Progress In Electromagnetics Research, PIER 44, 169–195, 2004

EFFECTIVE PERMITTIVITY OF A STATISTICALLY INHOMOGENEOUS MEDIUM WITH STRONG PERMITTIVITY FLUCTUATIONS

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Abstract—Most previous multiple-scattering theories for electromagnetic waves in strongly fluctuating media are limited by the assumption of statistical homogeneity of media. In the paper, a lossy electrically isotropic random medium is considered whose mean permittivity distribution, as well as the multipoint permittivity's moments are invariant under arbitrary rotations about and translations along a fixed symmetry axis, and are inhomogeneous in the radial direction. The goal of the paper is to calculate the effective permittivity operator (EPO) for such medium in the case of strong permittivity fluctuations. For this purpose, one has to eliminate the secular terms from the spectral representation of the-EPO in the basis set of waves suited to a statistically inhomogeneous medium. This is achieved via a renormalization approach which takes into proper account a delta function singularity of the spectral Green's function (rather than that of the spatial Green's function accounted for in the past) referring to a spatially inhomogeneous electrically anisotropic background medium. On this basis, the permittivity matrix of the background medium is explicitly found, a full perturbation series solution and a bilocal approximation for the

EPO are derived, the macroscopic properties of the spatially dispersive effective medium are studied, and a perturbative solution for the propagation constants of guided modes of the mean field is obtained.

1. INTRODUCTION

The effective permittivity plays a fundamental role in the electromagnetic (EM) theory of random media [1]. By knowledge of the effective permittivity it becomes possible to account for the "accumulating" effects which arise in the multiple scattering of EM waves in random media and which lead to strong distortions of the EM field. In earlier theoretical work most of the results have been obtained by the approach of Ref. [2] which invokes a volumetric integral equation of scattering and a bilocal approximation and is restricted to weakly fluctuating media. A new prospective line was initiated by the works [3, 4] where a renormalization approach was proposed referring to random media with strong permittivity fluctuations. Formally, it is grounded on taking into proper account the delta-function constituent of the spatial Green's function (GF). It was further developed in [5, 6] leading to contemporary strong fluctuation theory. An instance of generalization of this technique to random media with (bi)anisotropy of electric and magnetic properties is furnished by [7, 8]. The renormalization approach employed in all of the above mentioned studies is capable of handling statistically homogeneous media with small-scale perturbations only. It can also be applied to describe the mean-field properties of a “composite” random medium by dividing it into large subregions which can be approximately viewed as statistically homogeneous. This is always possible if the characteristic scale of random perturbations and the wavelength are much less than the dimensions of the aforementioned subregions. Clearly, said approximation does not work when the characteristic scale of regular inhomogeneity becomes comparable with that of random perturbations or a wavelength. In this case the multiple scattering of EM waves is essentially shaped by the regular inhomogeneity of the medium, and the model of a statistically inhomogeneous medium should be applied.

To accommodate, as regards the calculation of the effective permittivity operator (EPO), the statistical inhomogeneity of a random medium, one must recognize in full the following points [9]. First, a departure from statistical homogeneity gives rise to a completely different basis set of waves for the mean EM field as compared with the plane wave modes of [2–7]. Second, to achieve characterization of the mean-field properties of a random medium the very thing we need is not the EPO itself but its spectral representation in the appropriate basis set. Last, it is just the spectral-domain EPO that should be free from secular (in the meaning of [5]) terms to cover the case of strong permittivity fluctuations. All this has served as a starting point for the modification of the renormalization approach begun by work [9] and continued in [10, 11]. Cited attempts refer to the model of a statistically layered medium with uniaxial [9, 10] or general [11] anisotropy of EM properties.

In this paper which may be thought of as a natural extension of the works [9–11] we will develop a systematic multiple-scattering approach for a more sophisticated model of a statistically inhomogeneous medium. Namely, we here consider a strongly fluctuating medium which is *cylindrically inhomogeneous* in the statistical sense. The random media of such kind are distinguished by the fact that their electrophysical properties are, on the average, invariant under circular rotations about and longitudinal motions along a fixed symmetry axis, and may vary with distance from the aforementioned axis. This model is practically related to, for instance, design of artificial dielectrics for optical and radio wavebands, polariton propagation, plasma diagnostics, fiber optics etc. The problem, however, attracts our attention not only for practical reasons but also for the wealth of theoretical content pertinent to scattering phenomena in statistically inhomogeneous media.

