# DISCRETE HODGE-OPERATORS: AN ALGEBRAIC PERSPECTIVE

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**Abstract**—Discrete differential forms should be used to deal with the discretization of boundary value problems that can be stated in the calculus of differential forms. This approach preserves the topological features of the equations. Yet, the discrete counterparts of the metric-dependent constitutive laws remain elusive.

I introduce a few purely algebraic constraints that matrices associated with discrete material laws have to satisfy. It turns out that most finite element and finite volume schemes comply with these requirements. Thus convergence analysis can be conducted in a unified setting. This discloses basic sufficient conditions that discrete material laws have to meet in order to ensure convergence in the relevant energy norms.

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## 1. INTRODUCTION

The focus of this paper is on linear stationary boundary value problems that can be expressed in the calculus of differential forms (For

introduction/survey see [19, 33, 9, 10]). Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be some domain, whose (piecewise smooth) boundary $\partial\Omega$ is partitioned into $\Gamma_D$, $\Gamma_N$, and $\Gamma_M$. The generic boundary value problem stated in the language of differential forms is given by

$$d\,u = (-1)^l \boldsymbol{\sigma} \quad , \quad d\,\mathbf{j} = -\psi + \phi$$
$$\mathbf{t}_D u = 0 \quad \text{on } \Gamma_D \quad , \quad \mathbf{t}_N \mathbf{j} = 0 \quad \text{on } \Gamma_N$$
$$\mathbf{j} = \star_\alpha \boldsymbol{\sigma} \quad , \quad \psi = \star_\gamma u \quad \text{in } \Omega$$
$$\mathbf{t}_M \mathbf{j} = (-1)^{l-1} \star_\beta^\Gamma \mathbf{t}_M u \quad \text{on } \Gamma_M .$$

Here, $d$ is the exterior derivative of exterior calculus, and $u$, $\boldsymbol{\sigma}$, $\mathbf{j}$, $\psi$, and $\phi$ are differential forms of different orders. More precisely, if $u$ is an $l-1$-form, $l \in \mathbb{N}$, then $\sigma$ is an $l$-form, $\mathbf{j}$ an $m$-form, $m := n - l$, whereas both $\psi$ and $\phi$ are of order $m + 1$. In (1), $\phi$ can be regarded as source term, while we have to solve for the other forms $u$, $\boldsymbol{\sigma}$, $\mathbf{j}$, and $\psi$. Of course, knowledge of $u$ is sufficient.

The linear operators $\mathbf{t}_D$, $\mathbf{t}_N$, and $\mathbf{t}_M$ stand for the trace of differential forms onto $\Gamma_D$, $\Gamma_N$, and $\Gamma_M$, respectively [33, Sect. 1.10]. They preserve the order of the form and commute with exterior differentiation.

We pay special attention to the *Hodge-operators* $\star_\gamma$ and $\star_\beta$. They supply linear mappings of (continuous) $l$-forms into $n - l$-forms. In contrast to most other concepts in the theory of differential forms, *Hodge-operators* are not meaningful on their own, but they have to be spawned by Riemannian metrices. For the details I refer to [33, Sect. 1.4] or [7, Sect. 4.5]. Therefore, in (1) $\alpha$, $\gamma$ and $\beta$ designate uniformly positive metrices on $\Omega$ and $\Gamma_M$, respectively.

Translated into vector calculus (1) covers a wide array of boundary value problems. For instance, in three dimensions we recover standard second order elliptic boundary value problems in the case $l = 1$. Boundary value problems involving the double -**curl**-operator that arise in quasistatic electromagnetism are included as the case $l = 2$.

The formulation (1) is often dismissed as nice but moot, because people doubt whether the perspective of differential forms brings any tangible gains. The main message I want to send in this paper is that there is a substantial benefit both for the design, understanding, and error analysis of discretization procedures. I am by no means the first to make this point. I would like to mention Matiussi [35], Tonti [49], Dezin [24], Shashkov [30, 32], Chew [20, 48], and, most prominently, Bossavit [5, 8, 9, 7, 4, 3, 47]. Whereas the foundation is borrowed from these works, I am setting out to build upon it a *general unifying framework for the quantitative analysis of a wide class of finite element and finite volume schemes*. In a sense, the present paper supplements the

previous conceptual works by what is cherished as "rigorous analysis" and "abstract treatment". The reader can view it as an extension of A. Bossavit's paper [13], in which the concepts are exposed for the concrete case of Maxwell's equations. The benefit of abstraction is that fundamental relationships between different discretization schemes will be disclosed, widening the scope of techniques originally developed for only one type of method.

One insight can already be gleaned from (1). It is obvious that the top line in (1) and the boundary conditions on $\Gamma_D$ and $\Gamma_N$ do not involve any metric. They represent the topological relationships underlying the boundary value problems. I will refer to them as *equilibrium equations* (topological field equations in [48]). On top of that the two equilibrium equations are not directly coupled. One may view them as equations for different kinds of differential forms: "ordinary" forms and twisted forms [18, 7, 48, 13]. The link between the equilibrium equations is established by the *constitutive relations* (material laws, metric field equations [48]) that essentially depend on metrics.

Admittedly, there is a price tag on generality. The results I am getting may be weaker than those obtained through more specialized techniques. In addition, quite a few cumbersome details may still be left to work out for specific schemes. However, the sheer scope of the method compensates for these drawbacks. It can also serve as a reliable guide to the construction of appropriate schemes.

The paper is organized as follows: The next section reviews discrete differential forms, i.e., finite elements for differential forms. The third section introduces the key concept of discrete Hodge-operators and gives a purely algebraic characterization. Similar, though more restricted, approaches to the construction of discrete Hodge operators are pursued in [47] and [48, Sect. VII]. The fourth section studies examples of discrete Hodge operators. In particular, the focus is on finite volume methods. Using only a few basic algebraic properties of discrete Hodge operators, abstract bounds for the energies of the discretization errors are established in the fifth section. This is further elaborated in the case of concrete schemes in the sixth section.

## 2. DISCRETE DIFFERENTIAL FORMS

When solving (1) numerically, we hope to get a good approximation of all or some of the unknown differential forms that can be described by a finite number of real numbers. It is reasonable to insist that this approximation is a valid differential form itself. In short, as result of the computation we expect to get a discrete differential form.

DEFINITION 2.1 *A sequence of spaces $\mathcal{W}^l$, $0 \leq l \leq n$, provides discrete differential forms on $\Omega$, if*

- *the integral of any $\omega \in \mathcal{W}^l$ over any piecewise smooth $l$-dimensional oriented manifold is well defined.*
- $d\mathcal{W}^l \subset \mathcal{W}^{l+1}$ *for $0 \leq l < n$.*
- *all the spaces $\mathcal{W}^l$ have finite dimension.*
- *there is a linear mapping $I_h^l$ from $l$-forms onto $\mathcal{W}^l$ satisfying the commuting diagram property $dI_h^l\omega = I_h^{l+1}(d\omega)$ for all $l$-forms $\omega$, $0 \leq l < n$ (see e.g. [47, Sect. IV]).*

In the sequel I take for granted that the corresponding discrete differential forms comply with the boundary traces $\mathbf{t}_N\mathbf{j} = 0$ and $\mathbf{t}_D u = 0$ stated in (1).

In practice, the $\mathcal{W}^l$ are constructed as finite element spaces. In particular, the discrete differential forms are usually built upon some mesh (tessellation, cell complex) of $\Omega$ (cf. [48, Sect. IV])

DEFINITION 2.2 *For any $0 \leq k \leq n$ denote by $\mathcal{F}_k$ a collection of piecewise smooth oriented contractible $k$-dimensional submanifolds ($k$-facets) of $\bar{\Omega}$ such that*

- *for distinct facets their interiors are disjoint regardless of dimension.*
- *the intersection of the closures of any two facets of any dimensions coincides with the closure of one and only one other facet.*
- *the boundary of each $k$-facet, $1 \leq k \leq n$, is the union of a finite number of $k+1$-facets.*
- *the union of the closures of all $n$-facets is equal to $\bar{\Omega}$.*

*Then $\{\mathcal{F}_k\}_{k=0}^n$ forms a mesh $\mathcal{T}_h$ of $\Omega$, and $\mathcal{F}_n$ is the set of its cells.*

The boundary parts $\Gamma_N$, $\Gamma_D$, and $\Gamma_M$ are to be composed of entire faces of elements. Thus, by plain restriction of a mesh of $\Omega$ we get meshes of $\Gamma_N$, $\Gamma_D$, and $\Gamma_M$. By dropping all facets contained in the closure of $\Gamma_D$ ($\Gamma_N$) we end up with the so-called $\Gamma_D(\Gamma_N)$-active mesh. Examples of valid meshes are the customary finite element triangulations in the sense of [21].

