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SYNTHESIS OF OPTIMAL NARROW BEAM LOW SIDELOBE LINEAR ARRAY WITH CONSTRAINED LENGTH

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Abstract—The synthesis of optimal narrow beam low sidelobe linear array is addressed. Only the length of the array is constrained. The number, the positions and the weightings of the elements are left free. It is proven, that the optimal design is always an array with a small number of elements. One first demonstrates that among equally spaced linear arrays of given length, the *sparsest* Dolph-Chebyshev design, i.e., the one with the largest admissible inter-element distance, is the optimal one. Then, the restriction to equally spaced elements is removed, and the general problem is solved and discussed. It is shown that the *sparsest* Dolph-Chebyshev designs are optimal for array lengths in given specified intervals and close to optimal for all other lengths.

1. INTRODUCTION

The synthesis of linear arrays has been intensively investigated since the mid 20th century due to its many practical applications in radar, communication systems and more generally in signal processing.

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However, the basic question of determining the linear array of fixed length L, i.e., of fixed maximal distance between the center of phases of the two extreme elements, which radiates the optimal narrow beam low sidelobe pattern, seems to be still open.

It is only in the case of an array composed of equally spaced isotropic elements that the optimal excitations are known. Dolph was the first to propose a procedure that is optimal only for element spacings that are greater than half a wavelength ($d \geq \lambda/2$) [1]. One year later, Riblet [2] modified Dolph's method to obtain optimal designs also for $d < \lambda/2$ but only for an odd number of array elements. The extension to an even number of elements has been proposed much more recently by McNamara in [3] where a fairly exhaustive review of the works derived from Dolph original ideas is also given.

The design of unequally spaced arrays has also been and is still addressed by many authors [4–7]. In addition to the element weightings, the element spacing provides indeed another parameter to control the radiation pattern. However, the goals are different from the one studied here. In these studies, unequally spaced arrays are typically designed to reduce the number of array elements or to replace the amplitude tapering of equally spaced arrays by unequally spaced arrays with uniform amplitude, with no additional optimality goal.

In [8], the design of a narrow beam low sidelobe linear array with fixed number of elements is addressed using a global optimization routine. The computational load of these numerical techniques limits however their use to arrays with a small number of elements and optimality is not guaranteed.

Thus, while a considerable amount of work has been dedicated to array pattern synthesis, the canonical problem of finding the linear array of given length that radiates the pattern having the best tradeoff between sidelobe level and beam width has, to the best of our knowledge, not yet been addressed. This synthesis problem requires the simultaneous optimization of the array geometry (i.e., the number of elements and their positions) as well as the element weightings.

The next section describes this open problem and shows how to approximate it numerically while Section 3 deals with the synthesis of optimal equally spaced linear arrays. In Section 4, the general problem is solved using results from linear programming theory and the optimal solution is discussed. Finally, some numerical examples and comparisons between equally spaced arrays with optimal weightings and the optimal arbitrary arrays developed in this contribution are given to illustrate the obtained results.

2. PROBLEM DESCRIPTION

2.1. Problem Formulation and Discrete Approximation

One looks for the linear array of given length L (i.e., given distance between the phase center of the first and last element) that yields the pattern with a broadside magnitude normalized to unity having the minimal main beam width for a given maximal sidelobe level or equivalently the lowest sidelobe level ρ for a given half main beam width θ_s, as depicted in Fig. 1(a). This latter problem formulation will be considered in the sequel. Its solution yields the optimal array geometry and optimal associated weightings. The corresponding pattern will be called the optimal pattern.

Note that, as represented in Fig. 1(a), the beam width θ_s is the angular range in which the radiation pattern is not to be minimized. It thus differs from the standard half power beamwidth or from the definition used by Dolph [1] where the beam width is the angular range between the two first nulls.

In order to solve the optimal design problem, one approximates it by a specific (convex) optimization problem that can be solved using standard optimization routines. To do so, one considers a large number N of elements distributed over L. The linear array is represented in Fig. 1(b). This step, in constraining the positions of the elements, does not allow for an arbitrary geometry and the pattern associated with the solution of the optimization problem is thus only an approximation

