholtz's equation into another solution, one needs, in general, two mappings; one for \mathbf{E}_+ and \mathbf{E}'_+ , and one for \mathbf{E}_- and \mathbf{E}'_- . Second, the above result states that all solutions to Helmholtz's equation are, in some sense, similar to each other. One interpretation is that the contact structures for \mathbf{E}_+ and \mathbf{E}_- contain the necessary twisting for the field to radiate.

7. CONCLUSIONS

In this work we have studied contact and symplectic geometry and their relation to electromagnetics. Since contact and symplectic geometry has been found in numerous other areas of physics, this study is highly motivated. In Section 6, we have shown that known solutions to Helmholtz's equation always seem to induce contact structures. However, from the present work, we can not say whether the decomposed fields of an arbitrary solution to Maxwell's equations also induce contact structures. The problem is that the helicity

decomposition only assures that, say, for the electric field \mathbf{E} , we have $\int_{\mathbb{R}^3} \mathbf{E}_+ \cdot \nabla \times \mathbf{E}_+ d\mathbf{x} \geq 0$. To prove that $\ker \mathbf{E}_+^b$ is a contact structure, one should be able to conclude that $\mathbf{E}_+ \cdot \nabla \times \mathbf{E}_+ > 0$. Since the helicity decomposition is based on the Fourier transform, it can be very difficult to prove such local properties for the decomposed fields. Probably the most simple way to gain further insight into this problem, would be to perform numerical experiments.

However, if the decomposed fields in Maxwell's equations would always induce contact structures, it would be a very attractive result since it would give more "structure" to electromagnetism. If one can always assume that a solution splits into three components, and two of these would be contact structures, one can make much more assumptions, and possibly derive quite general results for solutions to Maxwell's equations. For instance, since contact geometry has been studied as a mathematical branch, there are many results, which could be applied directly to electromagnetism. As an example, we used Darboux's theorem in Section 6.3 to derive a local invariance result for solutions to Helmholtz's equations. By similar argumentation, we could use Darboux's theorem to show that all solutions to Maxwell's equations locally look like the standard contact structure. This could possibly be used to design a numerical solver for Maxwell's equations. For instance, if we compare the decomposed solutions in Figures 4–6 to the the figures of the standard structure in Figure 3, we see that they are very similar. That would suggest that in such a solver, one would not need too many elements to model the solution. However, how the 0-field should be modeled in such a solver is not quite clear.

Another motivation for studying contact and symplectic geometry in electromagnetism is that these structures are purely topological. In other words, they do not require an external structure such as a Riemannian metric. A very interesting result, which is related to this, is that both contact and symplectic structures induce their own internal "Hodge operators", i.e., mappings $\Omega^p(M^n) \rightarrow \Omega^{n-p}(M^n)$ [28]. In this work we have not studied these mappings. However, it is quite possible that using these mappings, one could formulate the constitutive equations. If that would be possible, it would yield an almost topological formulation for electromagnetism. In such a formulation, the only metrical dependence would be due to the helicity decomposition. An alternative approach would be to treat the decomposed fields as fundamental quantities of electromagnetism. If one further assumes that these are contact structures, and that the constitutive equations could be written using the induced Hodge operators, that would yield a completely topological formulation for electromagnetism.

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