For discrete differential forms on triangulations the first condition stated in definition (2.1) is equivalent to demanding that the traces of the discrete $l$-form on any $k$-dimensional facet, $l \leq k \leq n$ have to be well defined [27].

In the sequel we only consider (generalized) Whitney-forms, i.e. discrete differential $l$-forms, whose degrees of freedom are supplied by integrals over $l$-facets of the mesh. Mainly for the sake of simplicity, because the basic considerations carry over to higher order elements, as

well. However, we admit rather general meshes according to definition (2.2). The only restriction should be that any cell can be split into a few (curved) simplices. Hence, for complicated shapes of elements we can come up with composite Whitney-elements.

For each space $\mathcal{W}^l$ we pick the basis dual to the set of degrees of freedom and assume a numbering of the basis functions. Thus, a differential form $u_h \in \mathcal{W}^l$ is uniquely characterized by its coefficient vector $\vec{u} \in \mathcal{C}^l$, where we have abbreviated $\mathcal{C}^l := \mathbb{R}^{N_l}$, $N_l := \dim \mathcal{W}^l$ (number of active $l$-facets). I tag coefficient vectors by $\vec{\phantom{x}}$ and discrete differential forms by a subscript h. The latter can also be seen as a mapping assigning a differential form in $\mathcal{W}^l$ to some coefficient vector $\in \mathcal{C}^l$. This induces a matrix representation for all linear operators on the spaces $\mathcal{W}^l$, $l = 0, \ldots, n$. For instance, we write $\mathsf{D}^l \in \mathbb{R}^{N_{l+1} \times N_l}$ for the matrix representing the exterior derivative $d : \mathcal{W}^l \mapsto \mathcal{W}^{l+1}$. These matrices turn out to be the incidence matrices of the active mesh [5, 39, 41, 11], e.g., $\mathsf{D}^l$ is the incidence matrix of oriented active $l$-facets and active $l+1$-facets. Both from this background and $d \circ d = 0$ we can conclude $\mathsf{D}^l \mathsf{D}^{l-1} = 0$, $l = 1, \ldots, n$ [5, Ch. 5].

It goes without saying that the trace operators $\mathbf{t}_D$, $\mathbf{t}_N$, and $\mathbf{t}_M$ possess matrix representations, too. Those are particularly simple, because taking the trace of a discrete differential form on a part of the boundary just amounts to isolating the d.o.f. located there [27]. We chose the symbol $\mathsf{T}$ for these matrices.

Now, we are already able to come up with the discrete equilibrium equations

$$\mathsf{D}^{l-1}\vec{u} = (-1)^l \vec{\boldsymbol{\sigma}} \quad , \quad \mathsf{D}^m \vec{\mathbf{j}} = -\vec{\psi} + \vec{\phi} \,. \tag{1}$$

Here, $\vec{\phi}$ is the coefficient vector of some suitable interpolant $\in \mathcal{W}^{m+1}$ of the source term $\phi$. The reader should be aware that whatever features of the equations arise from the equilibrium equations alone are preserved in the discrete setting. For instance, for any oriented control volume $\Omega' \subset \Omega$

$$\int\limits_{\Omega'} (\boldsymbol{\sigma}_h \wedge \mathbf{j}_h + u_h \wedge \psi_h) = \int\limits_{\Omega'} u_h \wedge \phi_h + (-1)^l \int\limits_{\partial\Omega'} \boldsymbol{\sigma}_h \wedge \mathbf{j}_h \,.$$

In short, thanks to discrete differential forms we automatically achieve discrete models that inherit most of the global features of the original problem (cf. [34, 20] for a discussion of Maxwell's equations and [32] for discrete decomposition theorems).

*Remark.* The discrete equilibrium equations (1) could also have been derived as network or lattice equations [11, 20, 52, 15] by applying Stokes' theorem directly to facets of the mesh in the spirit of a finite

volume approach. Yet, my point is that discrete differential forms are indispensable when trying to assess the approximation properties of discretization schemes. This will be elucidated in the remaining sections.

## 3. DISCRETE HODGE OPERATORS

The Hodge operators defy a straightforward discretization in the spaces of discrete differential forms: Consider the example of a discrete 1-form $\omega$ in three dimensions ("edge elements"): Its vector representative $\mathbf{u}$ sports only tangential continuity [38] and so its normal components on interelement faces are not necessarily well defined. The Hodge operator belonging to the Euclidean metric leaves vector representatives unchanged, i.e. the (twisted) 2-form $\star\omega$ is also described by $\mathbf{u}$. If $F$ is a face, at which the normal component of $\mathbf{u}$ has a jump, it is not possible to evaluate the integral $\int_F \star\omega = \int_F \langle \mathbf{u}, \mathbf{n}_F \rangle \, dS$: This reveals that $\star\omega$ fails to supply a proper discrete 2-form on the same mesh.

Thus, when embracing discrete differential forms, one inevitably stumbles onto the issue of a discrete Hodge operator [47, 34, 23, 12]. Its construction is outside the scope of the canonical discrete exterior calculus, because the standard definition of the Hodge operator is a "strong concept", which relies on point values being well defined. However, conventional discrete differential forms, with the exception of discrete 0-forms, are inherently discontinuous at interelement faces. This situation is typical of finite element methods: Recall piecewise linear finite element whose derivative is also meaningless in a pointwise sense. We are forced to resort to "weak concepts" that involve variational principles. This is not a nuisance, but leaves us with ample choices (cf. the introduction of [11]).

One aspect of this freedom arises from the observation that the two equilibrium equations are only linked through the constitutive relations. In fact, they involve differential forms of different nature (ordinary and twisted). So there is no reason, why both equilibrium equations should be discretized via the same family of discrete differential forms:

In general, the two equilibrium equations can be discretized on different unrelated meshes of $\Omega$, called *primary mesh* $\mathcal{T}_h$ and *secondary mesh* $\tilde{\mathcal{T}}_h$.

I stress that absolutely no relationship between these two meshes is stipulated. The terms "primary" and "secondary" must not even insinuate some precedence. I adopt the convention that all symbols related to the secondary mesh will be labeled by a tilde. This applies

to matrices acting on coefficient vectors of discrete differential forms on the secondary mesh, too.

Basically, discrete versions of the Hodge operators occurring in (1) have to establish linear mappings between spaces of discrete $l$-forms and $n - l$-forms based on possibly different meshes. In other words, a discrete Hodge-operators can be described by a $N_m \times N_l$-matrix. Moreover, for continuous $l$-forms $\star_\alpha \circ \star_{1/\alpha} = (-1)^{ln-l} \cdot Id$ [33, Sect. 1.8], i.e. the Hodge operators are invertible. This must hold for discrete Hodge-operators, too.

To get down to specifics, remember that Hodge operators define inner products on space of continuous differential forms via $(\omega, \eta)_\alpha := \int_\Omega \omega \wedge \star_\alpha \eta$. Here $\wedge$ denotes the exterior product of differential forms, which generates an $l + k$-form when given an $l$-form and a $k$-form as arguments. The exterior product is a bilinear mapping. Admittedly, the inner product $(\omega, \eta)_\alpha$ is of little value, because it would yield a circular "definition" of a discrete Hodge operator. Yet, the relationship reveals the general pattern:

$$
\left.
\begin{aligned}
\star_\mu w &= \omega \\
&\Updownarrow \\
(-1)^{ln-l} \star_{1/\mu} \omega &= w
\end{aligned}
\right\}
\leftrightarrow
\left\{
\begin{aligned}
(w, \eta)_\mu &= \int_\Omega \omega \wedge \eta \quad \forall\ l\text{-forms } \eta \\
&\Updownarrow \\
(-1)^{ln-l} (\omega, v)_{1/\mu} &= \int_\Omega w \wedge v \quad \forall\ m\text{-forms } v
\end{aligned}
\right.
$$
(2)