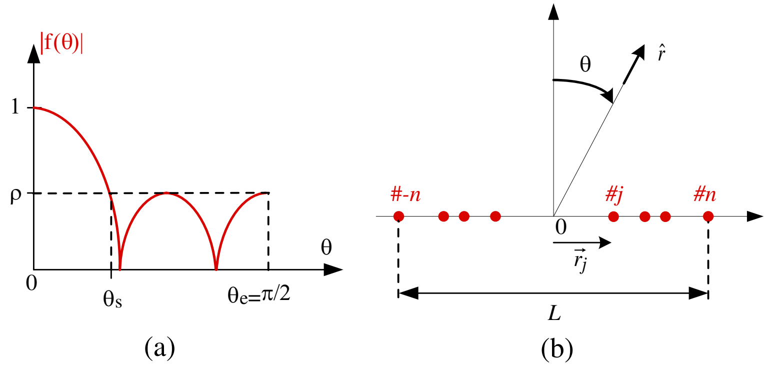

Figure 1. (a) Schematic view of the pattern synthesis problem. The magnitude of the far field pattern $|f(\theta)|$ is normalized to unity in the broadside direction ($|f(0)|=1$) whereas one minimizes its sidelobe level ρ for a fixed half beam width θ_s. (b) Geometry of the linear array of length L composed of N isotropic elements. The observation angle $\theta = 0$ corresponds to the broadside direction.

of the optimal pattern. It will nevertheless allow to guess the optimal geometry and it is then proven that this guessed geometry with its associated optimal weightings is indeed the sought-for optimal solution.

Due to the symmetry of the problem, it is easy to prove by contradiction that the optimal excitations can be chosen real and symmetric, i.e., that in the optimal solution both the geometry and the excitations of the array will be symmetric with respect to the middle of the linear array.

The steering vector $\mathbf{a}(\theta)$ of such a symmetric array can be condensed into a vector that is real and has dimension reduced to half:

$$\mathbf{a}(\theta)^T = \begin{cases} [1 \;\; 2\cos(2\pi r_2 \sin\theta) \ldots 2\cos(2\pi r_n \sin\theta)], & \text{if } N = 2n+1 \\ [2\cos(2\pi r_1 \sin\theta) \ldots 2\cos(2\pi r_n \sin\theta)], & \text{if } N = 2n \end{cases} \tag{1}$$

where $.^T$ denotes the transpose operator, θ is the observation angle with respect to broadside and $r_j = \|\vec{r_j}\|$ is the element position in wavelength with respect to the middle of the array.

Since a large number N of elements is considered, an element is always placed at the origin and $n+1$ weightings are thus to be determined. One then has $r_0 = 0$ and $r_n = \frac{L}{2}$.

The far field pattern f radiated in the direction θ is then:

$$f(\theta) = \mathbf{a}(\theta)^T \mathbf{w}, \tag{2}$$

where the j-th component w_j of $\mathbf{w}$, a column vector of dimension $n+1$, is the weighting of the element located at r_j.

As shown in [9, 10], the approximate synthesis problem can be translated into the following convex optimization problem:

$$\min_{\mathbf{w}} \; \rho, \text{ under } f(0) = 1, \; |f(\theta)| \leq \rho, \; \forall\, \theta \in [\theta_s, \tfrac{\pi}{2}] \text{ and } \mathbf{w} \geq 0, \tag{3}$$

where $\mathbf{w} \geq 0$ means $w_j \geq 0$, for all j and θ_s is the half beam width corresponding to the specific definition given earlier.

The directions θ in $[\theta_s, \pi/2]$ are then discretized by introducing m directions $\{\theta_i\}$, $i = 1, \ldots, m$. This amounts to introduce a second approximation into the formulation of the original problem.

While the steering vector (1) in the broadside direction is:

$$\mathbf{a}_0^T = \mathbf{a}(0)^T = [1 \;\; 2 \;\; \ldots \;\; 2]^T, \tag{4}$$

the other steering vectors are:

$$\mathbf{a}_i^T = \mathbf{a}(\theta_i)^T = [1 \; 2\cos(q_i r_1) \;\; \ldots \;\; 2\cos(q_i r_n)], \tag{5}$$

with $q_i = 2\pi \sin\theta_i$, $q_1 = q_s = 2\pi \sin\theta_s$ and $q_m = q_e = 2\pi$, where "e" stands for endfire.

One then builds the following matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & \dots & 2\cos\left(q_s r_j\right) & \dots & 2\cos\left(q_s \frac{L}{2}\right) \\ \vdots & & \vdots & & \vdots \\ 1 & \dots & 2\cos\left(q_i r_j\right) & \dots & 2\cos\left(q_i \frac{L}{2}\right) \\ \vdots & & \vdots & & \vdots \\ 1 & \dots & 2\cos\left(2\pi r_j\right) & \dots & \cos\left(2\pi \frac{L}{2}\right) \end{bmatrix}. \tag{6}$$

With this approximation, the synthesis problem (3) becomes then:

$$\min_{\mathbf{w}} \ \rho, \ \text{under } \mathbf{a}_0^T \mathbf{w} = 1, \ \|\mathbf{A}\mathbf{w}\|_\infty \leq \rho \text{ and } \mathbf{w} \geq 0, \tag{7}$$

where $\|\mathbf{A}\mathbf{w}\|_\infty \leq \rho$ is equivalent to $|\mathbf{a}_i^T \mathbf{w}| \leq \rho, \ \text{for } i = 1, \dots, m.$

