Progress In Electromagnetics Research B, Vol. 50, 315–329, 2013

NOVEL MULTI-DIMENSIONAL WCAWE TECHNIQUE FOR THE EFFICIENT CALCULATION OF RCS

Patrick J. Bradley*

RF & Microwave Research Group, School of Electrical Electronic and Communications Engineering, University College Dublin, Ireland

Abstract—In this paper, a novel moment-matching reduced order model technique termed the multi-dimensional well-conditioned asymptotic waveform evaluation (MDWCAWE) method is presented. The MDWCAWE method can be used to efficiently determine the radar cross section (RCS) of arbitrarily shaped objects, in both the frequency and angular domains simultaneously. Numerical results are given in order to demonstrate the accuracy and robustness of the MDWCAWE method. All scattering problems investigated in this work are formulated using the two-dimensional volume-surface electric field integral equation (EFIE). We consider problems involving scattering from both dielectric dispersive and conducting objects.

1. INTRODUCTION

The method of moments (MoM) is a key frequency domain technique in the analysis and design of microwave structures. Specifically, integral equation (IE) techniques can be used to efficiently model the electromagnetic interactions of composite structures, such as those comprised of perfect electrically conducting (PEC) surfaces and dielectric volumes, which are prevalent in many practical applications [1]. IE methods seek the solution of a coupled set of surface (S) and Volume (V) integral equations by enforcing the electric field boundary conditions on the PEC surfaces and throughout the dielectric volume.

Many problems, such as RCS computation [2] or analysis of reflection characteristics [3], require the determination of the system response at multiple frequencies and incident field angles. For such analysis, the equivalent current density must be calculated at each discrete parameter, which can become computationally prohibitive.

Received 11 February 2013, Accepted 27 March 2013, Scheduled 12 April 2013

* Corresponding author: Patrick J. Bradley (patrick.bradley@ucd.ie).

Model order reduction (MOR) approaches have been developed to reduce this computational complexity. The resulting reduced order model (ROM) is of very low-order. Thus permitting the calculation of the RCS at an almost arbitrary number of frequencies and angle samples, at a fraction of the original computational cost. Multi-parameter MOR approaches can be broadly characterised into two approaches, implicit and explicit moment matching. For the analysis of frequency dependent problems, explicit moment matching techniques such as the Padeé via AWE [4, 5], have received extensive attention. This technique approximates the response around an expansion point using a rational function by matching terms in a Taylor series expansion. However, this approach exhibits certain numerical stability problems and is inherently ill-conditioned, inevitably leading to stagnation in the moment matching process.

Until the middle of the last decade, implicit approaches such as Krylov subspace methods were limited to linearisation of the nonlinear dependence [6], which required an increase in the system matrix size and a substantial increase in memory requirements. A more elegant and efficient approach, the Second-order Arnoldi (SOAR) algorithm [7], resolved these problems and has led to several improvements and extensions in literature [8, 9].

In this paper, we focus on the EFIE formulation, as such, these approaches are inadequate to cope with the complex dependence on frequency. This is further complicated by the frequency dependent nature of the material properties and incident field vector. Not until the development of implicitly matching moments of the AWE method via a Krylov subspace approach, could higher-order ROM be produced based on EFIE formulations [10, 11]. This Galerkin AWE approach however was numerically unstable and required an augmentation to the orthonormalisation process through the introduction of correction terms [11]. This resulted in the WCAWE process, which enforces Galerkin conditions and introduces correction terms, to project the original system onto the subspace formed by the orthogonalised moments. This ensured the residual of the approximation is orthogonal to the moment subspace.

The objective of this paper is to propose a robust and stable extension of the WCAWE approach to accelerate the computation of the EFIE problem with explicit frequency and angular dependence, termed the multi-dimensional WCAWE (MDWCAWE) method. While single parameter methods are well established as detailed above, multi-parameter MOR techniques are less mature and are limited to Second-order Arnoldi [9, 12] and Galerkin AWE [2, 13, 14] approaches, which suffer from the same deficiencies as their single parameter counterparts.

In addition we outline a systemic methodology to include frequency dependent dielectric variations and cross derivative terms within our approximation which can have a significant impact on the validity of this approach as a design tool.