This suggests the following generic discrete form for a material law linking an ordinary $l$-form $w$ and a twisted $m$-form $\omega$:

$$
\left.
\begin{aligned}
\star_\mu w &= \omega \\
&\Updownarrow \\
(-1)^{ln-l} \star_{1/\mu} \omega &= w
\end{aligned}
\right\}
\xrightarrow{\text{discretize}}
\left\{
\begin{aligned}
\mathsf{M}^l_\mu \vec{w} &= \widetilde{\mathsf{K}}^l_m \vec{\omega} \\
&\textbf{or} \\
(-1)^{ln-l} \widetilde{\mathsf{M}}^m_{1/\mu} \vec{\omega} &= \mathsf{K}^m_l \vec{w} ,
\end{aligned}
\right.
$$
(3)

with yet obscure matrices $\mathsf{M}^l_\mu$, $\widetilde{\mathsf{M}}^m_{1/\mu}$, and $\mathsf{K}^m_l$, $\widetilde{\mathsf{K}}^l_m$. In an obvious fashion, the various indices are related to the order of differential forms on whose coefficient vectors the matrices act. Note that the discrete versions of the equivalent continuous material laws need not remain equivalent as explained in [48, Sect. VII]: In general we must reckon with

$$
\mathsf{M}^l_\mu \cdot (-1)^{ln-l} \widetilde{\mathsf{M}}^m_{1/\mu} \neq \mathsf{E}_{N_m} ,
$$

where $\mathsf{E}_{N_m}$ is the $N_m \times N_m$ identity matrix. An equality would be desirable, but seems to be rather elusive, as we will see in a moment. In sum, we can distinguish between two versions of the same material law. I dub the upper discrete material law "primary", the lower "secondary".

The matrices from (3) have to satisfy only a few *algebraic requirements*:

The **first** states that both $\mathsf{M}_\mu^l \in \mathbb{R}^{N_l,N_l}$ and $\widetilde{\mathsf{M}}_{1/\mu}^m \in \mathbb{R}^{\tilde{N}_m,\tilde{N}_m}$ are to be square, symmetric, and positive definite matrices, where $N_l := \dim \mathcal{W}^l$, $\tilde{N}_m := \dim \tilde{\mathcal{W}}^m$. This is natural if they are to give rise to inner products as explained above.

The **second** wants the "pairing matrices" $\mathsf{K}_l^m \in \mathbb{R}^{\tilde{N}_m,N_l}$ and $\widetilde{\mathsf{K}}_m^l \in \mathbb{R}^{N_l,\tilde{N}_m}$ to fulfill

$$\mathsf{K}_l^m = (-1)^{lm}(\widetilde{\mathsf{K}}_m^l)^T \quad \Longleftrightarrow \quad \widetilde{\mathsf{K}}_m^l = (-1)^{lm}(\mathsf{K}_l^m)^T \tag{4}$$

for all $0 \le l, m \le n$ such that $l + m = n$. Keep in mind that $\mathsf{K}_l^m$ and $\widetilde{\mathsf{K}}_m^l$ somehow approximate $\int_\Omega \omega \wedge \eta$ and $\int_\Omega w \wedge v$. Then the equation

$$\omega \wedge \eta = (-1)^{lk}\eta \wedge \omega, \quad \omega \ l\text{-form}, \eta \ k\text{-form} \tag{5}$$

immediately leads to (4).

The **third** requirement is the discrete counterpart of the integration by parts formula [33, Sect. 3.2]

$$\int_\Omega d\omega \wedge \eta + (-1)^l \int_\Omega \omega \wedge d\eta = \int_{\partial\Omega} \omega \wedge \eta \tag{6}$$

for $l$-forms $\omega$, $k$-forms $\eta$, $0 \le l, k < n - 1$, $l + k = n - 1$. Here, the boundary $\partial\Omega$ is endowed with the induced orientation. In the discrete setting this means

$$(\mathsf{D}^{l-1}\vec{u})^T\widetilde{\mathsf{K}}_m^l\vec{v} = (-1)^l \ \vec{u}^T\widetilde{\mathsf{K}}_{m+1}^{l-1}\widetilde{\mathsf{D}}^m\vec{v} + (\mathsf{T}_\Gamma^{l-1}\vec{u})^T\widetilde{\mathsf{K}}_{m,\Gamma}^{l-1}\widetilde{\mathsf{T}}_\Gamma^m\vec{v}. \tag{7}$$

for all $\vec{u} \in \mathcal{C}^{l-1}$, $\vec{v} \in \tilde{\mathcal{C}}^m$, and translates into

$$(\mathsf{D}^{l-1})^T\widetilde{\mathsf{K}}_m^l = (-1)^l \ \widetilde{\mathsf{K}}_{m+1}^{l-1}\widetilde{\mathsf{D}}^m + (\mathsf{T}_\Gamma^{l-1})^T\widetilde{\mathsf{K}}_{m,\Gamma}^{l-1}\widetilde{\mathsf{T}}_\Gamma^m \tag{8}$$

for all $0 < m, l < n$ with $l + m = n$. Here, we denote by $\widetilde{\mathsf{K}}_{m,\Gamma}^l$ a pairing matrix acting on degrees of freedom on the boundary part $\Gamma_M$. Using (4), (8) can be converted into

$$(\widetilde{\mathsf{D}}^m)^T\mathsf{K}_{l-1}^{m+1} = (-1)^{m+1}\mathsf{K}_l^m\mathsf{D}^{l-1} + (\widetilde{\mathsf{T}}_\Gamma^m)^T\mathsf{K}_{l-1,\Gamma}^m\mathsf{T}_\Gamma^{l-1}. \tag{9}$$

Let us assume that we have found a discrete Hodge operator according to the above specifications. Nevertheless, we cannot be sure that the resulting linear system of equation has a solution at all. Usually the number of unknowns and equations will not agree. To end up with a square linear system of equations we eliminate some of the unknowns by means of (8), (9) and the material laws. I first consider (1) and discuss some variants of choosing the discrete Hodge operators. I start with listing the formal discrete constitutive laws that might be used in the discretization of (1):

$$
\text{Primary:} \quad
\begin{cases}
\mathsf{M}_\alpha^l \vec{\sigma} &= \widetilde{\mathsf{K}}_m^l \vec{\mathbf{j}} & \text{(a)} \\
\mathsf{M}_\gamma^{l-1} \vec{u} &= \widetilde{\mathsf{K}}_{m+1}^{l-1} \vec{\psi} & \text{(b)} \\
\mathsf{M}_{\beta,\Gamma}^{l-1} \mathsf{T}_\Gamma^{l-1} \vec{u} &= (-1)^{l-1} \widetilde{\mathsf{K}}_{m,\Gamma}^{l-1} \widetilde{\mathsf{T}}_\Gamma^m \vec{\mathbf{j}} & \text{(c)}
\end{cases}
\tag{10}
$$

$$
\text{Secondary:} \quad
\begin{cases}
\widetilde{\mathsf{M}}_{1/\alpha}^m \vec{\mathbf{j}} &= (-1)^{mn-m} \mathsf{K}_l^m \vec{\sigma} & \text{(a)} \\
\widetilde{\mathsf{M}}_{1/\gamma}^{m+1} \vec{\psi} &= (-1)^{(l-1)(n-1)} \mathsf{K}_{l-1}^{m+1} \vec{u} & \text{(b)} \\
\widetilde{\mathsf{M}}_{1/\beta,\Gamma}^m \widetilde{\mathsf{T}}_\Gamma^m \vec{\mathbf{j}} &= (-1)^{(l-1)(n-1)} \mathsf{K}_{l-1,\Gamma}^m \mathsf{T}_\Gamma^{l-1} \vec{u} & \text{(c)}
\end{cases}
\tag{11}
$$

1. *Primary elimination:* Using only primary discrete Hodge operators (10a), (10c) and (1), (8) we get

$$
(\mathsf{D}^{l-1})^T \mathsf{M}_\alpha^l \mathsf{D}^{l-1} \vec{u} = (\mathsf{D}^{l-1})^T (-1)^l \mathsf{M}_\alpha^l \vec{\sigma} = (-1)^l (\mathsf{D}^{l-1})^T \widetilde{\mathsf{K}}_m^l \vec{\mathbf{j}} =
$$
$$
= \widetilde{\mathsf{K}}_{m+1}^{l-1} \widetilde{\mathsf{D}}^m \vec{\mathbf{j}} + (-1)^l (\mathsf{T}_\Gamma^{l-1})^T \widetilde{\mathsf{K}}_{m,\Gamma}^{l-1} \widetilde{\mathsf{T}}_\Gamma^m \vec{\mathbf{j}} =
$$
$$
= \widetilde{\mathsf{K}}_{m+1}^{l-1} (-\vec{\psi} + \vec{\phi}) - (\mathsf{T}_\Gamma^{l-1})^T \mathsf{M}_{\beta,\Gamma}^{l-1} \mathsf{T}_\Gamma^{l-1} \vec{u} =
$$
$$
= -\mathsf{M}_\gamma^{l-1} \vec{u} - (\mathsf{T}_\Gamma^{l-1})^T \mathsf{M}_{\beta,\Gamma}^{l-1} \mathsf{T}_\Gamma^{l-1} \vec{u} + \widetilde{\mathsf{K}}_{m+1}^{l-1} \vec{\phi} \, .
$$