2.2. Approximate Optimal Solution via a Linear Program

The solution of the approximate optimal design problem is found by solving (7), that can be transformed into a Linear Program (LP). Since this LP will be used in the proofs in Section 4, let us detail the LP in standard form equivalent to (7):

$$\min \mathbf{c}^T \mathbf{x}, \ \text{under } \mathbf{B}\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq 0. \tag{8}$$

To transform the inequality $\|\mathbf{A}\mathbf{w}\|_\infty \leq \rho$ of (7) into equalities as required in (8), one introduces two m-dimensional vectors $\mathbf{s}_1$ and $\mathbf{s}_2$. The inequality is then transformed into $2m$ equalities:

$$-\rho + \mathbf{a}_i^T \mathbf{w} + s_{1i} = 0 \text{ and } -\rho - \mathbf{a}_i^T \mathbf{w} + s_{2i} = 0, \ \text{for } i = 1, \dots, m \tag{9}$$

where s_{1i} and s_{2i}, the i-th component of $\mathbf{s}_1$ and $\mathbf{s}_2$, are the so-called slack variables required to be positive.

With $\mathbf{x}$ in (8), a column vector of dimension $1 + (1 + n) + 2m$:

$$\mathbf{x}^T = [\rho \ \mathbf{w} \ \mathbf{s}_1 \ \mathbf{s}_2] \tag{10}$$

that has indeed only positive components, the cost function $\min_{\mathbf{w}} \rho$ in (7) takes the appropriate form in (8) with $\mathbf{c}^T = [1 \ \ \mathbf{0}_{1,1+n+2m}]$.

By taking also into account $\mathbf{a}_0^T \mathbf{w} = 1$, the matrix $\mathbf{B}$ in (7) becomes:

$$\mathbf{B} = \left[\begin{array}{c|c|c|c} 0 & \mathbf{a}_0^T & \mathbf{0}_{1,m} & \mathbf{0}_{1,m} \\ \hline -\mathbf{1}_{m,1} & \mathbf{A} & \mathbf{I}_{m,m} & \mathbf{0}_{m,m} \\ \hline -\mathbf{1}_{m,1} & -\mathbf{A} & \mathbf{0}_{m,m} & \mathbf{I}_{m,m} \end{array}\right] \tag{11}$$

of dimension $(1 + 2m) \times (2 + n + 2m)$, where $\mathbf{1}_{m,n}$ and $\mathbf{0}_{m,n}$ are $m \times n$ matrices filled with respectively ones and zeros and $\mathbf{I}_{n,n}$ is the identity matrix of order n.

Before using these reformulations of the problem, a summary of the DC design that is another building block of the proofs to come is given.

3. OPTIMAL EQUALLY SPACED LINEAR ARRAY

For equally spaced linear arrays, applying the Dolph-Chebyshev method [1] allows to synthesize optimal patterns. However, for a given length L, there are several equally spaced arrays. It will be shown that among these arrays having the same length, the one built with the minimal number of elements, i.e., the array with the largest admissible inter-element distance d, is optimal.

Before establishing this result, a brief summary of Dolph design is presented.

3.1. Optimal Excitations for Equally Spaced Arrays

The pioneering work of Dolph [1] and the associated synthesis play a major role in the sequel. Therefore, the corresponding result and the notations that are used later are now presented.

The Dolph synthesis is a nice application of Chebyshev's classical theorem on the polynomials with minimal deviation from zero. It says that among all polynomials of degree N and leading coefficient equal to 1, the one with smallest maximal absolute value over the interval $[-1,\ 1]$ is $T_N(x)/2^{N-1}$, where T_N is the Chebyshev polynomial of degree N of the first kind:

$$T_N(y) = \begin{cases} \cos\left(N\cos^{-1}(y)\right), & \text{for } |y| \leq 1, \\ \cosh\left(N\cosh^{-1}(y)\right), & \text{for } |y| \geq 1. \end{cases} \tag{12}$$

In the present symmetric pattern synthesis context, an adequately chosen change of variables will transfer part or all of the $[-1, 1]$ interval, where the optimal equi-ripple property holds, to one of the sidelobe regions, the other one being taken care of by the evenness of the cosine function. More precisely for a linear array with N equally spaced elements, an inter-element distance d expressed in wavelength and a desired half beam width θ_s, the optimal pattern is [1]:

$$W(u) = \frac{T_{N-1}(x_0\cos(u))}{T_{N-1}(x_0)}, \tag{13}$$

with $u = \pi d\sin\theta$ and where it remains to characterize $x_0 > 1$.