2. VOLUME-SURFACE ELECTRIC FIELD INTEGRAL EQUATION FORMULATION

Consider an arbitrarily shaped two-dimensional composite structure, comprising PEC surfaces S and potentially inhomogeneous dielectric volumes V, that reside in free space with permittivity ϵ_b and permeability μ_b. The dielectric region V is characterised by its permittivity $\epsilon(\mathbf{r})$, conductivity $\sigma(\mathbf{r})$ and we assume constant permeability $\mu(\mathbf{r}) = \mu_b$. Inside the dielectric and on the surface of the PEC, the incident wave $\mathbf{E}^{inc}$ induces Vol. current $\mathbf{J}_{\mathrm{V}}$ and surface current $\mathbf{J}_{\mathrm{S}}$. The surface and Vol. current densities induced on S and in V generate scattered electric field $\mathbf{E}_{\Omega}^{sca}$ given by

$$\mathbf{E}_{\Omega}^{sca}(\mathbf{r}) = -\jmath k_b \eta_b \mathbf{A}_{\Omega}(\mathbf{r}), \quad \Omega = \mathrm{S} \text{ or } \mathrm{V} \tag{1}$$

where the magnetic vector potential is defined as

$$\mathbf{A}_{\Omega}(\mathbf{r}) = \int_{\Omega} \mathbf{J}_{\Omega}(\mathbf{r}') H_0^{(2)}\left(k_b \left|\mathbf{r} - \mathbf{r}'\right|\right) d\mathbf{r}'. \tag{2}$$

In Equations (1) and (2), the free-space Green's function is given by the zero-order Hankel function of the second kind. k_b and η_b are the wavenumber and impedance of the background medium, respectively. Enforcing the electric field boundary condition and taking account of the contribution of the scattered field generated by the surface and volume current densities, yields the EFIE [1]

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \mathbf{E}^{inc}(\mathbf{r}) + \mathbf{E}_{\mathrm{V}}^{sca}(\mathbf{r}) + \mathbf{E}_{\mathrm{S}}^{sca}(\mathbf{r}), \quad r \in \mathrm{V} \\ \hat{\mathbf{n}} \times \mathbf{E}^{inc}(\mathbf{r}) &= -\hat{\mathbf{n}} \times \mathbf{E}_{\mathrm{V}}^{sca}(\mathbf{r}) - \hat{\mathbf{n}} \times \mathbf{E}_{\mathrm{S}}^{sca}(\mathbf{r}), \quad r \in \mathrm{S} \end{aligned} \tag{3}$$

where $\mathbf{E}(\mathbf{r})$ denotes the total electric field, and $\hat{\mathbf{n}}$ is a unit vector normal to S. In V, $\mathbf{J}_{\mathrm{V}}$ and $\mathbf{E}(\mathbf{r})$ are related

$$\mathbf{J}_{\mathrm{V}} = \jmath\omega\epsilon_b(\epsilon_r(\mathbf{r}) - 1)\mathbf{E}. \tag{4}$$

Using m pulse basis functions and Dirac-Delta testing functions [15], the integral equations can be discretised by employing the method of moments. Taking due care to the treatment of the singularity associated with the Hankel function leads to the following linear system of equations,

$$\begin{bmatrix} \mathbf{Z}^{\mathrm{VV}} & \mathbf{Z}^{\mathrm{VS}} \\ \mathbf{Z}^{\mathrm{SV}} & \mathbf{Z}^{\mathrm{SS}} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{\mathrm{V}} \\ \mathbf{x}^{\mathrm{S}} \end{bmatrix} = \begin{bmatrix} \mathbf{b}^{\mathrm{V}} \\ \mathbf{b}^{\mathrm{S}} \end{bmatrix}, \quad \mathbf{Z}(f)\,\mathbf{x}(f,\theta) = \mathbf{b}(f,\theta) \tag{5}$$

where $\mathbf{Z}$ is a complex valued dense $n \times n$ matrix containing coupling information between the basis functions. $\mathbf{b}$ and $\mathbf{x}$ are the incident field vector and vector of unknown current density amplitudes, respectively, that are dependent on both frequency and angle. In order to solve the RCS over a frequency and angular range, Equation (5) must be solved for $\mathbf{x}$ repeatedly at each incident frequency and angle of excitation.

3. REVIEW OF MULTI-DIMENSIONAL ASYMPTOTIC WAVEFORM EVALUATION (MDAWE)

The MDAWE method approximates the frequency and angular response by expanding $\mathbf{x}(f,\theta)$ into a Taylor series around (f_o, θ_0)

$$\mathbf{x}(f,\theta) = \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} \mathbf{m}_{i,j}\,(f-f_0)^{i-1}\,(\theta-\theta_0)^{j-1}. \tag{6}$$