In the present paper, a harmonic time dependence $\exp(-i\omega t)$ of the sources and fields is assumed, with the corresponding time factor not printed. The whole of the three-dimensional space is referred to a cylindrical coordinate system ρ, φ, z, $(0 \le \rho < +\infty,\ 0 \le \varphi < 2\pi,\ -\infty < z < +\infty)$. Algebraic vectors $x = (\rho, \varphi, z)$ and $x' = (\rho', \varphi', z')$ correspond to the points of space characterized by the respective values of their cylindrical coordinates. For a vectorial function $\mathbf{A}(x)$, the values of the physical coordinates at point x in this coordinate system will be designated as $A_p(x)$ where the subscript p assumes the values 1, 2, 3 corresponding to the coordinates ρ, φ, z. A 3×3 matrix composed of elements B_{pq} is designated as $\underline{\underline{B}}$, and $\underline{C}$ stands for a column matrix $col[C_1,\ C_2,\ C_3]$. Finally, the symbol o is used to denote the matrix product.

2. STATEMENT OF THE PROBLEM

Let us assume that the entire space is filled with an inhomogeneous dielectric medium of permittivity $\varepsilon^{(r)}(x)$ which is a random function of spatial variable x. The EM field $\underline{E}^{(r)}(x)$, $\underline{H}^{(r)}(x)$ set up in the medium by the deterministic impressed sources of the electric type $\underline{J}(x)$ obeys Maxwell's equations

$$\nabla \times \underline{H}^{(r)}(x) + ik_0\varepsilon^{(r)}(x)\underline{E}^{(r)}(x) = \frac{4\pi}{c}\underline{J}^{(r)}(x), \tag{1}$$

$$\nabla \times \underline{E}^{(r)}(x) - ik_0\underline{H}^{(r)}(x) = 0, \tag{2}$$

the continuity condition for the tangential EM field at the interfaces, and the radiation condition at infinity. Here $k_0 = \omega/c$, c is the speed of light in vacuum, and the subscript "r" accompanies random quantities referring to the disordered medium. Note that ∇ should be interpreted as a matrix differential operation, since we have chosen to deal with the vector's components [in a cylindrical coordinate system] rather than the vector itself.

The EPO $\underline{\underline{\varepsilon}}^{(e)}$ which works on x is defined by the identity [1–11]

$$\left\langle \varepsilon^{(r)}(x)\underline{E}^{(r)}(x) \right\rangle \equiv \underline{\underline{\varepsilon}}^{(e)} \circ \left\langle \underline{E}^{(r)}(x) \right\rangle. \tag{3}$$

From Eq. (3) and Eqs. (1), (2) it follows that the equations obeyed by the mean EM field

$$\underline{E}(x) \equiv \left\langle \underline{E}^{(r)}(x) \right\rangle, \quad \underline{H}(x) \equiv \left\langle \underline{H}^{(r)}(x) \right\rangle \tag{4}$$

have the form characteristic of an "effective" deterministic medium with the nonlocal permittivity matrix $\underline{\underline{\varepsilon}}^{(e)}$, viz.

$$\nabla \times \underline{H}(x) + ik_0\underline{\underline{\varepsilon}}^{(e)} \circ \underline{E}(x) = \frac{4\pi}{c}\underline{J}(x), \tag{5}$$