With the change of variable $y = x_0\cos(u)$, as θ increases from broadside to endfire, y decreases. For $\theta = 0$, y is equal to x_0, which yields $W(0) = 1$ as expected. For $\theta = \theta_s$, y is chosen equal to 1 to enter the equi-ripple zone, which implies that x_0 satisfies:

$$\frac{1}{x_o} = \cos(\pi d\sin\theta_s) = \cos(u_s) \tag{14}$$

and this fixes the achievable level $\rho = W(u_s) = T_{N-1}(1)/T_{N-1}(x_o) = 1/T_{N-1}(x_o)$. As θ further increases, y further decreases, becomes negative but has to remain greater than -1 for $\theta = \theta_e = \pi/2$ to be still in the equi-ripple zone at endfire. This thus requires: $x_o \cos\left(\pi d \sin\frac{\pi}{2}\right) \geq -1$ or

$$d \leq d_{\max} = \frac{1}{\pi}\cos^{-1}\left(-\frac{1}{x_o}\right) = \frac{1}{1+\sin\theta_s}, \tag{15}$$

where the last equality follows from $\cos^{-1}(-\alpha) = \pi - \cos^{-1}(\alpha)$. For $d > d_{\max}$, a grating lobe appears at endfire. Note that, since in any reasonable design problem $\theta_s < \pi/2$, one always has $d_{\max} > \lambda/2$.

Using *de Moivre* formula, the pattern $W(u)$, which is a polynomial in $x_0 \cos(u)$ can be rewritten as a linear combination of $\cos(ku)$ where the real (and positive) coefficients are the excitations e_k to be applied to the elements. The real weighting to be applied to the elements located at $\pm(k-1)d$ with $k = 1, \ldots, n+1$ is then $e_k/2$.

3.2. Within Dolph Designs Parsimony Means Optimality

Given the length L and the wavelength λ, there are several equally spaced linear arrays that use this whole length. One usually takes d slightly larger than $\lambda/2$ but other values are allowed provided they satisfy (15).

Let $L = Nd_N = (N-1)d_{N-1}$ be two possible designs with respectively $N+1$ and N elements and $d_N < d_{N-1} \leq d_{\max}$. For simplicity, d_N is assumed to be greater than $\lambda/2$ but the result holds also otherwise.

With ρ_{N-1} the sidelobe level of the sparser N element array, one establishes that $\rho_{N-1} < \rho_N$, where ρ_{N-1} satisfies:

$$\rho_{N-1} = \frac{1}{T_{N-1}(x_{0,N-1})}. \tag{16}$$

One can equivalently establish that $T_{N-1}(x_{0,N-1}) > T_N(x_{0,N})$ or equivalently:

$$\begin{aligned}
\cosh\left((N-1)\cosh^{-1}(x_{0,N-1})\right) &> \cosh\left(N\cosh^{-1}(x_{0,N})\right)\\
(N-1)\cosh^{-1}\left(\frac{1}{\cos(\pi d_{N-1}\sin\theta_s)}\right) &> N\cosh^{-1}\left(\frac{1}{\cos(\pi d_N \sin\theta_s)}\right)\\
(N-1)\cosh^{-1}\left(\frac{1}{\cos\left(\frac{\pi L\sin\theta_s}{N-1}\right)}\right) &> N\cosh^{-1}\left(\frac{1}{\cos\left(\frac{\pi L\sin\theta_s}{N}\right)}\right)
\end{aligned} \tag{17}$$

$$\frac{N-1}{\pi L \sin\theta_s}\cosh^{-1}\left(\frac{1}{\cos\left(\frac{\pi L \sin\theta_s}{N-1}\right)}\right) > \frac{N}{\pi L \sin\theta_s}\cosh^{-1}\left(\frac{1}{\cos\left(\frac{\pi L \sin\theta_s}{N}\right)}\right)$$

or yet to prove that $h(y)$ increases as y increases, with:

$$y = \frac{\pi L \sin\theta_s}{N}, \qquad \text{and} \qquad h(y) = \frac{1}{y}\cosh^{-1}\left(\frac{1}{\cos y}\right) = \frac{1}{y}\ln\left(\frac{1+\sin y}{\cos y}\right).$$

Using (7), one can show that $y \in [0, \pi/2]$, let us evaluate the derivative h' of $h(y)$. After some manipulations one obtains:

$$h'(y) = -\frac{1}{y}\left(h(y) - \frac{1}{\cos y}\right).$$

To prove that $h' \geq 0$, let us show that $(1/\cos y) \geq h(y)$ for $y \in [0, \pi/2]$, or

$$\frac{y}{\cos y} \geq \ln\left(\frac{1+\sin y}{\cos y}\right)$$

where both sides are positive, start at zero for $y = 0$ and the derivative of the first member $(1/\cos y) + (y \sin y/\cos 2y)$ is greater than the derivative of the second member $(1/\cos y)$, hence the result.