By expanding the impedance matrix $\mathbf{Z}(f)$ and the excitation vector $\mathbf{b}(f,\theta)$ into a Taylor series

$$\sum_{i=0}^{\alpha} \mathbf{Z}^{[i]}\,(f-f_0)^i\,\mathbf{x} = \sum_{i=0}^{\alpha}\sum_{j=0}^{\beta} \mathbf{b}^{[i,j]}\,(f-f_0)^i\,(\theta-\theta_0)^j \tag{7}$$

and equating equal powers, a recursive relation for the moment vectors is obtained

$$\mathbf{m}_{i,j} = \mathbf{Z}^{-1}(f_0)\left[\frac{\mathbf{b}^{[i-1,j-1]}(f_0,\theta_0)}{(i-1)!\,(j-1)!} - \sum_{p=1}^{i-1}\frac{\mathbf{Z}^{[p]}(f_0)\,\mathbf{m}_{i-p,j}}{p!}\right] \tag{8}$$

were $\mathbf{Z}^{[p]}(f_0)$ denotes the pth derivative of $\mathbf{Z}(f)$ with respect to f evaluated at f_0, while

$$\mathbf{b}^{[i,j]}(f_0,\theta_0) = \left.\frac{\partial^{i+j}\mathbf{b}(f,\theta)}{\partial f^i \partial\theta^j}\right|_{f_0,\theta_0}. \tag{9}$$

Once the moment vectors are obtained, the RCS can be calculated using Equation (6). However, the use of this expansion is severely limited to the radius of convergence of the Taylor series. In such cases, a rational function approach is used to improve the accuracy of the numerical solution. The Padé representations have a larger radius of

convergence and therefore can provide broader extrapolation as they include poles as well as zeros in the response [16]. The MDAWE moment-matching multi-dimensional subspace, $\mathbf{M}_q$, generated from the recursive Equation (8), is given by the span of the columns of the matrix

$$\mathbf{M}_q = [\mathbf{m}_{1,1}, \mathbf{m}_{2,1}, \ldots \mathbf{m}_{\alpha,1}, \mathbf{m}_{1,2}, \ldots, \mathbf{m}_{\alpha-1,\beta}, \mathbf{m}_{\alpha,\beta}]. \tag{10}$$

It has been well documented [11, 16] that the process of sequentially evaluating $\mathbf{m}_{i,j}$ via Equation (8) is inherently ill-conditioned leading to instability in the computation of the Padé approximation. In a direct implementation, finite precision arithmetic causes each newly created moment vector $\mathbf{m}_{i,j}$ to converge towards the eigenvector that is associated with the dominant eigenvalue of $\mathbf{Z}(f_0)$. As a result, the moments $\mathbf{m}_{i,j}$ contain only information corresponding to one eigenvalue of $\mathbf{Z}(f_0)$, even for small values of i, j. Consequently, the solution of the Padé approximation becomes unattainable, thus restricting its use to approximations of relatively low-order.

4. MULTI-DIMENSIONAL WELL-CONDITIONED ASYMPTOTIC WAVEFORM EVALUATION (MDWCAWE)

To avoid the underlying numerical instability of the AWE approach the WCAWE technique was derived [11]. The WCAWE imposes an orthogonality relation upon the generated moment vectors, eliminating ill-conditioning and ensuring linear independence. In addition, the WCAWE introduces correction factors in the orthogonalisation process to ensure that the techniques preserve the moments of the original system. As a consequence, higher-order approximations can be constructed. In a similar vein to the WCAWE, the MDWCAWE process as outlined in Appendix A, constructs the columns of $\mathbf{M}_q$ from Equation (10) iteratively by utilizing a modified Gram-Schmidt process. This procedure is used to orthogonalise $\mathbf{v}_{i,j}$ onto the span

$$\tilde{\mathbf{V}}_{q-1} = [\tilde{\mathbf{v}}_{1,1}, \tilde{\mathbf{v}}_{2,1}, \ldots \tilde{\mathbf{v}}_{\alpha-1,\beta-1}]\,. \tag{11}$$

This is achieved by computing the orthogonal projection of $\tilde{\mathbf{v}}_{i,j}$ onto span$\{\mathbf{v}_{1,1}\ \mathbf{v}_{2,1}\ \ldots\ \mathbf{v}_{\alpha-1,\beta-1}\}$. This projection is subtracted from the original vector and the result is normalised to obtain $\mathbf{v}_{i,j}$. This is, by construction, orthogonal to all previously computed vectors $\tilde{\mathbf{v}}_{1,1}, \tilde{\mathbf{v}}_{2,1}, \ldots, \tilde{\mathbf{v}}_{\alpha-1,\beta-1}$ and has unit norm. In this way, the orthogonality of the basis vectors is guaranteed and the moment-matching process can be maintained. The resultant vector generated