We arrive at a linear system of equations

$$
(\mathsf{D}^{l-1})^T \mathsf{M}_\alpha^l \mathsf{D}^{l-1} \vec{u} + \mathsf{M}_\gamma^{l-1} \vec{u} + (\mathsf{T}_\Gamma^{l-1})^T \mathsf{M}_{\beta,\Gamma}^{l-1} \mathsf{T}_\Gamma^{l-1} \vec{u} = \widetilde{\mathsf{K}}_{m+1}^{l-1} \vec{\phi}
\tag{12}
$$

for the unknown coefficients $\vec{u}$ with a symmetric positive definite coefficient matrix. If $\gamma \neq 0$, it has a unique solution. Even if $\gamma = 0$, at least $\mathsf{D}^{l-1}\vec{u}$ can be uniquely determined. Two more important facts have to mentioned: To begin with, the secondary spaces do *only* affect the right hand side $\widetilde{\mathsf{K}}_{m+1}^{l-1} \vec{\phi}$ of the final system. In other words, the choice of the secondary mesh is totally irrelevant as regards the ultimate system matrix. Secondly, in the process of elimination we irretrievable lost information about the secondary unknowns $\sigma_h$ and $\psi_h$, unless the pairing matrices are invertible.

2. *Secondary elimination:* We exclusively rely on secondary discrete Hodge operators (11b), (11c) along with (1), (9):

$$\widetilde{\mathsf{M}}^m_{1/\alpha}\vec{\mathbf{j}} = (-1)^{n(m+1)}\mathsf{K}^m_l\mathsf{D}^{l-1}\vec{u}$$
$$= (-1)^{(n-1)(l-1)}((\widetilde{\mathsf{D}}^m)^T\mathsf{K}^{m+1}_{l-1}\vec{u} - (\widetilde{\mathsf{T}}^m_\Gamma)^T\mathsf{K}^m_{l-1,\Gamma}\mathsf{T}^{l-1}_\Gamma\vec{u})$$
$$= (\widetilde{\mathsf{D}}^m)^T\widetilde{\mathsf{M}}^{m+1}_{1/\gamma}\vec{\psi} - (\widetilde{\mathsf{T}}^m_\Gamma)^T\widetilde{\mathsf{M}}^m_{1/\beta,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\vec{\mathbf{j}}$$

Introducing the auxiliary unknown $\vec{\zeta} := \widetilde{\mathsf{M}}^{m+1}_{1/\gamma}\vec{\psi}$ we get the saddle point problem

$$\begin{pmatrix} \widetilde{\mathsf{M}}^m_{1/\alpha} + (\widetilde{\mathsf{T}}^m_\Gamma)^T\widetilde{\mathsf{M}}^m_{1/\beta,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma & -(\widetilde{\mathsf{D}}^m)^T \\ -\widetilde{\mathsf{D}}^m & -(\widetilde{\mathsf{M}}^{m+1}_{1/\gamma})^{-1} \end{pmatrix}\begin{pmatrix} \vec{\mathbf{j}} \\ \vec{\zeta} \end{pmatrix} = \begin{pmatrix} 0 \\ -\vec{\phi} \end{pmatrix}. \qquad (13)$$

As the diagonal blocks are positive and negative definite, respectively, this linear system has a unique solution (cf. [17]). In the case of vanishing $\gamma$ we just define the auxiliary unknown as $\vec{\zeta} := \mathsf{K}^{m+1}_{l-1}\vec{u}$ and still get uniqueness for $\vec{\mathbf{j}}$. Parallel to the purely primal case we observe that no trace of the primary discrete spaces is left. As above, in general we cannot solve for the primary unknowns $\vec{u}$ and $\vec{\sigma}$.

3. *Hybrid elimination:* Both primary and secondary discrete Hodge operators are used for the sake of eliminating unknowns. For example, if we use primary representations (10a), (10c) for the material laws $\mathbf{j} = \star_\alpha\,\sigma$ and $\mathbf{t}_M\mathbf{j} = (-1)^{l-1}\star^\Gamma_\beta\mathbf{t}_M u$, but the secondary version (11b) for $\psi = \star_\gamma\,u$ we get

$$(\mathsf{D}^{l-1})^T\mathsf{M}^l_\alpha\mathsf{D}^{l-1}\vec{u} + (\mathsf{T}^{l-1}_\Gamma)^T\mathsf{M}^{l-1}_{\beta,\Gamma}\mathsf{T}^{l-1}_\Gamma\vec{u} +$$
$$\widetilde{\mathsf{K}}^{l-1}_{m+1}(\widetilde{\mathsf{M}}^{m+1}_{1/\gamma})^{-1}\mathsf{K}^{m+1}_{l-1}\vec{u} = \widetilde{\mathsf{K}}^{l-1}_{m+1}\vec{\phi}.$$

Again, we have obtained a positive semidefinite system of linear equations for the unknown vector $\vec{u}$, in which both meshes are still visible. Yet, even if $\gamma \neq 0$ the system matrix need not be regular. At least a unique solution for $\vec{\sigma} := (-1)^l\mathsf{D}^{l-1}\vec{u}$ is guaranteed. On the other hand, the elimination might have squandered any information about $\vec{\mathbf{j}}$. If we resort to the secondary version of $\mathbf{t}_M\mathbf{j} = (-1)^{l-1}\star^\Gamma_\beta\mathbf{t}_M u$ instead, it results in

$$(\widetilde{\mathsf{K}}^l_m)^T(\mathsf{M}^l_\alpha)^{-1}\widetilde{\mathsf{K}}^l_m\vec{\mathbf{j}} = (-1)^{lm}\mathsf{K}^m_l(\mathsf{M}^l_\alpha)^{-1}\widetilde{\mathsf{K}}^l_m\vec{\mathbf{j}} = (-1)^{lm}\mathsf{K}^m_l\vec{\sigma}$$
$$= (-1)^{l(m+1)}\mathsf{K}^m_l\mathsf{D}^{l-1}\vec{u}$$
$$= (-1)^{(l+1)(m+1)}((\widetilde{\mathsf{D}}^m)^T\mathsf{K}^{m+1}_{l-1}u - (\widetilde{\mathsf{T}}^m_\Gamma)^T\mathsf{K}^m_{l-1,\Gamma}\mathsf{T}^{l-1}_\Gamma)$$

$$=(\widetilde{\mathsf{D}}^m)^T\widetilde{\mathsf{M}}^{m+1}_{1/\gamma}\vec{\psi} - (\widetilde{\mathsf{T}}^m_\Gamma)^T\widetilde{\mathsf{M}}^m_{1/\beta,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\vec{\mathbf{j}}$$

$$= -(\widetilde{\mathsf{D}}^m)^T\widetilde{\mathsf{M}}^{m+1}_{1/\gamma}(\widetilde{\mathsf{D}}^m\mathbf{j} - \vec{\phi}) - (\widetilde{\mathsf{T}}^m_\Gamma)^T\widetilde{\mathsf{M}}^m_{1/\beta,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\vec{\mathbf{j}}.$$

## 4. EXAMPLES

The most natural way to define the matrices in (3) is to define the inner products in (2) as weighted $L^2$-inner products for vector representatives of the forms w.r.t. the Euclidean metric. Where the discrete forms are continuous this is suggested by the definition of the Hodge-operator. As discontinuities are confined to sets of measure zero, they simply do not affect the integral.

Then plug in the bases of the spaces of discrete differential forms in the variational equations from (2). This may be dubbed the *finite element approach*: The matrices $\mathsf{M}^*_*$, $\widetilde{\mathsf{M}}^*_*$ become exact (weighted) mass matrices. I should point out that primary elimination yields (almost) the same system of linear equations (12) as the primal finite element Galerkin method. The only exception might be a modified right hand side. By secondary elimination we get the linear saddle point problem (13) of the dual mixed finite element Galerkin method (cf. [17]). Hitherto unknown problems arise from hybrid eliminations.

In the case of the finite element approach all the requirements stated for the matrices in the previous section are automatically satisfied. Since we are free to pick any primary and secondary mesh, the pairing matrices are not square in general. In particular, there is no reason, why they should be invertible and information about some eliminated unknowns cannot be recovered.