This concludes the demonstration that, among Dolph's arrays, the one built with the minimal number of elements is optimal. Such an array is from now on called *sparsest* Dolph design and for such an array the inter-element spacing is always greater than $\lambda/2$.

4. OPTIMAL LINEAR ARRAY

The synthesis of optimal pattern is now addressed. Since the problem is convex the solution is generically unique. The optimal solution depends upon the array length L and it will appear that the positive real axis will be paved with two types of alternating intervals. For L in the even intervals the optimal solution is of one sort, for L in the odd intervals the optimal solution is of a second sort. Two distinct proofs are provided for each type of intervals.

4.1. Overview of the Optimal Solution

The approximate optimal design can be obtained by solving the LP (8) where the array geometry is fully free except for the discretization of the localization of the potential elements.

It happens that the (true) optimal array has always a finite and small number of almost equally spaced elements. When solving (8) numerically, it might be difficult to recognize this feature of the true optimum, since the optimal element locations will strictly speaking never be among the proposed potential positions, the r_j's in (6). The solution of (8) will have in general almost equally spaced clusters of contiguous nonzero components in the w-part of the optimal $\mathbf{x}$.

As a matter of fact the solution of (8) will rigorously correspond to the optimal design only if one knows the optimal design beforehand, since one can then add intentionally the optimal element locations to the columns of $\mathbf{A}$ and the active directions (the q_i's where the optimal pattern reaches $\pm\rho$) to the rows of $\mathbf{A}$. This is the strategy that is used below in the proofs since the optimal solution, has been guessed and is thus known (at least when L is within one of the two types of intervals).

Let us present the guessed design that will be proved to be optimal. There always exists a positive integer k such that L falls within the interval $[(k-1)d_{\max},\ kd_{\max}]$ and this fixes the optimal number $N = k+1$ of elements. This interval is further divided into two sub-intervals. If L belongs to the upper sub-interval, then the optimal design is the *sparsest* Dolph design defined in Section 3.2. Otherwise, in the lower sub-interval, the elements are not equally spaced and their locations are difficult to characterize precisely.

The limit between the 2 sub-intervals is $kd_m(k)$, with $d_m(k)$ defined implicitly by:

$$\pi d_m(k) = \cos^{-1}\left(\cos\left(\frac{k-1}{k}\pi\right)\cos(\pi d_m(k)\sin\theta_s)\right). \qquad (18)$$

The physical interpretation of $d_m(k)$ can be seen in Fig. 2(b).

Loosely speaking, one can say that for $d = d_{\max}$ the optimality of the Chebyshev polynomial is fully exploited (see the definition of

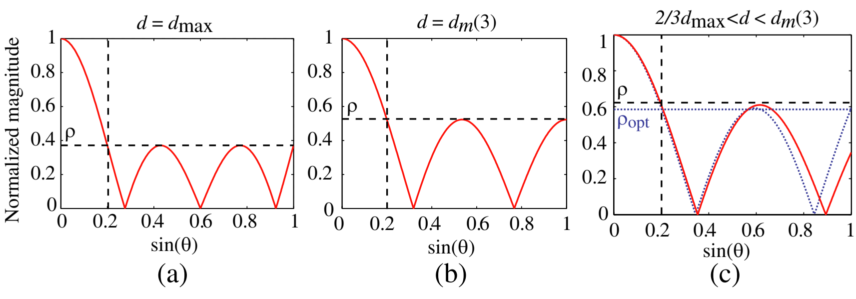

Figure 2. Far field patterns of a four element linear array of diminishing length $3d$ synthesized by the Dolph Chebyshev method, where $\sin\theta_s = 0.2$. As d decreases, ρ increases. In (c), the optimal pattern which, in this case, is different from the DC pattern, is added in dotted line.

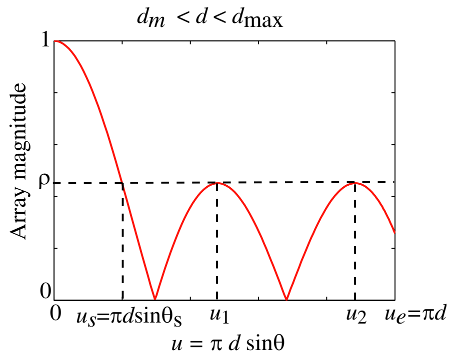

Figure 3. Far field pattern of a four element linear array synthesized by the Dolph Chebyshev method with some useful notations.