in Appendix A is given by

$$\tilde{\mathbf{v}}_{i,j}=\mathbf{Z}^{-1}\left[\sum_{p=1}^{i-1}\left(\mathbf{b}^{[p,j-1]}\mathbf{e}_1^T\mathbf{P}_{\mathbf{U}1}\left(i,p\right)\mathbf{e}_{i-p}\right)\right.$$
$$\left.-\mathbf{Z}^{[1]}\mathbf{v}_{i-1,j}-\sum_{p=2}^{i-1}\mathbf{Z}^{[p]}\mathbf{V}_{i-p,j}\mathbf{P}_{\mathbf{U}2}\left(i,p\right)\mathbf{e}_{i-p}\right] \tag{12}$$

where $\mathbf{e}_r$ is the $i \times 1$ vector with the rth component equal to unity and all others equal to zero. $\mathbf{V}_{[:,:,j]}$ and $\tilde{\mathbf{V}}_{[:,:,j]}$ are related by an $i \times i$ upper triangular, nonsingular matrix $\mathbf{U}_{[:,:,j]}$ (Appendix A) created by the coefficients of the Gram-Schmidt process

$$\mathbf{V}_{[:,:,j]} = \tilde{\mathbf{V}}_{[:,:,j]}\mathbf{U}^{-1}_{[:,:,j]} \quad \text{for} j = 1,\ldots,\beta. \tag{13}$$

The correction term in Equation (12) is given by

$$\mathbf{P}_{\mathbf{U}w}\left(i,p\right) = \prod_{t=w}^{p}\mathbf{U}^{-1}_{[t:i-p+t-1,t:i-p+t-1,j]} \tag{14}$$

where $w = 1$ or 2. $\mathbf{P}_{\mathbf{U}w}(i,p)$ is constructed from blocks extracted from the mapping matrix $\mathbf{U}$ created by the coefficients of the Gram-Schmidt process. The $\mathbf{P}_{\mathbf{U}w}(i,p)$ matrix tracks the mapping from one vector space to another in each of the jth subspaces in the MDWCAWE process.

5. FORMULATION OF THE MULTI-DIMENSIONAL REDUCED ORDER MODEL

Similar to the one-dimensional case MOR is achieved by projecting the system Equation (7) onto a lower dimensional subspace. After q steps of the MDWCAWE process, an approximation to $\mathbf{x}$ can be made in terms of q basis vectors of the form

$$\mathbf{x} \approx \mathbf{x}_q = \mathbf{V}_q\mathbf{a}_q = \sum_{n=1}^{q}\mathbf{v}_n\gamma_n \tag{15}$$

the $\mathbf{a}_q$ are chosen such that the approximation of Equation (15) minimises the residual

$$\mathbf{r}_q = \sum_{i=0}^{\alpha} \left(\mathbf{Z}^{[i]} \left(f - f_0 \right)^i \right) \sum_{n=1}^{q} \mathbf{v}_n \gamma_n - \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \left(\mathbf{b}^{[i,j]} \left(f - f_0 \right)^i \left(\theta - \theta_0 \right)^j \right). \tag{16}$$

This is conditional on the careful selection of the associated $\mathbf{a}_q$ specifically, that the residual vector is constrained to be orthogonal to q linearly independent vectors

$$\mathbf{r}_q \perp \mathbf{V}_q \tag{17}$$

by setting

$$\mathbf{a}_q = \left(\sum_{i=0}^{\alpha} \mathbf{V}_q^H \mathbf{Z}^{[i]} \mathbf{V}_q \left(f - f_0 \right)^i \right)^{-1} \left(\sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \mathbf{V}_q^H \mathbf{b}^{[i,j]} (f - f_0)^i (\theta - \theta_0)^j \right). \tag{18}$$

Ultimately, this process results in the approximation to the solution vector $\mathbf{x}$ for any frequency f and or angle θ in the range $f_{\min} \leq f \leq f_{\max}$, $\theta_{\min} \leq \theta \leq \theta_{\max}$, respectively, given by Equation (15). Clearly, Equation (15) can be used to efficiently solve over a wide range of frequencies and of angles as it requires the inversion of a matrix of order $q \ll n$ for each parameter pair. It can be shown that the reduced system, preserves the moments of the original system using the techniques presented within [17], ensuring that

$$\text{span} \left(\mathbf{V}_q \right) = \text{span} \left(\mathbf{M}_q \right). \tag{19}$$

It should be noted that in order to achieve additional bandwidth in either the frequency or angular domain, a multipoint MDWCAWE approach could be implemented similar to that of the WCAWE. Additionally, automated error controls are outlined in [17, 18] that can be used to determine the number of moment required such that the approximate solution can converge to a pre-specified tolerance.

