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TELEGRAPHIST'S EQUATIONS FOR RECTANGULAR WAVEGUIDES AND ANALYSIS IN NONORTHOGONAL COORDINATES

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Abstract—In our previous works, we have presented one differential method for the efficient calculation of the modal scattering matrix of junctions in rectangular waveguides. The formalism proposed relies on the Maxwell's equations under their covariant form written in a nonorthogonal coordinate system fitted to the structure under study. On the basis of a change of variables, we show in this paper that the curvilinear method and the generalized telegraphist's method lead to the same system of coupled differential equations.

1. INTRODUCTION

Rectangular waveguide junctions are widely used in the design of microwave components, such as curved waveguides, tapers, multiplexers, power dividers and filters for modern radar and satellite communications systems. A variety of analytical and numerical approaches have been developed for analyzing the discontinuities in rectangular waveguides [1–22].

In our previous works [1–4], we have presented a differential method for the efficient calculation of the scattering matrix of waveguide with varying cross-section. The microwave structures under consideration are H-plane and E-plane junctions in rectangular waveguides. An H-plane junction excited by the fundamental mode generates higher Transverse Electric modes in the input/output rectangular waveguides. In an E-plane discontinuity, the Longitudinal Section Electric modes are generated [9]. The generalized modal scattering matrix relates the amplitudes of outgoing modes to those of incoming modes.

The method proposed in references [1–3] for computing the generalized S-matrix lies on Maxwell's equations written in a non-orthogonal coordinates system and leads to a differential system having non-constant coefficients. This system represents an initial value problem and we show that the implementation requires several numerical integrations. During these integrations, unwanted solutions appear which can be important to consider with the true ones. It is essential to overcome this problem. For this, the junction is represented by several elementary transitions in series. The combination of elementary S-matrices by an iteration process [4] gives the overall multimodal scattering matrix of the junction under consideration. We show that this process ensures the stability of results. We have used this method to define the generalized scattering matrices of step discontinuities [1, 2], adapted sectoral horns [1, 2], broad-band resonator iris filters [3], power dividers and waveguide bends [4] and we have assessed simulation by comparison with published numerical and experimental results [1–4].

In the present paper, we compare the curvilinear coordinate formalism with the generalized Telegraphist's equation method in the case of an H-plane discontinuity but the analysis we will present can be applied to E-plane discontinuities. Section 2 describes essential steps of the curvilinear coordinate method. In Section 3, we present the generalized Telegraphist's equation method. The curvilinear coordinate method and the Telegraphist's method represent two different mathematical approaches. In this paper, we show on the basis of a change of variables that these methods lead to the same system of coupled differential equations and that from a theoretical point of view, they are equivalent.

2. CURVILINEAR COORDINATE METHOD

2.1. Geometry of H-plane Junctions

The H-plane junction under study is depicted in Fig. 1. This junction is connected to rectangular waveguides and can be a taper or a filter. l is the length of the transition. The cross-section of input and output waveguides are $a^{(i)} \times b$ and $a^{(o)} \times b$, where (i) and (o) denote quantities relative to input and output waveguides. Functions $x = f_1(z)$ and

$x = a^{(i)} + f_2(z)$ represent the perfectly conducting walls.

It is assumed that the dominant mode TE_{10} is incident upon the junction (1). The time factor in $\exp(j\omega t)$ is omitted. ω denotes the angular frequency, $k = \frac{2\pi}{\lambda}$ the wavenumber and λ the wavelength.

$$
\begin{aligned}
E_y^{(i)}(x,z) &= E_{0y}^{(i)} \sin\left(\frac{\pi x}{a^{(i)}}\right) \exp\left(-j\gamma_1 z\right) \\
H_x^{(i)}(x,z) &= H_{0x}^{(i)} \sin\left(\frac{\pi x}{a^{(i)}}\right) \exp\left(-j\gamma_1 z\right) \\
H_z^{(i)}(x,z) &= H_{0z}^{(i)} \cos\left(\frac{\pi x}{a^{(i)}}\right) \exp\left(-j\gamma_1 z\right) \\
E_x^{(i)} &= E_z^{(i)} = H_y^{(i)} = 0
\end{aligned}
\tag{1}
$$

where

$$\gamma_1 = \sqrt{k^2 - \frac{\pi^2}{{a^{(i)}}^2}} \tag{2}$$

For the junction under study, the y dimension undergoes no variation. So, the field dependence according to this variable is perfectly defined when we know the incident wave. In the case of H-plane polarization, the incident field does not depend on the y variable. Hence,

$$\frac{\partial}{\partial y} = 0 \tag{3}$$

2.2. Covariant Formalism of Maxwell's Equations

If there is no current density and no charge density, for a homogeneous and isotropic medium with permittivity ε_0 and permeability μ_0, the Maxwell-Faraday's equation and the Maxwell-Ampere's equation associated with the constitutive relations are expressed in a curvilinear coordinate system $\left(x^1, x^2, x^3\right)$ [22, 23] as follows:

$$
\left\{
\begin{aligned}
&\frac{\partial E_k}{\partial x^j} - \frac{\partial E_j}{\partial x^k} = -jkZ\sqrt{g}\sum_{j=1}^{3} g^{ij} H_j \\
&\frac{\partial ZH_k}{\partial x^j} - \frac{\partial ZH_j}{\partial x^k} = jk\sqrt{g}\sum_{j=1}^{3} g^{ij} E_j
\end{aligned}
\right. ; \tag{4}
$$

$$(i,j,k) = (1,2,3);\ (2,3,1);\ (3,1,2)$$

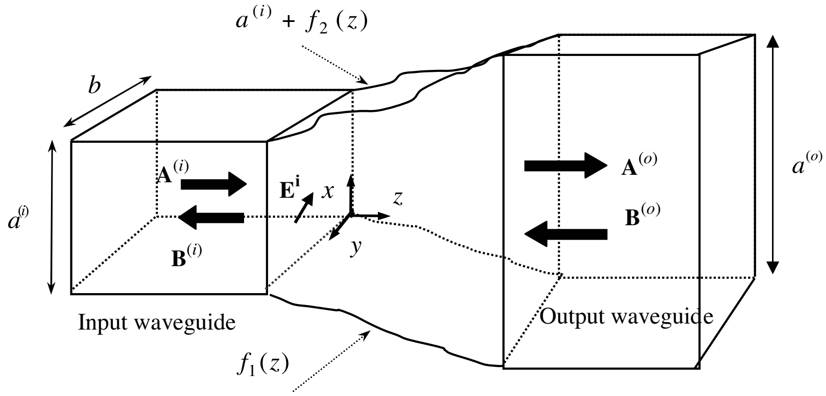

Figure 1. H-plane junction in rectangular waveguide. The incident electric field is parallel to the Oy axis. The incident magnetic field is in the plane xOz. Waveguide size with respect to the Oy axis is constant. The cross-section in input is: $S^{(i)} = a^{(i)} \times b$, in output: $S^{(o)} = a^{(o)} \times b$. l is the length of the junction.

where E_i and H_i are the covariant components of $\vec{E}$ and $\vec{H}$. g is the determinant of matrix tensor and g^{ij}, the contravariant metrical coefficients [23]. Z is the intrinsic impedance of free space:

$$Z = \sqrt{\mu_0/\varepsilon_0} \approx 120\pi. \tag{5}$$

2.3. Coordinate System – Matrix Tensor

The coordinate system (u, v, w) fitted to the junction geometry is obtained from the Cartesian system (x, y, z) by the following transformation:

$$\begin{aligned} u &= a^{(i)} \frac{x - f_1(z)}{a^{(i)} + f_2(z) - f_1(z)} \\ v &= y \qquad w = z \end{aligned} \tag{6}$$

In this new coordinate system, the perfectly conducting walls coincide with the following coordinate surfaces.

$$\begin{aligned} x = f_1(z) = f_1(w) &\quad \Leftrightarrow \quad u = 0 \\ x = a^{(i)} + f_2(z) = a^{(i)} + f_2(w) &\quad \Leftrightarrow \quad u = a^{(i)} \end{aligned} \tag{7}$$

The change from Cartesian components (V_x, V_y, V_z) of vector $\vec{V}$ to covariant components (V_u, V_v, V_w) is given by

$$\begin{pmatrix} V_u \\ V_v \\ V_w \end{pmatrix} = \mathbf{A} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{8}$$

where $\mathbf{A}$ denotes the Jacobian transformation matrix

$$\mathbf{A} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{pmatrix} = \begin{pmatrix} \dfrac{f_{21}(w)}{a^{(i)}} & 0 & 0 \\ 0 & 1 & 0 \\ u\dfrac{f'_{21}(w)}{a^{(i)}} + f'_1(w) & 0 & 1 \end{pmatrix} \tag{9}$$

$$\begin{aligned} f_{21}(w) &= f_{21}(z) = a^{(i)} + f_2(z) - f_1(z), \\ f'_{21}(w) &= \frac{df_{21}(z)}{dz} = f'_2(z) - f'_1(z) \end{aligned} \tag{10}$$

The change from covariant components (V_u, V_v, V_w) to contravariant components (V^u, V^v, V^w) is obtained by the matrix tensor $\mathbf{G}$ [2, 23] with:

$$\begin{pmatrix} V_u \\ V_v \\ V_w \end{pmatrix} = \mathbf{G} \begin{pmatrix} V^u \\ V^v \\ V^w \end{pmatrix} \tag{11}$$