A bijective relationship between primary and secondary unknowns is the rationale behind the second class of methods. It can be achieved by using a *dual* secondary mesh [11, 15].

DEFINITION 4.1 *Two meshes $\widetilde{\mathcal{T}}_h$ and $\mathcal{T}_h$ covering an $n$-dimensional manifold are called (topologically) dual to each other if $\mathsf{L}^T_l = (-1)^l\mathsf{L}_{n-l+1}$, $0 \le l < n$, where $\mathsf{L}_l$ and $\mathsf{L}_l$ are the incidence matrices of oriented $l$- and $l+1$-facets of $\mathcal{T}_h$ and $\widetilde{\mathcal{T}}_h$, respectively.*

In [48, Sect. V] the relationships between an externally oriented dual mesh and twisted differential forms is thoroughly discussed, but I will not dwell on this subject. Just note that for dual meshes the numbers of $l$-facets of one mesh and those of $n-l$-facets of the other mesh must be equal. More precisely, the secondary mesh $\widetilde{\mathcal{T}}_h$ is chosen such that

(i) the restriction of $\widetilde{\mathcal{T}}_h$ to the interior of $\Omega$ is dual to the entire mesh $\mathcal{T}_h$.

(ii) the restriction of $\tilde{\mathcal{T}}_h$ and $\mathcal{T}_h$ to the boundary $\partial\Omega$ are dual to each other.

Thanks to duality, we can assume a one-to-one correspondence between $l$-facets of $\mathcal{T}_h$ and interior $n-l$-facets of $\tilde{\mathcal{T}}_h$. Similarly, we may associate boundary $l$-facets, $0 \leq l < n$, of $\mathcal{T}_h$ and $n-1-l$-facets of $\tilde{\mathcal{T}}_h$ on $\partial\Omega$. If $\mathcal{T}_h$ is $\Gamma_D$-active, an $l$-facet of $\tilde{\mathcal{T}}_h$ is active, i.e., it bears a d.o.f., if one of the following alternatives applies:

(i) Either it is contained in $\partial\Omega \setminus \bar{\Gamma}_N$ and associated with an active primary $n-1-l$-facet of $\mathcal{T}_h$,

(ii) or it is located inside $\Omega$ and belongs to an active primary $n-l$-facet.

$\Gamma_D$ and $\Gamma_N$ may switch roles depending on which unknowns are discretized on the primary mesh. Figure 1 sketches an example of two dual grids in two dimensions.
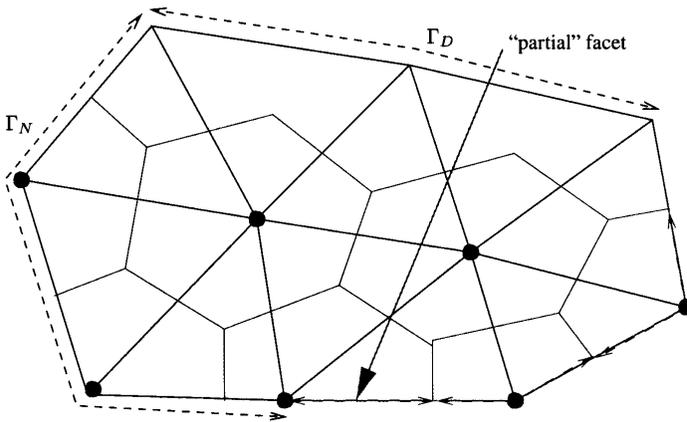


**Figure 1.** Primal and dual grid in two dimensions. The bullets represent active vertices of the primary grid, the edges $\longleftrightarrow$ active boundary faces of the secondary grid.

Figure 1 reveals that the dual mesh may fail to comply with the partitioning of the boundary. This can be cured by confining a degree of freedom to only a part of a boundary facet. If $\Gamma_M = \emptyset$, there is no active $n-1$-facet of $\tilde{\mathcal{T}}_h$ on the boundary. The numbering of interior $l$-facets of $\tilde{\mathcal{T}}_h$ is induced by the numbering of primary $n-l$-facets via duality. Boundary facets of $\tilde{\mathcal{T}}_h$ are numbered last. Then the intimate relationship between discrete exterior derivatives

and incidence matrices shows that for $0 < m, l < n$, $l + m = m$

$$(\mathsf{D}^{l-1})^T (\mathsf{E}_{N_l}, \mathsf{O}) = (-1)^l \mathsf{E}_{N_{l-1}} \widetilde{\mathsf{D}}^m + (\mathsf{T}_\Gamma^{l-1})^T \widetilde{\mathsf{T}}_\Gamma^m . \tag{14}$$

Here $\mathsf{E}_N$ stands for a $N \times N$ identity matrix, and $\mathsf{O}$ denotes a zero block of dimension $N_l \times \tilde{N}_m^\Gamma$, with $\tilde{N}_m^\Gamma$ the number of active boundary $m$-facets of $\widetilde{\mathcal{T}}_h$. A glance at (7) and (8) shows that we can choose

$$\widetilde{\mathsf{K}}_m^l := (\mathsf{E}_{N_l}, \mathsf{O}) \quad , \quad \widetilde{\mathsf{K}}_{m+1}^{l-1} := \mathsf{E}_{N_{l-1}} \quad , \quad \widetilde{\mathsf{K}}_{m,\Gamma}^{l-1} := \mathsf{E}_{\tilde{N}_m^\Gamma} , \tag{15}$$

and abide by requirement (8) at the same time. If $\Gamma_M = \emptyset$, the zero block disappears and the pairing matrices reduce to (signed) identity matrices. In any case, the dual unknowns $\vec{\mathbf{j}}$ and $\vec{\psi}$ can be calculated from $\vec{\sigma}, \vec{u}$, no matter which discrete Hodge-operator is used.

Examples for dual meshes are supplied by the usual covolumes (boxes) used in finite volume schemes [26]. Well known are the circumcentric dual Voronoi meshes of a Delauney tesselation and the barycentric dual meshes of the box method. This is why I chose to call the methods of this second class *generalized finite volume methods*. As a special subclass they include *covolume methods* distinguished by the use of diagonal approximate mass matrices and orthogonal dual meshes [40, 41].

Let us for simplicity assume $\Gamma_M = \emptyset$. Then the discrete material laws for generalized finite volume methods read

$$\mathsf{M}_\alpha^l \vec{\sigma} = \vec{\mathbf{j}} \quad \text{or} \quad (\widetilde{\mathsf{M}}_{1/\alpha}^m)^{-1} \vec{\sigma} = \vec{\mathbf{j}} , \tag{16}$$

$$\mathsf{M}_\gamma^{l-1} \vec{u} = \vec{\psi} \quad \text{or} \quad (\widetilde{\mathsf{M}}_{1/\gamma}^{m+1})^{-1} \vec{u} = \vec{\psi} . \tag{17}$$

In short, all discrete material laws can be viewed as both a primary and secondary version. This makes it possible to proceed with both primary and secondary elimination. We end up with linear systems of equations for primary or secondary unknowns only and draw an important conclusion:

COROLLARY 4.2 *Generalized finite volume methods combined with either primary or secondary elimination lead to linear systems of equations that also arise from a primal or mixed-dual finite element discretization employing some approximation of the mass matrices and source term.*

Thus, the study of generalized finite volume method can help itself to the powerful tools of finite element theory. This generalizes the results of [26, 1], where the case $n = 2$, $l = 1$ and its links with the primal Galerkin finite element method were thoroughly investigated. In [2, 42] the connection with a dual mixed Galerkin scheme with

lumped mass matrix was explored. Covolume schemes for Maxwell's equations [51, 52, 53, 50] can also be analyzed from this perspective [14]. Eventually, knowledge about the underlying discrete differential forms offers a recipe for the natural reconstruction of fields from the degrees of freedom. Thanks to the canonical transformations of discrete differential forms [27] this is useful even for distorted elements as in [46]. Ultimately, awareness of basic requirements for discrete Hodge operators reveals causes for instability of finite volume schemes and leads to remedies [43].

Another conclusion is that a finite volume method can be completely specified by prescribing some procedure to compute the (approximate) mass matrices and the right hand side. Conversely, Galerkin finite element schemes can be viewed as finite volume methods (cf [47]). Thus we learn how to recover approximations to quantities that have been dumped in the process of primary or secondary elimination (cf. [6]).