6. RECURSIVE FORMULATION OF THE MATRIX DERIVATIVES

Careful consideration needs to taken when differentiating Equation (3) in order to model the frequency dependant variations in ϵ. For our analysis we consider the CoFe alloy nano-particles (NPs) and nano-flakes (NFs) whose dielectric properties are tabulated in [19]. These alloys are traditionally used in tandem as a two/mulitlayered

radar absorber for perfect electric conductor (PEC) structures [19]. To account for these frequency dependant variations we fit a third-order polynomial to the individual dielectric properties resulting in an analytical expression independently fitted for both the real and imaginary permittivity of the form [18]

$$\epsilon'(f) = a'f^3 + b'f^2 + c'f + d' \tag{20}$$

$$\epsilon''(f) = a''f^3 + b''f^2 + c''f + d'' \tag{21}$$

where (a', b', c', d') and (a'', b'', c'', d'') are the polynomial constants for real and imaginary permittivity in each basis cell respectively.

Using Equations (20)–(21), the identities [20] and the following recursive relation when differentiating the Bessel and Hankel functions the ith, jth derivatives for the matrix entries can be derived

$$\Psi_v^{(q)} = \frac{1}{2^q}\sum_{p=0}^{q}(-1)^{(p+2)}\begin{pmatrix} q \\ p \end{pmatrix}\Psi_{v-q+2p} \tag{22}$$

where $\Psi_v^{(q)}$ denotes the qth derivative of $J, Y, H^{(1)}, H^{(2)}$ of order v [21].

$$Z_{n,m}^{VV[i]} = \frac{\eta\pi a_n}{2}\sum_{p=0}^{i}\binom{i}{p} J_1\left(k_{b0}a_n\right)^{[p]} H_0^{(2)}(k_{b0}\left|\mathbf{r}_n - \mathbf{r}_m\right|)^{[i-p]} \quad n \neq m \tag{23}$$

$$Z_{n,m}^{VV[i]} = \frac{\eta\pi a_n}{2} H_1^{(2)}\left(k_{b0}a_n\right)^{[i]} - C_i \qquad n = m \tag{24}$$

where

$$C_i = \frac{1}{k_{b0}\left(\epsilon_{ci}\left(k_{b0}\right) - \epsilon_0\right)}\left[\left(\jmath\eta\epsilon_{ci}\left(k_{b0}\right)\right)^{[i]} - \left[\sum_{j=1}^{i}\left[\sum_{p=0}^{j}\begin{pmatrix} j \\ p \end{pmatrix}\left(k_b\right)^{[p]}\left(\epsilon_{ci}\left(k_{b0}\right) - \epsilon_0\right)^{[j-p]}\right] C^{[i-j]}\right]\right] \tag{25}$$

and

$$\left(\jmath\eta\epsilon_{ci}\left(k_{b0}\right)\right)^{[i]} = \left(\jmath\eta\right)^{[i]}\left(\left(a_n'f^2 + b_n'f + c_n'\right)^{[i]} - \jmath\left(a_n''f^2 + b_n''f + c_n''\right)^{[i]}\right). \tag{26}$$

7. NUMERICAL RESULTS AND VALIDATIONS

To illustrate the validity of the volume-surface EFIE, we initially consider a simple two-dimensional coated conducting cylinder with a radius of 2λ (1.2 m), centered at the origin. The core of the cylinder has a radius of (0.6 m) and the thickness of the coating layer is (0.6 m), with a relative permittivity of $\epsilon_r = 4 - 4i$. The structure is

illuminated by a TM^z plane wave and the bi-static radar cross-section at 500 MHz is computed. The structure is discretised using a minimum of $\lambda/10$ discretisations per wavelength which results in a total of 12942 unknowns, given by 202 surface and 12740 volume basis functions. It can be seen in Figure 1 that there is good agrement between the MoM and the analytical Mie series solution.