The matrix tensor $\mathbf{G}$ is defined as follows:

$$\mathbf{G} = \mathbf{A}\mathbf{G_C}\mathbf{A^t} = \begin{pmatrix} g_{uu} & g_{uv} & g_{uw} \\ g_{vu} & g_{vv} & g_{vw} \\ g_{wu} & g_{wv} & g_{ww} \end{pmatrix} \tag{12}$$

where $\mathbf{G_C}$ is the matrix tensor associated with the Cartesian system (identity matrix, [23]). $\mathbf{A^t}$ denotes the adjoint matrix of $\mathbf{A}$. Considering the expressions of $\mathbf{A}$ and $\mathbf{A^t}$, we can write:

$$\mathbf{G} = \begin{pmatrix} \left(\dfrac{f_{21}(w)}{a^{(i)}}\right)^2 & 0 & \dfrac{f_{21}(w)}{a^{(i)}}\left(\dfrac{uf'_{21}(w)}{a^{(i)}} + f'_1(w)\right) \\ 0 & 1 & 0 \\ \dfrac{f_{21}(w)}{a^{(i)}}\left(\dfrac{uf'_{21}(w)}{a^{(i)}} + f'_1(w)\right) & 0 & \left(\dfrac{uf'_{21}(w)}{a^{(i)}} + f'_1(w)\right)^2 + 1 \end{pmatrix} \tag{13}$$

and

$$\mathbf{G}^{-1} = \begin{pmatrix} g^{uu} & g^{uv} & g^{uw} \\ g^{vu} & g^{vv} & g^{vw} \\ g^{wu} & g^{wv} & g^{ww} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\left(uf'_{21}(w)+a^{(i)}f'_1(w)\right)^2+a^{(i)^2}}{f_{21}(w)^2} & 0 & -\dfrac{uf'_{21}(w)+a^{(i)}f'_1(w)}{f_{21}(w)} \\ 0 & 1 & 0 \\ -\dfrac{uf'_{21}(w)+a^{(i)}f'_1(w)}{f_{21}(w)} & 0 & 1 \end{pmatrix} \quad (14)$$

The determinant g is a strictly positive w-function.

$$g = \det(\mathbf{G}) = \left(\frac{f_{21}(w)}{a^{(i)}}\right)^2 \quad (15)$$

2.4. Continuity of Fields

From Equations (8) and (9), we can make several observations:

- Covariant component V_v is equal to Cartesian component V_y.
- Component V_u is proportional to Cartesian component V_x.
- Component V_w is tangential to perfectly conducting walls $u = 0$ and $u = a^{(i)}$.

Moreover, in accordance with the Jacobian transformation matrix, continuity relations in planes $z = w = 0$ and $z = w = l$ between covariant components and Cartesian ones are given as follows:

At $z = w = 0$ with $u = x$,

$$\begin{aligned} V_x^{(i)}(x, z) &= V_u(u, w) \\ V_y^{(i)}(x, z) &= V_v(u, w) \end{aligned} \quad (16)$$

At $z = w = l$ with $u = a^{(i)}\left(x - f_1(l)\right)/a^{(o)}$,

$$\begin{aligned} V_x^{(o)}(x, z) &= \frac{a^{(i)}}{a^{(o)}} V_u(u, w) \\ V_y^{(o)}(x, z) &= V_v(u, w) \end{aligned} \quad (17)$$

2.5. Fields Components inside the Junction

As a result of (3), we show that the covariant components E_v and H_v are non-coupled (18).

$$\begin{aligned} &\frac{\partial}{\partial u}\left(\sqrt{g}\left(g^{uu}\frac{\partial \Psi}{\partial u}+g^{uw}\frac{\partial \Psi}{\partial w}\right)\right) \\ +\ &\frac{\partial}{\partial w}\left(\sqrt{g}\left(g^{wu}\frac{\partial \Psi}{\partial u}+g^{ww}\frac{\partial \Psi}{\partial w}\right)\right)+\sqrt{g}k^2\Psi=0 \end{aligned} \tag{18}$$

where $\Psi = E_v$ or $\Psi = H_v$. Furthermore, we show that components E_u, H_u, E_w and H_w can be expressed in terms of components E_v and H_v only (19). From the knowledge of E_v and H_v, the four other components can be deduced.

$$\begin{aligned} ZH_u &= \frac{1}{k^2}\left(-jk\sqrt{g}g^{ww}\frac{\partial E_v}{\partial w}-jk\sqrt{g}g^{wu}\frac{\partial E_v}{\partial u}\right) \\ ZH_w &= \frac{1}{k^2}\left(jk\sqrt{g}g^{wu}\frac{\partial E_v}{\partial w}+jk\sqrt{g}g^{uu}\frac{\partial E_v}{\partial u}\right) \\ E_u &= \frac{1}{k^2}\left(jk\sqrt{g}g^{ww}\frac{\partial ZH_v}{\partial w}+jk\sqrt{g}g^{wu}\frac{\partial ZH_v}{\partial u}\right) \\ E_w &= \frac{1}{k^2}\left(-jk\sqrt{g}g^{wu}\frac{\partial ZH_v}{\partial w}-jk\sqrt{g}g^{uu}\frac{\partial ZH_v}{\partial u}\right) \end{aligned} \tag{19}$$

So, if the incident field coming from the rectangular guide possesses only a single component in the y direction, the second component will not be generated. The dominant mode TE_{10} does not have the y magnetic component. From (8), (18) and (19), we can then deduce:

$$H_v = E_u = E_w = 0 \tag{20}$$

and from (18) et (19), we show that the transverse components E_v and H_u verify differential system (21).