$$\nabla \times \underline{E}(x) - ik_0\underline{H}(x) = 0. \tag{6}$$

Within the framework of the present paper, the random function $\varepsilon^{(r)}$ is assumed to fulfill the following fundamental conditions: i) its first statistical moment, i.e., the mean permittivity profile $\langle\varepsilon^{(r)}(x)\rangle$ does not depend upon φ, z and is a function of ρ variable only, ii) the statistical moments of higher order, i.e., the quantities $\langle\varepsilon^{(r)}(x_1)\varepsilon^{(r)}(x_2)\dots\varepsilon^{(r)}(x_n)\rangle$, $n = 2, 3, \dots$, depend on $x_1, x_2, \dots x_n$ through $\rho_1, \rho_2, \dots \rho_n$ and the differences $\varphi_1-\varphi_2$, $\varphi_2-\varphi_3, \dots \varphi_{n-1}-\varphi_n$ and $z_1 - z_2$, $z_2 - z_3, \dots z_{n-1} - z_n$ only. In other words, we consider a random field $\varepsilon^{(r)}(x)$ whose multipoint probability distributions of all orders are invariant under arbitrary rotations about the z-axis and translations along this axis. It is natural to term such media as cylindrically inhomogeneous, or cylindrically-layered in the statistical sense. Under these conditions, it is almost intuitively obvious that the EPO represents an integral operator

$$\begin{aligned}\underline{\underline{\varepsilon}}^{(e)} &= \int_0^{+\infty} \rho' d\rho' \int_0^{2\pi} d\varphi' \int_{-\infty}^{+\infty} dz' \underline{\underline{\varepsilon}}^{(e)}(\rho, \rho', \varphi-\varphi', z-z')\dots \\ &\equiv \int dV' \underline{\underline{\varepsilon}}^{(e)}(\rho, \rho', \varphi-\varphi', z-z')\dots \end{aligned} \tag{7}$$

whose kernel $\underline{\underline{\varepsilon}}^{(e)}(\rho, \rho', \varphi-\varphi', z-z')$ depends on x and x' through ρ, ρ' and the difference variables $\varphi-\varphi'$ and $z-z'$. [This property follows rigorously from subsequent Eqs. (24) and (26)]. Let the impressed sources $\underline{J}(x)$ be a spatial harmonic of the form

$$\underline{J}(x) = \underline{J}(\rho)\exp[i(n\varphi + hz)], \tag{8}$$

where n is an arbitrary integer, h is an arbitrary complex quantity, and $\underline{J}(\rho)$ is a given spectral source amplitude. Then the mean EM field, in view of Eqs. (5)–(7), will have the form of a spatial harmonic as well:

$$\begin{aligned}\underline{E}(x) &= \underline{E}(\rho)\exp[i(n\varphi + hz)], \\ \underline{H}(x) &= \underline{H}(\rho)\exp[i(n\varphi + hz)],\end{aligned} \tag{9}$$

where $\underline{E}(\rho)$, $\underline{H}(\rho)$, are the spectral field amplitudes. In verifying this claim, we are assisted by taking note of the fact that

$$\begin{aligned}\underline{\underline{\varepsilon}}^{(e)} \circ \underline{E}(x) &= \exp[i(n\varphi + hz)] \int_0^{+\infty} \underline{\underline{\varepsilon}}^{(e)}(\rho, \rho', n, h) \circ \underline{E}(\rho')\rho' d\rho' \\ &\equiv \exp[i(n\varphi + hz)]\underline{\underline{\varepsilon}}^{(e)}(n, h) \circ \underline{E}(\rho).\end{aligned} \tag{10}$$

Here $\underline{\underline{\varepsilon}}^{(e)}(n, h)$ is a spectral counterpart of the EPO, and

$$\underline{\underline{\varepsilon}}^{(e)}(\rho, \rho' n, h) = \mathcal{F}\underline{\underline{\varepsilon}}^{(e)}(\rho, \rho', \varphi-\varphi', z-z')$$

$$\equiv \int_0^{2\pi} d\varphi \int_{-\infty}^{+\infty} dz \exp\{-i[n(\varphi-\varphi')+h(z-z')]\}$$
$$\underline{\underline{\varepsilon}}^{(e)}(\rho,\rho',\varphi-\varphi',z-z'). \quad (11)$$

Spatial harmonics (9) form a natural basis set of waves for the class of media under study, since a general mean field can be represented as a proper superposition of waves like (9). The spectral-domain EPO $\underline{\underline{\varepsilon}}^{(e)}(n,h)$ provides an exhaustive characterization of the random medium properties with respect to the mean field (9). The calculation of said operator constitutes the main goal of the present paper. In the next Section, we will prepare the mathematical tools which are indispensable for manipulating the random EM fields in a (statistically) cylindrically-layered medium.