*Remark.* The discussion illustrates the basic limitation of finite volume schemes to lowest order. If we had decided to use higher order discrete differential forms, the matrices of the exterior derivative could not have been identified as incidence matrices. Then it is very hard to come up with a suitable secondary mesh rendering the pairing matrices square and invertible.

Also the *method of support operators* (mimetic finite differences) [28, 44] can be seen as a special finite volume approach to the construction of discrete Hodge operators. Using only discrete differential forms on a primary grid [30], it focuses on Hodge codifferentials $d_* := (-1)^{nl-1} \star d \star$ taking $l$-forms to $l - 1$-forms. [33, Sect. 3.2]. The construction of their discrete counterparts $\mathsf{D}_*^l : \mathcal{C}^l \mapsto \mathcal{C}^{l-1}$ is based on the variational characterization $\left(\vec{u}, \mathsf{D}^{l-1}\vec{v}\right)_0 = \left(\mathsf{D}_*^l\vec{u}, \vec{v}\right)_0$ for all $\vec{u} \in \mathcal{C}^l$, $\vec{v} \in \mathcal{C}^{l-1}$. Special approximations of the inner products are employed to this end [29]. This policy has been successfully pursued for a wide range of problems, e.g. in [31, 45].

## 5. ABSTRACT ERROR ANALYSIS

We have learned that often the error analysis can be carried out in a Galerkin setting involving variational crimes (cf. [16, Ch. 6]). Yet, this is not possible for all combinations of discrete material laws. Therefore, the error analysis presented in this section forgoes the Galerkin option.

I start from the premises that it is most natural to use the energy norm when gauging the discretization error. An ambiguity arises, because, given discrete solutions $u_h, \boldsymbol{\sigma}_h, \mathbf{j}_h, \psi_h$ we can either examine

continuous energy norms of the error, e.g.

$$\|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{\mathcal{E}}^2 := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_\alpha^2 + \|u - u_h\|_\gamma^2 + \|u - u_h\|_\beta^2 \;,$$

$$\|(\mathbf{j} - \mathbf{j}_h, \psi - \psi_h)\|_{\tilde{\mathcal{E}}}^2 := \|\mathbf{j} - \mathbf{j}_h\|_{1/\alpha}^2 + \|\psi - \psi_h\|_{1/\gamma}^2 + \|\mathbf{j} - \mathbf{j}_h\|_{1/\beta}^2 \;,$$

or discrete energy norms of the following *nodal errors*

$$\delta\vec{u} := \vec{u}^* - \vec{u}, \quad \vec{u}^* := I_h^{l-1} u \qquad , \qquad \delta\vec{\sigma} := \vec{\sigma}^* - \vec{\sigma}, \quad \vec{\sigma}^* := I_h^l \sigma \;,$$

$$\delta\vec{\mathbf{j}} := \vec{\mathbf{j}}^* - \vec{\mathbf{j}}, \quad \vec{\mathbf{j}}^* := \tilde{I}_h^m \mathbf{j} \qquad , \qquad \delta\vec{\psi} := \vec{\psi}^* - \vec{\psi}, \quad \vec{\psi}^* := \tilde{I}_h^{m+1} \psi \;.$$

What are meaningful discrete energy norms heavily hinges on the choice of the discrete material laws.

First, we tackle 1 when only primary discrete Hodge operators from (10) are used. This fixes the relevant discrete energy norm

$$|(\vec{u}, \vec{\sigma})|_{\mathcal{E}}^2 := |\vec{\sigma}|_\alpha^2 + |\vec{u}|_\gamma^2 + |\vec{u}|_\beta^2 \;,$$

$$|\cdot|_\alpha^2 := \left\langle \mathsf{M}_\alpha^l \cdot, \cdot \right\rangle \;, \; |\cdot|_\gamma^2 := \left\langle \mathsf{M}_\gamma^{l-1} \cdot, \cdot \right\rangle \;, \; |\cdot|_\beta^2 := \left\langle \mathsf{M}_{\beta,\Gamma}^{l-1} \mathsf{T}_\Gamma^{l-1} \cdot, \mathsf{T}_\Gamma^{l-1} \cdot \right\rangle \;.$$

Next, in the spirit of [41, 15], we observe that thanks to the commuting diagram property of the nodal projection the discrete equilibrium laws are free of consistency errors:

$$\mathsf{D}^{l-1}\delta\vec{u} = (-1)^l \,\delta\vec{\sigma} \quad , \quad \widetilde{\mathsf{D}}^m \delta\vec{\mathbf{j}} = -\delta\vec{\psi} + \delta\vec{\phi} \;. \tag{18}$$

In what follows I assume that $\delta\vec{\phi} = 0$, i.e. $\vec{\phi} = \tilde{I}_h^{m+1}\phi$. If this is not the case, one additional term enters the error bounds established below. In contrast to (18), consistency errors lurk in the discrete material laws (cf. [48, Sect. VII] and [51])

$$\mathsf{M}_\alpha^l \delta\vec{\sigma} = \widetilde{\mathsf{K}}_m^l \delta\vec{\mathbf{j}} + \vec{R}_l \;, \tag{19}$$

$$\mathsf{M}_\gamma^{l-1} \delta\vec{u} = \widetilde{\mathsf{K}}_{m+1}^{l-1} \delta\vec{\psi} + \vec{R}_{l-1} \;, \tag{20}$$

$$\mathsf{M}_{\beta,\Gamma}^{l-1} \mathsf{T}_\Gamma^{l-1} \delta\vec{u} = (-1)^{l+1} \widetilde{\mathsf{K}}_{m,\Gamma}^{l-1} \widetilde{\mathsf{T}}_\Gamma^m \delta\vec{\mathbf{j}} + \vec{R}_\Gamma \;, \tag{21}$$

with some residuals $\vec{R}_l \in \mathcal{C}^1$, $\vec{R}_{l-1} \in \mathcal{C}^0$, and $\vec{R}_\Gamma \in \mathcal{C}_\Gamma^0$. Based on (18) and (7), we can estimate the discrete energy of the nodal errors

$$\left\langle \mathsf{M}_\alpha^l \delta\vec{\sigma}, \delta\vec{\sigma} \right\rangle + \left\langle \mathsf{M}_\gamma^{l-1} \delta\vec{u}, \delta\vec{u} \right\rangle + \left\langle \mathsf{M}_{\beta,\Gamma}^{l-1} \mathsf{T}_\Gamma^{l-1} \delta\vec{u}, \mathsf{T}_\Gamma^{l-1} \delta\vec{u} \right\rangle =$$

$$= \left\langle \widetilde{\mathsf{K}}_m^l \delta\vec{\mathbf{j}} + \vec{R}_l, (-1)^l \,\mathsf{D}^{l-1} \delta\vec{u} \right\rangle + \left\langle \widetilde{\mathsf{K}}_{m+1}^{l-1} \delta\vec{\psi} + \vec{R}_{l-1}, \delta\vec{u} \right\rangle +$$

$$+ \left\langle (-1)^{l-1} \widetilde{\mathsf{K}}_{m,\Gamma}^{l-1} \widetilde{\mathsf{T}}_\Gamma^m \delta\vec{\mathbf{j}} + \vec{R}_\Gamma, \delta\vec{u} \right\rangle$$

$$= \left\langle \vec{R}_l, \delta\vec{\sigma} \right\rangle + \left\langle \vec{R}_{l-1}, \delta\vec{u} \right\rangle + \left\langle \vec{R}_\Gamma, \delta\vec{u} \right\rangle \;.$$

Please note that the second terms on both sides do not occur in the case $\gamma = 0$ and an error estimate for $u$ remains elusive. Hardly surprising, because there might not be a unique solution for $\vec{u}$, unless special properties of $\mathsf{D}^{l-1}$ (injectivity) are known. By the Cauchy-Schwarz inequality

$$|(\delta\vec{u}, \delta\vec{\sigma})|_{\mathcal{E}} \leq \left|(\mathsf{M}_\alpha^l)^{-1}\vec{R}_l\right|_\alpha + \left|(\mathsf{M}_\gamma^{l-1})^{-1}\vec{R}_{l-1}\right|_\gamma + \left|(\mathsf{M}_{\beta,\Gamma}^{l-1})^{-1}\vec{R}_\Gamma\right|_\beta .$$