$$\left\{\begin{aligned} \frac{\partial E_v}{\partial w} &= D(u,w)\frac{\partial E_v}{\partial u}+jkC(w)ZH_u \\ \frac{\partial H_u}{\partial w} &= \frac{\partial H_w}{\partial u}+\frac{jk}{ZC(w)}E_v \\ H_w &= \frac{j}{kZ}C(w)\frac{\partial E_v}{\partial u}+D(u,w)H_u \end{aligned}\right. \tag{21}$$

where

$$D(u,w) = -\frac{g^{wu}}{g^{ww}} = \frac{uf'_{21}(w) + a^{(i)} f'_1(w)}{f_{21}(w)} \tag{22}$$

$$C(w) = \frac{a^{(i)}}{f_{21}(w)} \tag{23}$$

2.6. Fields Components inside the Rectangular Waveguides

An H-plane junction that is excited by the mode TE_{10} only generates three covariant components E_v, H_u and H_w. A similar result is obtained if the incident mode is of the type TE_{n0}. Therefore, the reflected and transmitted fields in rectangular waveguides are given by linear combinations of independent modes TE_{n0}.

$$\begin{aligned} E_{ny}^{(g)}(x,z) &= \left(W_n^{(g+)} \exp\left(-j\gamma_n^{(g)} z\right) + W_n^{(g-)} \exp\left(+j\gamma_n^{(g)} z\right)\right) \sin\left(\frac{n\pi x}{a^{(g)}}\right) \\ H_{nx}^{(g)}(x,z) &= \left(Y_n^{(g+)} \exp\left(-j\gamma_n^{(g)} z\right) + Y_n^{(g-)} \exp\left(+j\gamma_n^{(g)} z\right)\right) \sin\left(\frac{n\pi x}{a^{(g)}}\right) \end{aligned} \tag{24}$$

$$\begin{aligned} H_{nz}^{(g)}(x,z) &= j\frac{n\pi}{kZa^{(g)}} \left(W_n^{(g+)} \exp\left(-j\gamma_n^{(g)} z\right)\right. \\ &\quad \left. + W_n^{(g-)} \exp\left(+j\gamma_n^{(g)} z\right)\right) \cos\left(\frac{n\pi x}{a^{(g)}}\right) \end{aligned} \tag{25}$$

$$E_{nx}^{(g)} = E_{nz}^{(g)} = H_{ny}^{(g)} = 0 \tag{26}$$

Superscripts $(+)$ and $(-)$ denotes a wave moving in directions $+z$ and $-z$, respectively. The propagation constant $\gamma_n^{(g)}$ defines the nature of the mode: a propagating mode if $\gamma_n^{(g)}$ is real, an evanescent mode if $\gamma_n^{(g)}$ is imaginary.

$$\gamma_n^{(g)} = \left(k^2 - \left(\frac{n\pi}{a^{(g)}}\right)^2\right)^{1/2} \quad \text{where} \quad \mathrm{Imag}\left(\gamma_n^{(g)}\right) \leq 0 \tag{27}$$

The amplitudes $W_n^{(g\pm)}$ and $Y_n^{(g\pm)}$ are linked by the impedance $Z_n^{(g)}$ of the TE_{n0} mode:

$$Y_n^{(g\pm)} = \mp\frac{W_n^{(g\pm)}}{Z_n^{(g)}} \tag{28}$$

$$Z_n^{(g)} = \frac{kZ}{\gamma_n^{(g)}} \tag{29}$$

The normalized complex amplitudes $(A_n^{(g)}; B_n^{(g)})$ [1] are defined as follows:

$$A_n^{(g)} = \frac{W_n^{(g+)}}{\sqrt{Z_n^{(g)}}} \exp\left(-j\gamma_n^{(g)} z\right)$$
$$B_n^{(g)} = \frac{W_n^{(g-)}}{\sqrt{Z_n^{(g)}}} \exp\left(+j\gamma_n^{(g)} z\right) \tag{30}$$

Amplitude $A_n^{(g)}$ is associated with a wave moving in direction $+z$, $B_n^{(g)}$ in direction $-z$, respectively (Fig. 1). The dimension of the normalized complex amplitudes is a square root of a power. The multimodal scattering matrix relates the normalized complex amplitudes $\left(A_n^{(i)}; B_n^{(o)}\right)$ of incoming modes to the amplitudes $\left(B_n^{(i)}; A_n^{(o)}\right)$ of outgoing modes.

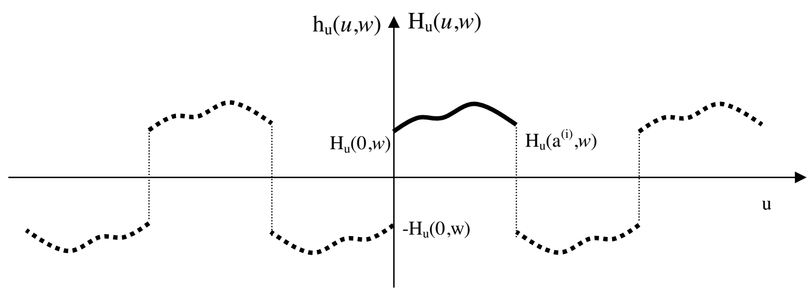

Figure 2. Periodic function $h_u(u,w)$ and local function $H_u(u,w)$. $h_u(u,w)$ is obtained by an even analytical extension of the local function $H_u(u,w)$ on $[-a^{(i)}, 0]$. Function $h_u(u,w)$ is periodic with period $2a^{(i)}$.