$d_{\max}$ (15) and Fig. 2(a)) where the pattern at endfire is strictly equal to ρ. For d slightly smaller but greater or equal to $d_m(k)$, there is some apparent unused freedom (see in Fig. 3), since the value of the pattern at endfire is smaller than ρ. This apparent freedom is however unexploitable for symmetry or periodicity reasons. For $d = d_m(k)$ in Fig. 2(b), exactly one sidelobe has disappeared and the pattern at endfire becomes precisely equal to ρ. For $d < d_m(k)$, see Fig. 2(c), there is again some freedom at endfire and it is now exploitable, meaning that one can do better than DC (dotted pattern in Fig. 2(c)) by moving the elements and changing their weightings.

4.2. Proof

The exact (non equally spaced) locations of the elements in the first sub-interval are difficult to characterize analytically and it will only be proven that the *sparsest* Dolph design is non optimal in this case. In the second sub-interval, it will be established that the sparsest Dolph design is optimal.

Since the proof is quite intricate, only the case $k = 3$ is considered. This case highlights all the main points and is simple enough to permit a somehow detailed presentation. Some familiarity with the basics of the linear programming theory is assumed [11]. For $k = 3$, the two sub-intervals are then $[2d_{\max}, 3d_m(3)[$ and $[3d_m(3), 3d_{\max}]$ and the associated inter-element distances d for equally spaced arrays are $[\frac{2}{3}d_{\max}, d_m(3)[$ and $[d_m(3), d_{\max}]$.

- In a first step, it will be verified that for an equally spaced linear array with $N = 4$ elements, the solution of the adequately tailored LP is identical to the Dolph design.

- In a second step, one then proves that if $\frac{2}{3}d_{\max} < d < d_m(3)$, this solution is no longer optimal if an additional element is permitted (at a specific location) which establishes the announced result.
- In a third step, one establishes that for $d_m(3) \leq d \leq d_{\max}$, this Dolph solution remains optimal even if an additional elements at any arbitrary location is permitted. In a LP context, this establishes the result.

4.2.1. First Step

From (13) and the fact that $T_3(y) = 4y^3 - 3y$, it follows that the optimal pattern for an equally spaced linear array with $N = 4$ equally spaced elements is [1]:

$$\begin{aligned} W(u) &= \frac{1}{T_3(x_0)}\left(4{x_0}^3\cos^3 u - 3x_0\cos u\right) \\ &= \frac{3{x_0}^2-3}{4{x_0}^2-3}\cos u + \frac{{x_0}^2}{4{x_0}^2-3}\cos 3u \end{aligned} \tag{19}$$

and this result remains valid as long as $d \leq d_{\max}$.

The far field patterns of this array are plotted in Fig. 2 for different values of d. For $\sin\theta_s = 0.2$, one gets $d_{\max} = 0.833$ (15) and $d_m(3) \simeq 0.652$ (18).

- For $d = d_{\max}$ (Fig. 2(a)), the pattern has two full sidelobes and at θ_e (endfire), the pattern attains exactly ρ.
- For $d = d_m(3)$ (Fig. 2(b)), one full sidelobe has disappeared and at θ_e the pattern attains exactly ρ.
- For $\frac{2}{3}d_{\max} < d < d_m(3)$ (Fig. 2(c)), there is one sidelobe and at endfire the pattern is smaller than ρ.

Some notations used in the following developments are indicated in Fig. 3.

The weightings in (19) together with the optimal ρ will now be recovered as the optimum of a quite simple LP that corresponds to (7) or (8) when only two potential elements are proposed at locations $r_1 = d/2$ and $r_2 = 3d/2$. The matrix $\mathbf{A}$ has thus two columns and it remains to define its active rows, those for which $W(u)$ attains its maximal value $\pm\rho$ (16,14). By definition of the beam width, there is $u_s = \pi d \sin\theta_s = \arccos(1/x_0)$ and also the values of u for which the derivative $W'(u)$ is zero. One can check (see Fig. 2(c)) that for $d < d_m(3)$ there is only one value $u_1 = \arccos(1/2x_0)$ and one can now define the tailored LP that allows to recover the Dolph-Chebyshev pattern and actually prove that this pattern is optimal for this geometry.

The tailored LP is:

$$\min_{\mathbf{z}}\ [1\ 0\ 0]\mathbf{z},\ \text{under}\ \mathbf{Qz} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \text{and}\ \mathbf{z} = [\rho\ \ w_1\ \ w_2]^T \geq 0, \qquad (20)$$

with

$$\mathbf{Q} = \begin{bmatrix} 0 & 2 & 2 \\ -1 & \frac{2}{x_0} & \frac{8}{x_0{}^3} - \frac{6}{x_0} \\ -1 & -\frac{1}{x_0} & -\frac{1}{x_0{}^3} + \frac{3}{x_0} \end{bmatrix}. \qquad (21)$$

It has only one admissible point that is thus the optimum:

$$\mathbf{z} = \mathbf{Q}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \left[\frac{1}{4x_0{}^3 - 3x_0}\ \ \frac{1}{2}\frac{3x_0{}^2 - 3}{4x_0{}^2 - 3}\ \ \frac{1}{2}\frac{x_0{}^2}{4x_0{}^2 - 3}\right]^T \qquad (22)$$

and one recovers exactly ρ (16) and the weightings w_1 and w_2 in (19). In this LP approach to the synthesis, one must now further check that the sidelobes remain smaller than ρ over the whole sidelobe region.