Similar considerations apply, when solely secondary discrete Hodge operators from (11) are employed. Then the suitable discrete energy norm is given by

$$\left|(\vec{\mathbf{j}}, \vec{\psi})\right|_{\tilde{\mathcal{E}}}^2 := \left|\vec{\mathbf{j}}\right|_{1/\alpha}^2 + \left|\vec{\psi}\right|_{1/\gamma}^2 + \left|\vec{\mathbf{j}}\right|_{1/\beta}^2 ,$$

$$|\cdot|_{1/\alpha}^2 := \left\langle \widetilde{\mathsf{M}}_{1/\alpha}^m \cdot, \cdot \right\rangle , \quad |\cdot|_{1/\gamma}^2 := \left\langle \widetilde{\mathsf{M}}_{1/\gamma}^{m+1} \cdot, \cdot \right\rangle , \quad |\cdot|_{1/\beta}^2 := \left\langle \widetilde{\mathsf{M}}_{1/\beta,\Gamma}^m \cdot, \cdot \right\rangle .$$

Partly retaining the notations for the consistency errors, we can write

$$\widetilde{\mathsf{M}}_{1/\alpha}^m \delta\vec{\mathbf{j}} = (-1)^{mn-m} \mathsf{K}_l^m \delta\vec{\sigma} + \vec{R}_m \tag{22}$$

$$\widetilde{\mathsf{M}}_{1/\gamma}^{m+1} \delta\vec{\psi} = (-1)^{(l-1)(n-1)} \mathsf{K}_{l-1}^{m+1} \delta\vec{u} + \vec{R}_{m+1} \tag{23}$$

$$\widetilde{\mathsf{M}}_{1/\beta,\Gamma}^m \widetilde{\mathsf{T}}_\Gamma^m \delta\vec{\mathbf{j}} = (-1)^{(l-1)(n-1)} \mathsf{K}_{l-1,\Gamma}^m \mathsf{T}_\Gamma^{l-1} \delta\vec{u} + \vec{R}_\Gamma , \tag{24}$$

with $\vec{R}_m \in \tilde{\mathcal{C}}^m$, $\vec{R}_{m+1} \in \tilde{\mathcal{C}}^{m+1}$, and $\vec{R}_\Gamma \in \tilde{\mathcal{C}}_\Gamma^m$. Since (18) remains valid, the following identity is established as above:

$$\left\langle \widetilde{\mathsf{M}}_{1/\alpha}^l \delta\vec{\mathbf{j}}, \delta\vec{\mathbf{j}} \right\rangle + \left\langle \widetilde{\mathsf{M}}_{1/\gamma}^{l-1} \delta\vec{\psi}, \delta\vec{\psi} \right\rangle + \left\langle \widetilde{\mathsf{M}}_{1/\beta,\Gamma}^{l-1} \widetilde{\mathsf{T}}_\Gamma^m \delta\vec{\mathbf{j}}, \widetilde{\mathsf{T}}_\Gamma^m \delta\vec{\mathbf{j}} \right\rangle =$$

$$= \left\langle \vec{R}_m, \delta\vec{\mathbf{j}} \right\rangle + \left\langle \vec{R}_{m+1}, \delta\vec{\psi} \right\rangle + \left\langle \vec{R}_\Gamma, \delta\vec{\mathbf{j}} \right\rangle .$$

Other combinations of discrete material laws are treated alike. For the sake of brevity I am not elaborating on this. In sum, estimating the consistency errors of the material laws is the key to controlling discrete energy norms of nodal errors. The analysis of finite volume methods [40, 41, 37] is often content with this goal, but I am not (cf. the discussion in [15, Sect. III]). Discrete energies might lack any physical meaning, so that the focus should be on the exact energy norm. In the case of primary discrete Hodge operators

$$\|(u - u_h, \sigma - \sigma_h)\|_{\mathcal{E}} \leq \|(u - u_h^*, \sigma - \sigma_h^*)\|_{\mathcal{E}} + \|(\delta u_h, \delta\sigma_h)\|_{\mathcal{E}}$$

tells us that it is essential to have *stability*

$$\|(u_h, \sigma_h)\|_{\mathcal{E}} \leq C \, |(\vec{u}, \vec{\sigma})|_{\mathcal{E}} \quad \forall \vec{u} \in \mathcal{C}^{l-1}, \, \vec{\sigma} \in \mathcal{C}^l , \tag{25}$$

in order to get information about the energy of the total discretization error. Here, the constant $C > 0$ should be independent of as much geometric parameters of the mesh as possible. In the case of secondary discrete constitutive laws, we proceed as above, replacing $\|\cdot\|_{\mathcal{E}}$ by $\|\cdot\|_{\tilde{\mathcal{E}}}$ and $|\cdot|_{\mathcal{E}}$ by $|\cdot|_{\tilde{\mathcal{E}}}$.

In addition, the energy of the *projection error* $\|(u - u_h^*, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}$ has to be bounded. This is the objective of asymptotic finite element interpolation estimates. Those rely on a Sobolev space setting and are large based on the Bramble-Hilbert lemma and affine equivalence techniques depending on families of quasiuniform and shape regular meshes [16, Ch. 4], [21, Ch. 3]. In particular, for results on discrete 2-forms in two and three dimensions the reader should consult [17]. Estimates for discrete 1-forms $(n = 3)$ can be found in [38, 25, 22, 36]. It is important to be aware that all estimates hinge on assumptions on the smoothness of the continuous solutions.

In the case of finite difference or finite volume methods, one might object that the spaces of discrete differential forms are "artificial" and so is the notion of a total discretization error. Yet, an approximation of the total energy must always be available and the error in the approximation of the total energy is well defined at any rate. For this error we get in the case of primary discrete Hodge operators

$$
\begin{aligned}
\|(u, \boldsymbol{\sigma})\|_{\mathcal{E}}^2 - |(\vec{u}, \vec{\boldsymbol{\sigma}})|_{\mathcal{E}}^2 &= \\
&= \|(u, \boldsymbol{\sigma})\|_{\mathcal{E}}^2 - \|(u_h^*, \boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}^2 + \|(u_h^*, \boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}^2 - \\
&\quad - |(\vec{u}^*, \vec{\boldsymbol{\sigma}}^*)|_{\mathcal{E}}^2 + |(\vec{u}^*, \vec{\boldsymbol{\sigma}}^*)|_{\mathcal{E}}^2 - |(\vec{u}, \vec{\boldsymbol{\sigma}})|_{\mathcal{E}}^2 \\
&\leq \|(u - u_h^*, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*)\|_{\mathcal{E}} \|(u + u_h^*, \boldsymbol{\sigma} + \boldsymbol{\sigma}_h^*)\|_{\mathcal{E}} + \\
&\quad + |(\vec{u}^* - \vec{u}, \vec{\boldsymbol{\sigma}}^* - \vec{\boldsymbol{\sigma}})|_{\mathcal{E}} |(\vec{u}^* - \vec{u}, \vec{\boldsymbol{\sigma}}^* - \vec{\boldsymbol{\sigma}})|_{\mathcal{E}} + \\
&\quad + \|(u_h^*, \boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}^2 - |(\vec{u}^*, \vec{\boldsymbol{\sigma}}^*)|_{\mathcal{E}}^2 \ .
\end{aligned}
$$

Even if the discretization error, the nodal error, and the projection error tend to zero, the error in the energy need not, owing to the quantity $\|(u_h^*, \boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}^2 - |(\vec{u}^*, \vec{\boldsymbol{\sigma}}^*)|_{\mathcal{E}}^2$. It can be regarded as a consistency error in the approximation of the mass matrices. Hence, the quality of the approximation of the discrete energy can serve as an acid test for the efficacy of a discretization scheme.

*Remark.* Primary Hodge operators do not permit us to get error estimates for dual quantities. However, the generalized finite volume methods are an exception. For instance, in the case $\Gamma_M = \emptyset$ the bound for $|(\delta\vec{u}, \delta\vec{\boldsymbol{\sigma}})|_{\mathcal{E}}$ also applies to

$$
\left\langle (\widetilde{\mathsf{M}}_\alpha^l)^{-1}\delta\vec{\mathbf{j}}, \delta\vec{\mathbf{j}} \right\rangle + \left\langle (\widetilde{\mathsf{M}}_\gamma^{l-1})^{-1}\delta\vec{\psi}, \delta\vec{\psi} \right\rangle + \left\langle (\widetilde{\mathsf{M}}_{\beta,\Gamma}^{l-1})^{-1}\widetilde{\mathsf{T}}_\Gamma^m\delta\vec{\mathbf{j}}, \widetilde{\mathsf{T}}_\Gamma^m\delta\vec{\mathbf{j}} \right\rangle \ .
$$

This paves the way for coming to terms with $\|(\mathbf{j} - \mathbf{j}_h, \psi - \psi_h)\|_{\widehat{\mathcal{E}}}^2$.