3. RENORMALIZED EQUATION OF SCATTERING

We here first concentrate on the distributional nature of the spectral-domain matrix GF for wave equation in an inhomogeneous anisotropic medium, and then substitute a conventional integral equation of scattering with a renormalized equation in a new field variable. What distinguishes our renormalization approach from previous versions used in [3–6] is that it is the delta function singularity of the spectral-domain GF rather than that of the spatial-domain GF which is accounted for in the renormalized formulation. Also, in order to meet the requirement (37) in the next Section, we have to work with an electrically anisotropic background medium in spite of electrical isotropy of the random medium. This is a direct consequence of the statistical inhomogeneity assumption as contrasted to [9–11] where electrical anisotropy of the background medium arises due to statistical and/or electrical anisotropy of the random medium.

Let us introduce an inhomogeneous anisotropic background medium whose permeability equals 1 and the permittivity tensor in the cylindrical coordinate system assumes a diagonal form

$$\underline{\underline{\varepsilon}}(\rho) = diag[\varepsilon_{||}(\rho)\varepsilon_{\perp}(\rho),\varepsilon_{\perp}(\rho)], \quad (12)$$

where $\varepsilon_{||}(\rho),\varepsilon_{\perp}(\rho)$ are piecewise-smooth functions of variable ρ. Physically, this model describes a locally uniaxial medium whose principal dielectric constants vary with distance from the symmetry axis $\rho = 0$. The matrix $\underline{\underline{G}}(x,x')$ for this medium is defined as solution to an equation

$$\nabla\times\nabla\times\underline{\underline{G}}(x,x')-k_0^2\underline{\underline{\varepsilon}}(\rho)\circ\underline{\underline{G}}(x,x') = \underline{\underline{I}}\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')/\rho, \quad (13)$$

which depicts an outgoing wave as $\rho \to +\infty$. Here $\underline{\underline{I}}$ is the unity matrix, and δ is a one-dimensional Dirac delta function. It is expressible as a double inverse Fourier transform of the spectral-domain Green's matrix $\underline{\underline{G}}(\rho, \rho', n, h)$:

$$\begin{aligned} \underline{\underline{G}}(x, x') &= \mathcal{F}^{-1}\underline{\underline{G}}(\rho, \rho', n, h) \\ &\equiv (2\pi)^{-2} \sum_{n=-\infty}^{n=+\infty} \int_{-\infty}^{+\infty} \exp\{i[n(\varphi - \varphi') \\ &\quad + h(z - z')]\}\underline{\underline{G}}(\rho, \rho', n, h)dh. \end{aligned} \tag{14}$$

Here, it is presumed that the poles and the branch points of $\underline{\underline{G}}(\rho, \rho', n, h)$ as function of variable h lie off the real axis in the complex h-plane (see Appendix A for more details). The spectral GF is a generalized function of variables ρ, ρ' which includes a regular term $\underline{\underline{G}}'(\rho, \rho', n, h)$ and a singular term proportional to $\delta(\rho - \rho')$:

$$\underline{\underline{G}}(\rho, \rho', n, h) = \underline{\underline{G}}'(\rho, \rho', n, h) - \underline{\underline{I}}_{\|} \frac{\delta(\rho - \rho')}{k_0^2 \rho' \varepsilon_{\|}(\rho')}. \tag{15}$$

In this formula $\underline{\underline{I}}_{\|} = diag[1, 0, 0]$, and the quantity $\underline{\underline{G}}'(\rho, \rho', n, h)$ represents an integrable function of each of the variables ρ, ρ' which experiences a step-like discontinuity at $\rho = \rho'$ and assumes finite limiting values as $\rho \to \rho' \pm 0$. Its explicit expression is derived in Appendix B. A spatial-domain counterpart of representation (15) acquires, in view of Eq. (14), the following characterization:

$$\underline{\underline{G}}(x, x') = \underline{\underline{G}}'(x, x') - \underline{\underline{I}}_{\|} \frac{\delta(\rho - \rho')\delta(\varphi - \varphi')\delta(z - z')}{k_0^2 \rho' \varepsilon_{\|}(\rho')}, \tag{16}$$

with $\underline{\underline{G}}'(x, x') = \mathcal{F}^{-1}\underline{\underline{G}}'(\rho, \rho', n, h)$ being a generalized function. Note that in contrast to previous authors [3–6] we do not associate $\underline{\underline{G}}'(x, x')$ with the principal-value part of the spatial GF. In fact, its precise mathematical nature is unimportant in future reasoning.