4.2.2. Second Step

To establish the non optimality of this solution, when other elements are allowed one creates the column vector $\mathbf{c}(.)$ associated with such an additional element to be added to $\mathbf{Q}$ in (21).
For this purpose, let us introduce a parameter $\delta \in [0, 3]$, then for an additional element at position $r(\delta) = \delta d/2 > 0$ say, the first component in c belongs to a_0 and is thus equal to two, the second component is $2 \cos q_s r(\delta)$ (6) and thus substituting q_s and $r(\delta)$ and keeping δ and u_s, the second component becomes $2 \cos u_s \delta$ and similarly for the third component. This yields:

$$\mathbf{c}(\delta) = \begin{bmatrix} 2 \\ 2\cos \delta u_s \\ -2\cos \delta u_1 \end{bmatrix} \qquad (23)$$

and one can check that the column 2 and 3 in Q are simply $c(1)$ and $c(3)$ since $r_1 = d/2$ and $r_2 = 3d/2$.

It will now be proven that the point $\mathbf{z}$ in (22) is non optimal if an additional element is allowed at any location $\delta \in [0, 1]$. To simplify the evaluation, let us introduce a element at location $\delta = 0^+$ whose associated column $\mathbf{c}(0^+)$ is $[2\ \ 2\ -2]^T$. For this element, the Lagrange multiplier associated with its non-negative weighting constraint is:

$$[-1\ \ 0\ \ 0]\mathbf{Q}^{-1}\ \mathbf{c}(\delta) = \frac{(x_0 - 1)(6 - 8x_0 - 8x_0{}^2)}{12x_0{}^3 - 9x_0}. \qquad (24)$$

This scalar can be shown to be negative (since $x_0 > 1$) and this establishes the non optimality of $\mathbf{z}$ in (22).

4.2.3. Third Step

Let us now consider an inter-element distance d satisfying $d_m(3) \leq d \leq d_{\max}$. There are some modifications to be brought to the reduced LP introduced in first step since there is now an additional constraint that becomes active (see Fig. 2(b)), namely $u_2 = \arccos(-1/2x_0)$ (see Fig. 3), where $W'(u) = 0$. This adds a fourth component to $\mathbf{c}$ (23) and a fourth row to $\mathbf{Q}$:

$$\mathbf{Q} = \begin{bmatrix} 0 & 2 & 2 \\ -1 & \frac{2}{x_0} & \frac{8}{{x_0}^3} - \frac{6}{x_0} \\ -1 & -\frac{1}{x_0} & -\frac{1}{{x_0}^3} + \frac{3}{x_0} \\ -1 & -\frac{1}{x_0} & -\frac{1}{{x_0}^3} + \frac{3}{x_0} \end{bmatrix}, \tag{25}$$

and the reduced LP becomes:

$$\min_{\mathbf{z}} \ [1\ 0\ 0]\mathbf{z}, \ \text{under } \mathbf{Qz} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \text{and } \mathbf{z} \geq 0. \tag{26}$$

Since the fourth constraint is identical to the third, there is still an unique admissible point, the optimal solution is still (22) and identical to the Dolph design. From a LP theory point of view, this solution is however degenerate and the optimality conditions are therefore more difficult to specify. One has to add a column to $\mathbf{Q}$ to make it square again, i.e., to come back to a basic optimal solution for which the same condition as above applies. This condition which was necessary and sufficient before, is then only sufficient, but this is coherent with our purpose. To establish that (22) is the optimum even if other elements are available, it suffices to show that $t(\delta) = [-1 \ \ 0 \ \ 0 \ \ 0]\mathbf{Q}^{-1}\mathbf{c}(\delta) \geq 0$ for all $\delta \in [0, 3]$ with:

$$\mathbf{c}(\delta) = \begin{bmatrix} 2 \\ 2\cos\delta u_s \\ -2\cos\delta u_1 \\ 2\cos\delta u_2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ \cos\left(\delta\cos^{-1}\left(\frac{1}{x_0}\right)\right) \\ -\cos\left(\delta\cos^{-1}\left(\frac{1}{2x_0}\right)\right) \\ \cos\left(\delta\cos^{-1}\left(-\frac{1}{2x_0}\right)\right) \end{bmatrix}. \tag{27}$$

The column to be added is difficult to identify. If for a given column, $t(\delta)$ does not remain positive, no conclusion can be drawn but in the opposite case, step three is established.