## 6. ESTIMATION OF CONSISTENCY ERRORS

Let us study the consistency error term $\vec{R}_l$ from (19) and find bounds for the relevant primary norm $\left|(\mathsf{M}_\alpha^l)^{-1}\vec{R}_l\right|_\alpha$. As a consequence of (18), it is immediate that

$$\left|(\mathsf{M}_\alpha^l)^{-1}\vec{R}_l\right|_\alpha^2 = \left\langle \mathsf{M}_\alpha^l(\vec{\sigma}^* - \vec{\zeta}), \vec{\sigma}^* - \vec{\zeta} \right\rangle , \tag{26}$$

where I set $\vec{\zeta} := (\mathsf{M}_\alpha^l)^{-1}\widetilde{\mathsf{K}}_m^l\vec{\mathbf{j}}^*$. Note that by the definition of the weak solution (cf. (2)) $(\boldsymbol{\sigma}, \boldsymbol{\eta})_\alpha = \int_\Omega \mathbf{j} \wedge \boldsymbol{\eta}$ for all $\boldsymbol{\eta} \in \mathcal{H}_l(d, \Omega)$. A few formal manipulations yield

$$\left|(\mathsf{M}_\alpha^l)^{-1}\vec{R}_l\right|_\alpha = \sup_{\vec{\boldsymbol{\eta}} \in \mathcal{C}^l} \frac{\left\langle \mathsf{M}_\alpha^l(\vec{\sigma}^* - \vec{\zeta}), \vec{\boldsymbol{\eta}} \right\rangle}{|\vec{\boldsymbol{\eta}}|_\alpha} =$$

$$= \sup_{\vec{\boldsymbol{\eta}} \in \mathcal{C}^l} \frac{1}{|\vec{\boldsymbol{\eta}}|_\alpha} \left( \left\langle \mathsf{M}_\alpha^l\vec{\sigma}^*, \vec{\boldsymbol{\eta}} \right\rangle - \left\langle \widetilde{\mathsf{K}}_m^l\vec{\mathbf{j}}^*, \vec{\boldsymbol{\eta}} \right\rangle \right)$$

$$= \sup_{\vec{\boldsymbol{\eta}} \in \mathcal{C}^l} \frac{1}{|\vec{\boldsymbol{\eta}}|_\alpha} \left( \left\langle \mathsf{M}_\alpha^l\vec{\sigma}^*, \vec{\boldsymbol{\eta}} \right\rangle - (\vec{\sigma}^*, \boldsymbol{\eta}_h)_\alpha + (\vec{\sigma}^* - \boldsymbol{\sigma}, \boldsymbol{\eta}_h)_\alpha + \right.$$

$$\left. + \int_\Omega \mathbf{j} \wedge \boldsymbol{\eta}_h - \int_\Omega \mathbf{j}_h^* \wedge \boldsymbol{\eta}_h + \int_\Omega \mathbf{j}_h^* \wedge \boldsymbol{\eta}_h - \left\langle \widetilde{\mathsf{K}}_m^l\vec{\mathbf{j}}^*, \vec{\boldsymbol{\eta}} \right\rangle \right)$$

$$\leq \sup_{\vec{\boldsymbol{\eta}} \in \mathcal{C}^l} \frac{\left\langle \mathsf{M}_\alpha^l\vec{\sigma}^*, \vec{\boldsymbol{\eta}} \right\rangle - (\vec{\sigma}^*, \boldsymbol{\eta}_h)_\alpha}{|\vec{\boldsymbol{\eta}}|_\alpha} + \sup_{\vec{\boldsymbol{\eta}} \in \mathcal{C}} \frac{\int_\Omega \mathbf{j}_h^* \wedge \boldsymbol{\eta}_h - \left\langle \widetilde{\mathsf{K}}_m^l\vec{\mathbf{j}}^*, \vec{\boldsymbol{\eta}} \right\rangle}{|\vec{\boldsymbol{\eta}}|_\alpha} +$$

$$+ \sup_{\vec{\boldsymbol{\eta}} \in \mathcal{C}^l} \frac{\|\boldsymbol{\eta}_h\|_\alpha}{|\vec{\boldsymbol{\eta}}|_\alpha} \cdot \left( \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*\|_\alpha + \|\mathbf{j} - \mathbf{j}_h^*\|_{1/\alpha} \right)$$

The first two terms are typical consistency errors, as they occur in estimates for inexact finite element schemes [16, Ch. 8]. The factor in front of the third term reflects the stability of the approximate mass matrix $\mathsf{M}_\alpha^l$, whereas the third term itself incorporates approximation errors of the nodal projection. In the case of an exact Galerkin approach, the consistency terms and the stability factor can be dropped. Then, in combination with the results of the previous section, a standard finite element error estimate pops up.

There is a more direct approach to bounding the norm from (26), which is particularly useful in the case of diagonal finite volume methods (cf. [40, 41]). So let us assume that $\widetilde{\mathsf{K}}_m^l$ is an identity matrix and $\mathsf{M}_\alpha^l$ is diagonal. For an $l$-facet $F$ let $\tilde{F}$ stand for its associated dual $n - l$-facet. Remember that the components of vectors in $\mathcal{C}^l$ can be indexed by the $l$-facets in $\mathcal{F}_l$. By definition of $\vec{R}_l$

$$\left|(\mathsf{M}_\alpha^l)^{-1}\vec{R}_l\right|_\alpha^2 = \sum_{F \in \mathcal{F}_l} m_F \vec{R}_{l,F}^2 \,, \quad \vec{R}_{l,F} := m_F^{-1}\vec{\sigma}_F^* - \vec{\mathbf{j}}_{\tilde{F}}^* \,, \qquad (27)$$

with $m_F$ the diagonal element of $\mathsf{M}_\alpha^l$ belonging to $F$, and a subscript $F$ acting as a selector for vector components. Plugging in the canonical degrees of freedom for Whitney forms, we arrive at

$$\vec{R}_{l,F} = m_F^{-1}\int_F \boldsymbol{\sigma} - \int_{\tilde{F}} \mathbf{j} = m_F^{-1}\int_F \boldsymbol{\sigma} - \int_{\tilde{F}} \star_\alpha \boldsymbol{\sigma} \,. \qquad (28)$$

In the spirit of finite difference methods, the final term may be tackled based on a Taylor's expansion of $\boldsymbol{\sigma}$. If possible, it should be around a suitable point in space, provided by the intersection of $F$ and $\tilde{F}$. Sufficient smoothness of $\boldsymbol{\sigma}$ is tacitly assumed. An alternative to Taylor's expansion are Bramble-Hilbert techniques, which impose less stringent requirements on smoothness. However, shape-regularity of the meshes is indispensable then [40].

It is worth noting that (28) offers a prescription for a viable choice of $\mathsf{M}_\alpha^l$. For instance, one could try to fix all $m_F$ such that $\vec{R}_{l,F}, F \in \mathcal{F}_l$, vanishes for any constant $\boldsymbol{\sigma}$. However, the space of constant $l$-forms has dimension $\binom{n}{l}$. As $0 < l < n$, this objective cannot be achieved in general. Consider the case of a constant metric $\alpha$ and flat facets. If $\boldsymbol{\sigma}$ is constant, too, (28) means

$$\vec{R}_{l,F} = m_F^{-1}\mathrm{vol}_\alpha(F)\boldsymbol{\sigma}(\mathbf{t}_1,\dots,\mathbf{t}_l) - \mathrm{vol}_\alpha(\tilde{F})\boldsymbol{\sigma}(\mathbf{n}_1,\dots,\mathbf{n}_l) \,, \qquad (29)$$

where $\{\mathbf{t}_1,\dots,\mathbf{t}_l\}$ and $\{\mathbf{n}_1,\dots,\mathbf{n}_l\}$ are $\alpha$-orthonormal (oriented) bases of the tangent space of $F$ and of the orthogonal complement of the tangent space of $\tilde{F}$. Only if $\mathrm{Span}\,\{\mathbf{t}_1,\dots,\mathbf{t}_l\} = \mathrm{Span}\,\{\mathbf{n}_1,\dots,\mathbf{n}_l\}$, i.e. if $F$ and $\tilde{F}$ are $\alpha$-orthogonal, we can make $\vec{R}_{l,F}$ vanish for all alternating $l$-linear forms $\boldsymbol{\sigma}$. This highlights the necessity of *orthogonal dual meshes*, if diagonal approximate mass matrices are desired. (cf. [15, Sect. III]).

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