The solution for the random electric field vector can be written in terms of the matrix GF by regarding the random medium as a perturbation of the background medium [2–6]:

$$\underline{E}^{(r)}(x) = \underline{E}^{(b)}(x) + k_0^2 \int \underline{\underline{G}}(x, x') \circ \left[\varepsilon^{(r)}(x')\underline{\underline{I}} - \underline{\underline{\varepsilon}}(\rho')\right] \circ \underline{E}^{(r)}(x')dV'. \tag{17}$$

Here $\underline{E}^b(x)$ is the electric field created in the background medium by the impressed sources $\underline{J}(x)$. By taking care of the representation (16) and introducing a new field quantity $\underline{F}(x)$ and random perturbation $\underline{\underline{\xi}}(x)$,

$$\underline{F}(x) = diag\left[\varepsilon^{(r)}(x)/\varepsilon_{\|}(\rho), 1, 1\right] \circ \underline{E}^{(r)}(x), \tag{18}$$

$$\underline{\underline{\xi}}(x) = diag\left[\xi_{\|}(x), \xi_{\perp}(x), \xi_{\perp}(x)\right], \tag{19}$$

$$\xi_{\|}(x) = \varepsilon_{\|}(\rho)[1 - \varepsilon\|(\rho)/\varepsilon^{(r)}(x)],\ \xi_{\perp}(x) = \varepsilon^{(r)}(x) - \varepsilon_{\perp}(\rho), \tag{20}$$

equation (17) now takes the renormalized form

$$\underline{F}(x) = \underline{E}^{(b)}(x) + k_0^2 \int \underline{\underline{G}}(x, x') \circ \underline{\underline{\xi}}(x') \circ \underline{F}(x') dV' \tag{21}$$

$$\equiv \underline{E}^{(b)}(x) + k_0^2 \underline{\underline{G}}' \circ \underline{\underline{\xi}} \circ \underline{F}(x). \tag{22}$$

Below, in much the same way as the strong fluctuation theory for statistically homogeneous media [3–7], this new distributional equation is applied to the effective permittivity calculation for a statistically inhomogeneous medium.

4. CALCULATIONS

To derive the spectral-domain EPO we apply a two-step procedure [3–7] which consists first of solving for the effective perturbation operator $\underline{\underline{\xi}}^{(e)}$ which is defined by the identity

$$\left\langle \underline{\underline{\xi}}(x) \circ \underline{F}(x) \right\rangle \equiv \underline{\underline{\xi}}^{(e)} \circ \langle \underline{F}(x) \rangle, \tag{23}$$

and then making use of the relation

$$\underline{\underline{\varepsilon}}^{(e)} - \underline{\underline{\varepsilon}} = \underline{\underline{\xi}}^{(e)} \circ \left(\frac{1}{\varepsilon_{\|}} \underline{\underline{\varepsilon}}^{(e)} \circ \underline{\underline{I}}_{\|} + \underline{\underline{I}}_{\perp} \right), \tag{24}$$

$\underline{\underline{I}}_{\perp} = \underline{\underline{I}} - \underline{\underline{I}}_{\|}$, which links $\underline{\underline{\xi}}^{(e)}$ with $\underline{\underline{\varepsilon}}^{(e)}$. Eq. (24) can easily be arrived at by inserting into Eq. (23) the expressions

$$\begin{aligned} \left\langle \underline{\underline{\xi}}(x) \circ \underline{F}(x) \right\rangle &= \left(\underline{\underline{\varepsilon}}^{(e)} - \underline{\underline{\varepsilon}} \right) \circ \left\langle \underline{E}^{(r)}(x) \right\rangle, \\ \langle \underline{F}(x) \rangle &= \left(\frac{1}{\varepsilon_{\|}} \underline{\underline{\varepsilon}}^{(e)} \circ \underline{\underline{I}}_{\|} + \underline{\underline{I}}_{\perp} \right) \circ \left\langle \underline{E}^{(r)}(x) \right\rangle, \end{aligned} \tag{25}$$

consequent to Eqs. (18)–(20).