In the present case, the column that works is the column obtained when taking the derivative of $\mathbf{c}(\delta)$ with respect to δ evaluated at $\delta = 1$

$$\frac{\partial}{\partial \delta}\mathbf{c}(\delta)|_{\delta=1} = \begin{bmatrix} 0 \\ -2u_s \sin(u_s) \\ 2u_1 \sin(u_1) \\ -2u_2 \sin(u_2) \end{bmatrix}, \tag{28}$$

this choice corresponds to an additional element (column) that is infinitely close to the element 1 located at $\delta = 1$. When this column is added to $\mathbf{Q}$ in (25) to make it square of order 4, the function $t(\delta)$ is zero by construction for $\delta = 1$ and $\delta = 3$ and remains greater than zero for all $\delta \in [0, 3]$. To establish this result analytically is cumbersome and the details are not presented. For a specific value of θ_s, one can easily evaluate $d_{\max}$, x_0, u_s, u_1 and u_2 and check the positivity numerically.

4.3. Illustrations

For a half beam width θ_s of 20° and various L, radiation patterns are computed applying the *sparsest* Dolph design (SD) and the (quasi) optimal array obtained from (7) with large m and n. For both cases, the algebraic value of the maximum sidelobe levels ρ are plotted in Figs. 4 and 5 as a function of the array length L.

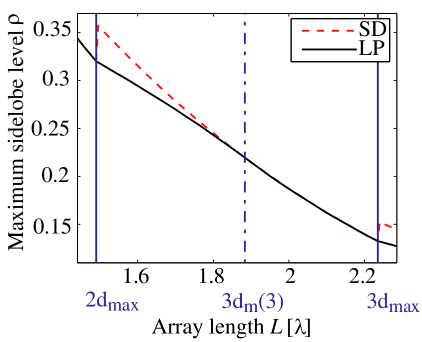

Figure 4. Maximum sidelobe level as a function of the array length for a half beam width θ_s of 20°. The results applying the *sparsest* Dolph design (SD) and the optimal array obtained by LP are compared for L ranging between $2d_{\max}$ and $3d_{\max}$. They are superimposed for $L \in [3d_m(3), 3d_{\max}]$.

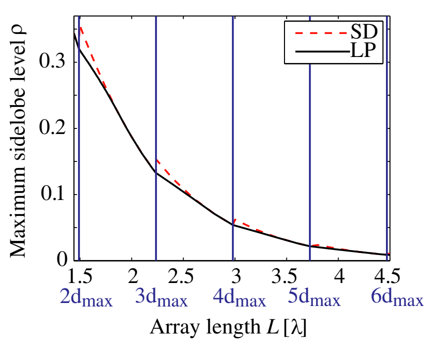

Figure 5. Maximum sidelobe level as a function of the array length for a half beam width θ_s of 20°. The results applying the *sparsest* Dolph design (SD) and the optimal array obtained by LP are compared for L ranging between $2d_{\max}$ and $6d_{\max}$.

It turns out looking at Fig. 5 that the results obtained by both approaches are close. A zoom on the performances for L ranging between $2d_{\max}$ and $3d_{\max}$, i.e., for a four element array, enables one however to distinguish the sub-intervals of L where the SD design is optimal (between $3d_m(3)$ and $3d_{\max}$) or not (between $2d_{\max}$ and $3d_m(3)$). The discrepancies between ρ found by SD and LP are about 1 dB. For broader beam widths, more important discrepancies are obtained. When $\theta_s = 30°$, the maximum sidelobe levels synthesized by SD design are indeed up to 4 dB higher than the one obtained using LP strategy when L is slightly higher $3d_{\max}$.

5. SUMMARY AND CONCLUSION

The synthesis of the linear array of given length that radiates the optimal narrow beam low sidelobe pattern has been addressed. It is proven that the optimal array is very close to a Dolph-Chebyshev array. More precisely, the optimal array has always a small number of almost equally spaced elements of maximal inter-element spacing.

To summarize the proposed procedure, the steps to synthesize the optimal linear array of length L are, for a specified half beam width θ_s, the following:

(i) From θ_s, one deduces $d_{\max}$ (15).

(ii) From L, one determines the optimal element number $N = k + 1$ that is such as $L \in [(k-1)d_{\max}, kd_{\max}]$.

(iii) With k and θ_s, one computes $d_m(k)$ (18).

(iv) The optimal array is the *sparsest* Dolph design for $L \in [kd_m(k), kd_{\max}]$ and an unequally spaced array whose geometry and weightings can be numerically found by Linear Program for $L \in [(k-1)d_{\max}, kd_m(k)]$.

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