In the second example, a similar numerical experiment is conducted for a coated conducting cylinder of radius 0.138 m composed of two concentric layers of dielectric material with outer radius 2λ (0.133 m). In particular, the outer layer is comprised of CoFe nano particles (NPs) while the inner layer consists of CoFe nano flakes (NFs) where the thickness is 0.033 m. The structure is discretised using $n = 2460$ cells and the mono-static RCS frequency and angular response is computed over the range $f = 4 : 4.5$ GHz and $\theta = 175 : 185$, for increments of 0.1 GHz and 1°, respectively, using the MDWCAWE method. The expansion points for both f and θ are chosen to be in the centre of each band of interest. Selecting 18 moments with respect to frequency and 10 moments with respect to angle, results in a reduced order model of dimension $q = 180$. As seen from Figure 2 the MDWCAWE techniques outperform the MDAWE with Padé approximation using the same number of moments, duplicating the reference solution over the entire band of frequencies to within a 0.1% relative error for $\theta = 0$. These results are not unexpected as the WCAWE has been previously shown to be a more accurate technique than the AWE with Padé. To fully illustrate the capability of the

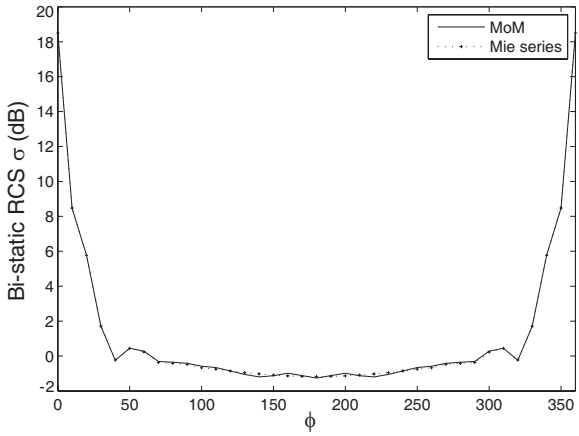

Figure 1. Example 1: Bi-static RCS comparing the MoM against the analytical Mie series solution.

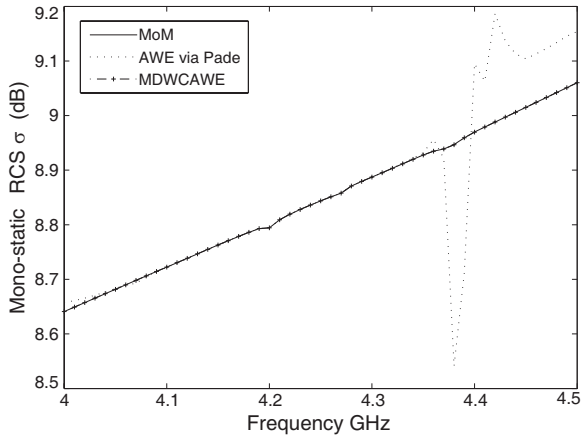

Figure 2. Example 2: RCS as a function of frequency comparing the MDWCAWE and the MDAWE with Padé against the MoM solution for $\theta = 0$.

MDWCAWE, the frequency and angular response of the MDWCAWE compared against the MoM solution is given in Figure 3 while the associated % relative error is illustrated in Figure 4. In order to achieve a wider bandwidth, a balance must be struck between using more moments albeit at an rapidly increasing computational cost or implementing a multipoint approach where additional expansion points are placed over the frequency and angular domains [18].

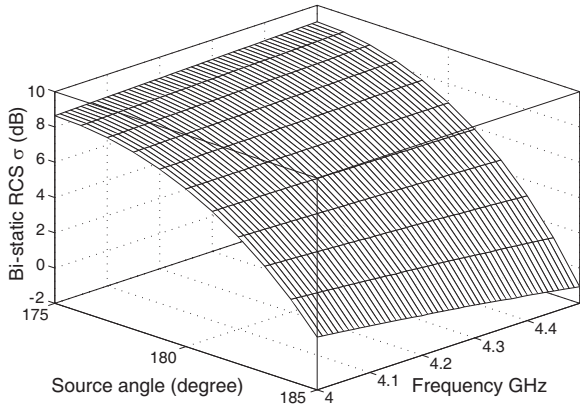

Figure 3. Example 2: RCS as a function of frequency and angle comparing the MDWCAWE against the MoM solution.

The total CPU times for MWCAWE compared against the MoM in shown in Table 1, which demonstrate the capability of this approach. This table clearly shows the efficiency obtained using the proposed method, achieving a speed-up factor of 3.92 compared to the MoM,

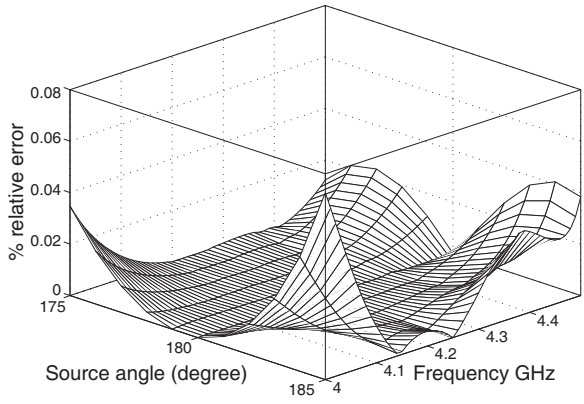

Figure 4. Example 2: Percentage relative error for Figure 3.