Taking note of the formal solution to Eq. (22) $\underline{F}(x) = (\underline{\underline{I}} - k_0^2 \underline{\underline{G}}' \circ \underline{\underline{\xi}})^{-1} \circ \underline{E}^{(b)}(x)$ in identity (23) affords a means to determining $\underline{\underline{\xi}}^{(e)}$. There follows, by elimination of $\underline{E}^{(b)}(x)$,

$$\underline{\underline{\xi}}^{(e)} = \left\langle \underline{\underline{\xi}} \cdot \left(\underline{\underline{I}} - k_0^2 \underline{\underline{G}}' \circ \underline{\underline{\xi}}\right)^{-1} \right\rangle \circ \left\langle \left(\underline{\underline{I}} - k_0^2 \underline{\underline{G}}' \circ \underline{\underline{\xi}}\right)^{-1} \right\rangle^{-1}. \tag{26}$$

An inspection of Eq. (26) reveals that the effective perturbation operator admits characterization $\underline{\underline{\xi}}^{(e)} = \int dV' \underline{\underline{\xi}}^{(e)}(\rho, \rho', \varphi - \varphi', z - z') \ldots$. The substitution of integral representations for operators $\underline{\underline{\varepsilon}}^{(e)}, \underline{\underline{\xi}}^{(e)}$ followed by Fourier transformation relative to angular and longitudinal variables yields an integral equation counterpart to Eq. (24)

$$\underline{\underline{\varepsilon}}^{(e)}(\rho, \rho', n, h) - \underline{\underline{\varepsilon}}(\rho) \frac{\delta(\rho - \rho')}{\rho'} = \underline{\underline{\xi}}^{(e)}(\rho, \rho', n, h) \circ \underline{\underline{I}}_{\perp}$$
$$+ \int_0^{+\infty} \underline{\underline{\xi}}^{(e)}(\rho, \rho'', n, h) \circ \underline{\underline{\varepsilon}}^{(e)}(\rho'', \rho', n, h) \circ \underline{\underline{I}}_{\parallel} \varepsilon_{\parallel}^{-1}(\rho'') \rho'' d\rho'', \tag{27}$$

where $\underline{\underline{\xi}}^{(e)}(\rho, \rho', n, h) = \mathcal{F} \underline{\underline{\xi}}^{(e)}(\rho, \rho', \varphi - \varphi', z - z')$. Note that in the case of statistically homogeneous media an equation analogical to (24) could be replaced [3–7] by an algebraic equation via a three-dimensional Fourier transformation. By contrast, we have to deal with an integral equation, said complication being in keeping with a more sophisticated model of random medium analyzed in the paper.

Solution (26) for $\underline{\underline{\xi}}^{(e)}$ does not put an end to our work, since it contains the inverse operators which are not easy to find. Thus we resort to a perturbation technique by expanding the right-hand side in Eq. (26) in powers of $\underline{\underline{\xi}}$. This in conjunction with a customary assumption [3–7] that the perturbation is small on average, $\langle \underline{\underline{\xi}} \rangle = 0$, which assures the fastest rate of convergence of the resulting expansions, gives $\underline{\underline{\xi}}^{(e)}$ the following representation:

$$\underline{\underline{\xi}}^{(e)} = \sum_{m=2}^{+\infty} \underline{\underline{\theta}}^{(m)}, \tag{28}$$

where $\underline{\underline{\theta}}^{(m)} \sim \underline{\underline{\xi}}^m$ are defined by the recurrence formulae [11]