2.7. Boundary Conditions on Perfectly Conducting Walls and Expansion in Fourier's Series

The electric component $E_v(u,w)$ is parallel to horizontal perfectly conducting walls. So, $E_v(u,w)$ checks a Dirichlet's condition in $u = 0$ and $u = a^{(i)}$. So, $E_v(u,w)$ is expanded in a sine series as follows:

$$e_v(u,w) = \sum_{n=1}^{+\infty} V_n(w)\ \sin(\alpha_n u) \tag{31}$$

where

$$\alpha_n = \frac{n\pi}{a^{(i)}} \tag{32}$$

The restriction of Fourier series $e_v(u,w)$ on $[0; a^{(i)}]$ expresses the distribution of $E_v(u,w)$ on the same closed interval.

$$E_v(u,w) = e_v(u,w) \text{ for } u \in \left[0;\ a^{(i)}\right] \tag{33}$$

In most cases, the magnetic component $H_u(u,w)$ is not normal to the perfectly conducting walls. Thus it makes a jump in $u = 0$ and $u = a^{(i)}$. However, we don't represent the function $H_u(u,w)$ with a cosine series. For H-plane junctions, the Cartesian components $H_x^{(g)}$ and $E_y^{(g)}$ are expanded on $\sin\left(\frac{n\pi x}{a^{(g)}}\right)$ functions as shown in Equation (24). In order to make the solving of field continuity relations easier in the planes $z = 0$ and $z = l$ (16)–(17), $H_u(u,w)$ is also expanded on $\sin\left(\frac{n\pi x}{a^{(g)}}\right)$ basis functions:

$$h_u(u,w) = \sum_{n=1}^{+\infty} I_n(w)\sin(\alpha_n u) \tag{34}$$

The sine series $h_u(u,w)$ is zero when $u = 0$ and $u = a^{(i)}$ (Figure 2). This expansion does not give access to the jumps made by the magnetic component $H_u(u,w)$ on the perfectly conducting walls. Thus, the expansion $h_u(u,w)$ becomes identified with $H_u(u,w)$ in the open interval $]0; a^{(i)}[$ and not in the closed interval $[0; a^{(i)}]$. The first equation of system (21) allows us to express the values of $H_u(u,w)$ on the walls $u = 0$ and $u = a^{(i)}$ in term of the Fourier's coefficients $V_n(w)$:

$$\begin{aligned} H_u(u=0,w) &= -\frac{f_1'(w)}{jkZ}\sum_{n=1}^{\infty}\alpha_n V_n(w) \\ H_u(u=a^{(i)},w) &= -\frac{f_2'(w)}{jkZ}\sum_{n=1}^{\infty}\alpha_n(-1)^n V_n(w) \end{aligned} \tag{35}$$

The second magnetic component $H_w(u,w)$ is always tangential to boundary surfaces. Function $H_w(u,w)$ is non zero when $u = 0$ and $u = a^{(i)}$. So, $H_w(u,w)$ is expanded in a cosine series as follows:

$$h_w(u,w) = \sum_{n=0}^{+\infty} H_n(w)\cos(\alpha_n u) \tag{36}$$

with

$$H_w(u,w) = h_w(u,w) \ \text{ for } \ u \in \left[0\,;a^{(i)}\right] \tag{37}$$

If the upper and lower walls are not bent, the previous expansions in Fourier's series represent combinations of rectangular waveguides eigen-modes TE_{n0}. This property is not satisfied if the magnetic component $H_u(u,w)$ is expanded on cosine basis functions.

2.8. Initial Value Problem

The Fourier's series $e_v(u,w)$, $h_u(u,w)$ and $h_w(u,w)$ are of period $2a^{(i)}$. $e_v(u,w)$ and $h_w(u,w)$ are continuous and at least once derivable with respect to u. Consequently, theirs derivatives with respect to u become identified with the derivatives of theirs expansions in Fourier's series. This identification does not make any sense for $h_u(u,w)$ but the derivative of this function does not appear in equation (21). Substituting geometric function $D(u,w)$ with its expansion in sine series (38), expansions $e_v(u,w)$, $h_u(u,w)$ and $h_w(u,w)$ are solutions of Equation (21). With this expansion (38), left and right-hand sides of (21) are both either odd functions or even functions in terms of u.

$$d(u,w) = \sum_{q=-\infty}^{+\infty} d_q(w)\sin\left(\alpha_q u\right) \tag{38}$$

with

$$D(u,w) = d(u,w) \ \text{ for } \ u \in]0;a^{(i)}[\tag{39}$$

and

$$d_{q>0}(w) = \frac{f_1'(w) - (-1)^q f_2'(w)}{\alpha_q f_{21}(w)};\ d_{q<0}(w) = -d_{q>0}(w);\ d_0(w) = 0 \tag{40}$$

After projecting equations of system (21) on trigonometric functions, several calculations lead to a set of partial differential equations (41) relating coefficients V_n and I_n.