Table 1. CPU time analysis — n_f — number of frequency samples, n_θ — number of angle source samples, t_q = CPU time in seconds to generate moments, t_z = CPU time in seconds to generate the coupling matrix derivatives, t_b = CPU time in seconds to generate the incident vector derivatives, t_s = average CPU time in seconds to solve for RCS at each frequency using a GMRES with tolerance 10^{-6}, t_t = total CPU time in seconds to generate and solve case study problem.

Legend	MoM	MDWCAWE
Order	$n = 2124 + 336 = 2460$	$q = 18 * 10 = 180$
n_f	51	51
n_θ	11	11
$t_q(s)$	-	643.98
$t_z(s)$	-	1418.14
$t_b(s)$	-	4.20
$t_s(s)$	25.2	2.74
$t_t(s)$	14137.2	3604.1
$e_q(\%)$	-	< 0.1
Speed-up	-	3.92

while only incurring a relative error within 0.1% over the entire frequency and angular domains. It should be noted that the main computational overhead is due to the generation of the derivatives, however, as the derivatives need only be calculated once, there is minimal computational expense required for all subsequent solutions.

8. CONCLUSION

In this paper, a multi-dimensional version of the WCAWE model order reduction technique was developed that can produce both a broad frequency and angular band approximation simultaneously. In addition, we have provided a reduced order model that can accurately account for the frequency dependent dielectric parameters. Examples are presented which demonstrate that the MDWCAWE can produce a numerically stable and robust high-order approximation that provides the flexibility needed to efficiently handle the short comings of the MDAWE. Notably the presented approach outperforms the existing techniques in accuracy over the entire parameter range while showing a significant reduction in CPU time compared to the original MoM system.

APPENDIX A. MULTI-DIMENSIONAL WELL-CONDITIONED ASYMPTOTIC WAVEFORM EVALUATION ALGORITHM (MDWCAWE)

Main() {
  for $j = 1, \ldots, \beta$
    $\tilde{\mathbf{v}}_{1,j} = \mathbf{Z}^{-1}\mathbf{b}^{[0,j-1]}$
    $\mathbf{U}_{[1,1,j]} = \|\tilde{\mathbf{v}}_{1,j}\|_2$
    $\mathbf{v}_{1,j} = \tilde{\mathbf{v}}_{1,1}\mathbf{U}^{-1}_{[1,1,j]}$
    if $j \neq 1$
      Orthogonalise()
    end if
    for $i = 2, \ldots, \alpha$
      $\tilde{\mathbf{v}}_{i,j} = \mathbf{Z}^{-1}\left[\sum_{p=1}^{i-1}\left(\mathbf{b}^{[p,j-1]}\mathbf{e}_1^T\mathbf{P}_{\mathbf{U}1}(i,p)\mathbf{e}_{i-p}\right)\right.$
      $\left.-\mathbf{Z}^{[1]}\mathbf{v}_{i-1,j} - \sum_{p=2}^{i-1}\mathbf{Z}^{[p]}\mathbf{V}_{i-p,j}\mathbf{P}_{\mathbf{U}2}(i,p)\mathbf{e}_{i-p}\right]$

```
            Orthogonalise()
            U_[i,i,j] = ||ṽ_{i,j}||_2
            v_{i,j} = ṽ_{i,j} U^{-1}_[i,i,j]
        end i.
        V = [V, v_{:,j}]
    end j.
}
Orthogonalise() {
    for γ = 1, ..., j
        if γ = j
            h = i − 1
        else
            h = i
        end if
        for ζ = 1, ..., h
            U_[ζ,i,γ] = v^H_{ζ,γ} ṽ_{i,j}
            ṽ_{i,j} = ṽ_{i,j} − U_[ζ,i,γ] v_{ζ,γ}
        end ζ
    end γ
}
```

REFERENCES

1. Ewe, W. B., L. W. Li, and M. S. Leong, "Fast solution of mixed dielectric/conducting scattering problem using volume-surface adaptive Integral method," *IEEE Trans. on Antennas and Propagation*, Vol. 52, No. 11, 3071–3077, 2004.
2. Tong, C., P. Yan, Z. Huang, and F. Geng, "Simultaneous extrapolation of RCS of a radar target in both angular and frequency domains based on bivariate Padé approximant technique," *Proc. Asia-Pacific Microwave Proceedings*, Vol. 3, 1–4, 2005.
3. Gudu, T. and L. Alatan, "Use of asymptotic waveform evaluation technique in the analysis of multilayer structures with doubly periodic dielectric gratings," *IEEE Trans. on Antennas and Propagation*, Vol. 57, No. 9, 2641–2649, 2009.
4. Olsson, K. H. A., "Model order reduction with rational krylov methods," Ph.D. Thesis, KTH Computer Science and

Communication, Stockholm, Sweden, 2005.