$$\underline{\underline{\theta}}^{(2)} = \left\langle \underline{\underline{v}} \circ \underline{\underline{\xi}} \right\rangle, \quad \underline{\underline{\theta}}^{(3)} = \left\langle \underline{\underline{v}}^2 \circ \underline{\underline{\xi}} \right\rangle, \tag{29}$$

$$\underline{\underline{\theta}}^{(m)} = \left\langle \underline{\underline{v}}^{m-1} \circ \underline{\underline{\xi}} \right\rangle - \sum_{k=2}^{m-2} \left\langle \underline{\underline{v}}^k \right\rangle \circ \underline{\underline{\theta}}^{(m-k)}, \ (m = 4, 5, \ldots), \tag{30}$$

$$\underline{\underline{v}} = k_0^2 \underline{\underline{\xi}} \circ \underline{\underline{G}}', \tag{31}$$

(the quantity $\underline{\underline{v}}$ should not be confused with the operator $\underline{v}$ given by Eq. (B6) in Appendix B). Relying on the expression (28), solution to the equation (24) for the EPO can be exhibited in the explicit form:

$$\underline{\underline{\varepsilon}}^{(e)} - \underline{\underline{\varepsilon}} = \sum_{m=2}^{+\infty} \underline{\underline{\nu}}^{(m)}, \tag{32}$$

with

$$\underline{\underline{\nu}}^{(m)} = \underline{\underline{\theta}}^{(m)} \circ \left(\frac{1}{\varepsilon_{\|}} \underline{\underline{\varepsilon}} \circ \underline{\underline{I}}_{\|} + \underline{\underline{I}}_{\perp} \right), \quad (m = 2, 3), \tag{33}$$

$$\underline{\underline{\nu}}^{(m)} = \underline{\underline{\theta}}^{(m)} \circ \left(\frac{1}{\varepsilon_{\|}} \underline{\underline{\varepsilon}} \circ \underline{\underline{I}}_{\|} + \underline{\underline{I}}_{\perp} \right) + \sum_{k=2}^{m-2} \underline{\underline{\theta}}^{(k)} \circ \frac{1}{\varepsilon_{\|}} \underline{\underline{\nu}}^{(m-k)} \circ \underline{\underline{I}}_{\|}, \quad (m = 4, 5, \ldots). \tag{34}$$

As a final step, one can easily rewrite expansions (28), (32) in a spectral-domain version:

$$\underline{\underline{\xi}}^{(e)}(\rho, \rho', n, h) = \sum_{m=2}^{+\infty} \underline{\underline{\theta}}^{(m)}(\rho, \rho', n, h), \tag{35}$$

$$\underline{\underline{\varepsilon}}^{(e)}(\rho, \rho', n, h) - \underline{\underline{\varepsilon}}(\rho) \frac{\delta(\rho - \rho')}{\rho'} \equiv \underline{\underline{\nu}}(\rho, \rho', n, h) = \sum_{m=2}^{+\infty} \underline{\underline{\nu}}^{(m)}(\rho, \rho', n, h). \tag{36}$$

so that a perturbational scheme for determining the spectral-domain EPO is thereby completed.

On recalling the definition of $\underline{\underline{\xi}}$ given by Eqs. (19), (20) we find that the requirement $\langle \underline{\underline{\xi}} \rangle = 0$ allows an explicit determination of the permittivity of the background medium,

$$\frac{1}{\varepsilon_{\|}(\rho)} = \left\langle \frac{1}{\varepsilon^{(r)}(x)} \right\rangle, \quad \varepsilon_{\perp}(\rho) = \left\langle \varepsilon^{(r)}(x) \right\rangle. \tag{37}$$

This is to be contrasted to the strong fluctuation approach of [3–7] where the background medium's permittivity is unavailable in closed form. Eq. (37) clearly evidences that the background medium appears to be anisotropic (namely — locally-uniaxial, with radially-oriented optic axis), since in general $\varepsilon_{\|} \neq \varepsilon_{\perp}$. It is useful to note that the right-hand side in Eq. (37) is independent of φ, z due to the (statistically) cylindrically-layered character of the random medium. The physical contents of $\underline{\underline{\varepsilon}}$ is revealed by Eqs. (32), (36) after noting that the sums on the right determine a purely non-local contribution to the EPO, thus delivering $\underline{\underline{\varepsilon}}$ the meaning of a local component of the EPO.