For $1 \leq n \leq N$

$$\begin{cases} \dfrac{\partial V_n(w)}{\partial w} = \displaystyle\sum_{q=1}^{N} P_{nq}(w)V_q(w) + jkZC(w)I_n(w) \\ \dfrac{\partial I_n(w)}{\partial w} = \dfrac{1}{jkZC(w)}\left(\left(\dfrac{n\pi}{f_{21}(w)}\right)^2 - k^2\right)V_n(w) + \displaystyle\sum_{q=1}^{+\infty} Q_{nq}(w)I_q(w) \end{cases} \tag{41}$$

with

$$P_{nq} = \alpha_q \left(d_{q+n}(w) - d_{q-n}(w)\right) \tag{42}$$

$$Q_{nq} = -\alpha_n \left(d_{q+n}(w) + d_{q-n}(w)\right) \tag{43}$$

From (40), (42) and (43), we show that the elements of matrices **P** and **Q** are equal except for the diagonal elements that have opposite signs.

For $q \neq n$,

$$P_{nq} = Q_{nq} = \frac{2nq}{n^2 - q^2} \frac{f_1'(w) - (-1)^{q+n} f_2'(w)}{f_{21}(w)} \tag{44a}$$

and

$$P_{nn} = -Q_{nn} = \frac{f_1'(w) - f_2'(w)}{2f_{21}(w)} \tag{44b}$$

The numerical solution of system (41) requires a truncation order N. Then the covariant components inside the junction are described by only N coefficients (V_n and I_n) and the fields inside the rectangular waveguides by N independent modes TE_{n0}. Differential system (41) has non-constant coefficients and represents an initial value problem.

The eigen-modes TE_{n0} are independent. This allows us to define $2N$ independent initial conditions. Inside the input waveguide, each combination is characterized by amplitudes $\left(W_n^{(i)}; Y_n^{(i)}\right)$. Relations of continuity (16) allows amplitudes $(V_n; I_n)$ to be defined in $z = 0$. A numerical integration leads to the amplitudes $(V_n; I_n)$ in $z = l$. Relations of continuity (17) give the generalized voltages $W_n^{(o)}$ and the generalized currents $Y_n^{(o)}$ associated with the output waveguide. The normalized complex amplitudes are derived from relation (30). The fourth order Runge-Kutta's method is chosen for numerical integrations from $w = z = 0$ to $w = z = l$. In references [2, 3], we have shown that the amplitudes $(V_n; I_n)$ and $(V_n^*; -I_n^*)$ are solution of system (41). Using this property, we determine the multimodal scattering matrix with only N numerical integrations. Moreover, the differential system (41) contains the structure geometry. As shown in references [2, 3], electromagnetic effects of symmetric can be easily deduced and the system size reduced and a saving in computation time obtained [2, 3].

3. GENERALIZED TELEGRAPHIST'S METHOD AND WAVEGUIDES

3.1. Transverse Components – Longitudinal Components

Vectors $\vec{E}$ and $\vec{H}$ are expanded as follows:

$$\begin{aligned}\vec{E} &= \vec{E}_T + E_z\vec{u}_z\\ \vec{H} &= \vec{H}_T + H_z\vec{u}_z\end{aligned}\tag{45}$$

where $\vec{E}_T$ and $\vec{H}_T$ represent the transverse components, E_z and H_z, the longitudinal components. Maxwell-Faraday's equation and Maxwell-Ampere's equation associated with the constitutive relations yield [5, 6]:

$$\begin{aligned}\frac{\partial\vec{E}_T}{\partial z} &= -jkZ\left(\vec{H}_T\wedge\vec{u}_z\right)+\vec{\nabla}_T\cdot E_z\\ \frac{\partial\vec{H}_T}{\partial z} &= j\frac{k}{Z}\left(\vec{E}_T\wedge\vec{u}_z\right)+\vec{\nabla}_T\cdot H_z\end{aligned}\tag{46}$$

and

$$\begin{aligned}E_z &= \frac{Z}{jk}\vec{\nabla}_T\cdot\left(\vec{H}_T\wedge\vec{u}_z\right)\\ H_z &= -\frac{1}{jkZ}\vec{\nabla}_T\cdot\left(\vec{E}_T\wedge\vec{u}_z\right)\end{aligned}\tag{47}$$

where $\wedge$ denotes the vectorial product and $\vec{\nabla}_T$, the transverse Hamilton's operator:

$$\vec{\nabla}_T = \vec{u}_x\frac{\partial}{\partial x}+\vec{u}_y\frac{\partial}{\partial y}\tag{48}$$