5. Grimme, E. J., "Krylov projection methods for model reduction," Ph.D. Thesis, Coordinated-Science Laboratory, University of Illinois at Urbana-Champaign, USA, Urbana-Champaign, IL, 1997.
6. Salimbahrami, B. and B. Lohmann. "Order reduction of large scale second-order systems using Krylov subspace methods," *Linear Algebra and Its Applications*, Vol. 415, Nos. 2–3, 385–405, 2006.
7. Bai, Z. and Y. Su, "SOAR: A second-order Arnoldi method for the solution of the quadratic eigenvalue problem," *SIAM J. Matrix Anal. Appl.*, Vol. 26, No. 3, 640–659, Mar. 2005.
8. Koyama, T. and S. Govindjee. "Moment matching theorems for dimension reduction of higher order dynamical systems via higher order Krylov subspaces," Department of Civil and Environmental Engineering, University of California, 2008.
9. Li, Y.-T., Z. Bai, Y. Su, and X. Zeng. "Model order reduction of parameterized interconnect networks via a two-directional Arnoldi process," *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, Vol. 27, No. 9, 1571–1582, Sept. 2008.
10. Bradley, P., C. Brennan, M. Condon, and M. Mullen. "Accelerated source-sweep analysis using a reduced-order model approach," *IEEE Trans. on Antennas and Propagation*, Vol. 59, No. 11, 4360–4363, Nov. 2011.
11. Slone, R. D., R. Lee, and J.-F. Lee, "Well-conditioned asymptotic waveform evaluation for finite elements," *IEEE Trans. on Antennas and Propagation*, Vol. 51, No. 9, 2442–2447, Sep. 2003.
12. Farle, O. and R. Dyczij-Edlinger, "Numerically stable moment matching for linear systems parameterized by polynomials in multiple variables with applications to finite element models of microwave structures," *IEEE Trans. on Antennas and Propagation*, Vol. 58, No. 11, 3675–3684, Nov. 2010.
13. Farle, O., V. Hilla, P. Ingelstrma, and R. Dyczij-Edlinger, "Multiparameter polynomial order reduction of linear finite element models," *Mathematical and Computer Modelling of Dynamical Systems*, Vol. 14, No. 5, 421–434, 2008.
14. Ahmadloo, M. and A. Dounavis. "Parameterized model-order reduction for efficient eigenanalysis of dielectric waveguide structures," *IEEE Trans. on Microwave Theory and Techniques*, Vol. 56, No. 12, 2851–2858, 2008.
15. Peterson, A. F., S. L. Ray, and R. Mittra. *Computational Methods*

for Electromagnetics, IEEE Press Series on ElectromagneticWave Theory, 1st Edition, Wiley-IEEE Press, Piscataway, New Jersey, USA, 1997.

16. Fasenfest, B., J. D. Rockway, N. J. Champagne, and R. M. Sharpe, "A generalized fast frequency sweep algorithm for coupled circuit EM simulations," *Proc. IEEE Antennas and Propagation Society International Symposium*, Vol. 4, 3944–3947, Jun. 2004.
17. Slone, R. D., "Fast frequency sweep model order reduction of polynomial matrix equations resulting from finite element discretizations," Ph.D. Thesis, Ohio State Univ., Columbus, 2002.
18. Bradley, P., C. Brennan, and M. Condon, "Efficient wideband electromagnetic scattering computation for frequency dependent lossy dielectrics using WCAWE," *IEEE Trans. on Antennas and Propagation*, Vol. 57, No. 10, 3274–3282, Oct. 2009.
19. Gong, Y. X., R. Mittra, L. Zhen, and W. Z. Shao, "Design of a two-layer ultra-wideband microwave absorber," *IEEE Antennas and Propagation Society International Symposium*, 1–4, Jul. 2010.
20. Jorba, A. and M. Zou, "A software package for the numerical integration of odes by means of high-order taylor methods," *Experimental Mathematics Exp.*, Vol. 14, 99–117, 2005.
21. Abramowitz, M. and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th edition, Dover, New York, 1964.