In the case of TE polarization, $E_z = 0$ and then we obtain the following system of coupled differential equations:

$$\frac{\partial\vec{E}_T}{\partial z} = -jkZ\left(\vec{H}_T\wedge\vec{u}_z\right)\tag{49a}$$

$$\frac{\partial\vec{H}_T}{\partial z} = j\frac{k}{Z}\left(\vec{E}_T\wedge\vec{u}_z\right)-\frac{1}{jkZ}\vec{\nabla}_T\cdot\left[\vec{\nabla}_T\cdot\left(\vec{E}_T\wedge\vec{u}_z\right)\right]\tag{49b}$$

3.2. Expression of Transverse Components

The generalized telegraphist's method leads to expand the transverse components on a basis of orthogonal functions [5, 6]:

$$\begin{aligned}\vec{E}_T &= \sum_q U_q(z)\,\vec{e}_q(x,y,z)\\ \vec{H}_T &= \sum_q J_q(z)\,\vec{h}_q(x,y,z)\end{aligned} \tag{50}$$

For a rectangular waveguide excited by the fundamental mode, the eigen-modes TE_{n0} give the orthogonal functions $\vec{e}_q$ and $\vec{h}_q$:

$$\vec{e}_q(x,y,z) = \sin\left(\frac{n\pi\,(x-f_1(z))}{a}\right)\vec{u}_y \tag{51}$$

$$\vec{h}_q(x,y,z) = \sin\left(\frac{n\pi\,(x-f_1(z))}{a}\right)\vec{u}_x \tag{52}$$

with

$$\vec{h}_q(x,y,z) = \vec{e}_q(x,y,z) \wedge \vec{u}_z \tag{53}$$

and

$$\frac{2}{ab}\int_\Sigma \vec{e}_q(x,y,z)\vec{e}_n(x,y,z)dxdy = \frac{2}{ab}\int_\Sigma \vec{h}_q(x,y,z)\vec{h}_n(x,y,z)dxdy = \delta_{qn} \tag{54}$$

δ_{qn} is the Kronecker's symbol and $\Sigma = a \times b$, the rectangular cross-section of the microwave component at distance z (where $a = f_{21}(z)$). For a rectangular cross-section, functions $\vec{h}_q$ (or $\vec{e}_q$) are solutions of the following eigen-values problem:

$$\vec{\nabla}_T \cdot \left[\vec{\nabla}_T \cdot \vec{h}_q\right] + \left(\frac{n\pi}{a}\right)^2 \vec{h}_q = 0 \tag{55}$$

The orthonormality relation (54) permits a formal determination of coefficients $U_q(z)$ and $J_q(z)$ with:

$$\begin{aligned}U_n(z) &= \frac{2}{ab}\int_\Sigma \vec{E}_T(x,y,z)\vec{e}_n(x,y,z)dxdy\\ J_n(z) &= \frac{2}{ab}\int_\Sigma \vec{H}_T(x,y,z)\vec{e}_n(x,y,z)dxdy\end{aligned} \tag{56}$$

3.3. Initial Condition Problem

Integrating over Σ the scalar product of Equation (49a) with $\vec{e}_n(x,y,z)$, and of Equation (49b) with $\vec{h}_n(x,y,z)$, we obtain:

$$\frac{\partial U_n(z)}{\partial z} = \sum_q R_{nq} U_q(z) + jkZ J_n(z) \tag{57a}$$

$$\frac{\partial J_n(z)}{\partial z} = \sum_q R_{nq} J_q(z) + \frac{jk}{Z} U_n(z) - \frac{1}{jkZ} \frac{2}{ab} \int_\Sigma \vec{\nabla}_T \cdot \left[\vec{\nabla}_T \cdot \left(\vec{E}_T \wedge \vec{u}_z \right) \right] \vec{h}_n dx dy \tag{57b}$$

where for $q \neq n$,

$$R_{nq} = -\frac{2}{f_{21}(z)} \int_{f_1(z)}^{f_2(z)} \vec{e}_n \frac{\partial \vec{e}_q}{\partial z} dx = \frac{2nq}{n^2 - q^2} \frac{f_1'(z) - (-1)^{q+n} f_2'(z)}{f_{21}(z)} \tag{58a}$$

and

$$R_{nn} = \frac{f_1'(z) - f_2'(z)}{2 f_{21}(z)} = -\frac{C'(z)}{2C(z)} \tag{58b}$$

Taking into account Equations (55) and (56), the two-dimensional Gauss's theorem gives the integral of Equation (57b) in the following form:

$$\frac{2}{ab} \int_\Sigma \vec{\nabla}_T \cdot \left[\vec{\nabla}_T \cdot \left(\vec{E}_T \wedge \vec{u}_z \right) \right] \vec{h}_n dx dy$$
$$= -\left(\frac{\pi n}{f_{21}(z)} \right)^2 U_n + \frac{2}{ab} \oint_\Gamma \vec{h}_n \left(\frac{\partial \vec{E}_T}{\partial N} \wedge \vec{u}_z \right) dl - \frac{2}{ab} \oint_\Gamma \left(\vec{E}_T \wedge \vec{u}_z \right) \frac{\partial \vec{h}_n}{\partial N} dl \tag{59}$$

where Γ is the rectangular curve of the microwave component at distance z. $\frac{\partial}{\partial N}$ denotes the normal derivative and $\vec{u}_N$ the unit vector normal to Γ. Remembering that $\frac{\partial}{\partial N} = \vec{u}_N \vec{\nabla}_T$ and taking into account expression (47), the first curve integral becomes:

$$\oint_\Gamma \vec{h}_n \left(\frac{\partial \vec{E}_T}{\partial N} \wedge \vec{u}_z \right) dl = -jkZ \oint_\Gamma \vec{h}_n \vec{u}_N H_z dl \tag{60}$$

For $x = f_1(z)$ and $x = f_2(z) + a^{(i)}$, $\vec{u}_N = \vec{u}_x$ but, from expression (52), $\vec{h}_n(x, y, z) = 0$. For $y = 0$ and $y = b$, $\vec{u}_N = \vec{u}_y$ and $\vec{u}_N \vec{h}_n = 0$. Then the first curve integral is zero:

$$\oint_\Gamma \vec{h}_n \left(\frac{\partial \vec{E}_T}{\partial N} \wedge \vec{u}_z \right) dl = 0 \tag{61}$$

Component $\vec{E}_T$ is parallel to the unit vector $\vec{u}_y$. Then, $\vec{E}_T$ is zero on the perfectly conducting walls $x = f_1(z)$ and $x = f_2(z) + a^{(i)}$. In the same way, from expression (52), $\frac{\partial \vec{h}_n}{\partial N} = \frac{\partial \vec{h}_n}{\partial y} = 0$ on the walls $y = 0$ and $y = b$. Then the second curve integral is also zero.

$$\oint_\Gamma \left(\vec{E}_T \wedge \vec{u}_z \right) \frac{\partial \vec{h}_n}{\partial N} dl = 0 \tag{62}$$

Substituting expressions (59), (61) and (62) into differential system (57), we obtain for $1 \leq n \leq N$:

$$\begin{cases} \dfrac{\partial U_n(z)}{\partial z} = \displaystyle\sum_{q=1}^{N} R_{nq}(z) U_q(z) + jkZ J_n(z) \\ \dfrac{\partial J_n(z)}{\partial z} = \dfrac{1}{jkZ} \left(\left(\dfrac{n\pi}{f_{21}(z)} \right)^2 - k^2 \right) U_n(z) + \displaystyle\sum_{q=1}^{+\infty} R_{nq}(z) J_q(z) \end{cases} \tag{63}$$

The system of coupled differential Equation (63) has non-constant coefficients and represents an initial condition problem. Using the numerical processing presented in Section 2.8, the resolution of the differential system provides the multimodal scattering matrix.

4. CONCLUSIONS

The curvilinear coordinate method leads to the non-constant coefficients differential system (41), the Telegraphists method to system (63). Using the following change of variables,

$$\begin{aligned} U_n(z) &= V_n(w) \\ J_n(z) &= C(w) I_n(w) \end{aligned} \tag{64}$$

system (63) takes the following form:

$$\begin{cases} \dfrac{\partial V_n(z)}{\partial z} = \displaystyle\sum_{q=1}^{N} R_{nq}(z)V_q(z) + jkZC(z)I_n(z) \\ \dfrac{\partial I_n(z)}{\partial z} = \dfrac{1}{jkZC(z)}\left(\left(\dfrac{n\pi}{f_{21}(z)}\right)^2 - k^2\right)U_n(z) \\ \qquad\qquad +\displaystyle\sum_{q=1}^{+\infty}\left(R_{nq}(z) - \dfrac{C'(z)}{C(z)}\delta_{nn}\right)I_q(z) \end{cases} \tag{65}$$

Matrices **P** (44) and **R** (58) are identical and matrices **Q** (44) and **R** (58) are equal except for diagonal elements that have opposite signs.

For $q \neq n$,

$$R_{nq} = Q_{nq} \tag{66a}$$

and

$$R_{nn} = -Q_{nn} \tag{66b}$$

According to (58b) and (67b), we can write:

$$R_{nn}(z) = Q_{nn}(z) + \frac{C'(z)}{C(z)} \tag{67}$$

Taking into account matrix relations (66) and expression (67), we show that systems (41) and (65) are equivalent with $z = w$:

$$\begin{cases} \dfrac{\partial V_n(z)}{\partial z} = \displaystyle\sum_{q=1}^{N} P_{nq}(z)V_q(z) + jkZC(z)I_n(z) \\ \dfrac{\partial I_n(z)}{\partial z} = \dfrac{1}{jkZC(z)}\left(\left(\dfrac{n\pi}{f_{21}(z)}\right)^2 - k^2\right)U_n(z) + \displaystyle\sum_{q=1}^{+\infty} Q_{nq}(z)I_q(z) \end{cases} \tag{68}$$

A similar demonstration can be used in the case of an E-plane discontinuity excited by LSE1n modes. From a theoretical point of view, the curvilinear coordinate method using for the analysis of microwave components of rectangular cross-section is then equivalent to the Telegraphists method.

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