Having truncated the infinite series (28), (32) or (35), (36), there results a workable approximate representation for $\underline{\underline{\xi}}^{(e)}$, $\underline{\underline{\varepsilon}}^{(e)}$. In particular, retaining there the first nonvanishing term yields the bilocal approximation [2]: $\underline{\underline{\xi}}^{(e)} \simeq \underline{\underline{\theta}}^{(2)}$, $\underline{\underline{\varepsilon}}^{(e)} \simeq \underline{\underline{\varepsilon}} + \underline{\underline{\theta}}^{(2)}$, with the spectral-domain characterization

$$\underline{\underline{\theta}}^{(e)}(\rho,\rho',n,h) \simeq \underline{\underline{\theta}}^{(2)}(\rho,\rho',n,h) =$$

$$(k_0/2\pi)^2 \sum_{n'=-\infty}^{n'=+\infty} \int_{-\infty}^{+\infty} dh' \int_{-\infty}^{+\infty} dz \int_0^{2\pi} d\varphi \exp\{-i[(n-n')\varphi + (h-h')z]\}$$

$$\left\langle \underline{\underline{\xi}}(\rho,\varphi,z) \circ \underline{\underline{G}}'(\rho,\rho',n',h') \circ \underline{\underline{\xi}}(\rho',0,0) \right\rangle, \tag{38}$$

$$\underline{\underline{\nu}}(\rho,\rho',n,h) \simeq \underline{\underline{\xi}}^{(e)}(\rho,\rho',n,h), \tag{39}$$

where a designation $\underline{\underline{\xi}}(\rho,\varphi,z) \equiv \underline{\underline{\xi}}(x)$ is used. Representation (38) is valid under assumption that the singularities of $\underline{\underline{G}}'(\rho,\rho',n',h')$ as function of complex variable h' [which coincide with those of $\underline{\underline{G}}'(\rho,\rho',n',h')$ are shifted off the real-axis path of integration into the complex h'-plane, e.g., by introducing infinitesimal losses into an originally lossless background medium. Explicit formulae for the components of matrix $\underline{\underline{\theta}}^{(2)}(\rho,\rho',n,h)$ are given in Appendix C. We remark in passing that the statistical topology of random perturbations enters the bilocal approximation through spectral densities $B_{uv}(\rho,\rho',n,h)$ which are defined as Fourier transforms,

$$B_{uv}(\rho,\rho',n,h) = (2\pi)^{-2}\mathcal{F}B_{uv}(\rho,\rho',\varphi-\varphi',z-z'), \tag{40}$$

of correlation functions

$$\langle \xi_u(x)\xi_v(x')\rangle = B_{uv}(\rho,\rho',\varphi-\varphi',z-z'), \quad (u,v=\perp,\|). \tag{41}$$

The truncation of series (28), (32), (35) or (36) to a finite number of terms implies that contribution of the terms discarded is negligible. Let us examine legitimacy of this procedure in more detail. For calculational simplicity we shall consider a situation where the random function $\underline{\underline{\xi}}(x)$ is Gaussian. Due to the well known properties [1] of Gaussian random fields the terms in the aforementioned series with m odd vanish, and, in the remaining terms, the statistical moments of random perturbations are expressible through correlation functions (41). At this stage it is convenient to introduce a positive constantσ which is intended to measure the intensity of random perturbations, and to make a further simplifying assumption that the correlation functions are independent of the angular and longitudinal variables and tend to zero sufficiently fast as the distance between ρ and ρ' exceeds l — a characteristic scale of random perturbations in radial direction. This is tantamount to the claim that

$$B_{uv}(\rho, \rho', \varphi - \varphi', z - z') = \sigma^2 \Phi_{uv}[\rho + \rho', (\rho - \rho')/l], \tag{42}$$

where $\Phi_{uv}[\lambda, \mu]$ are the functions of the order of unity which assume zero values when $|\mu| > 1$. Then by recourse to Eqs. (29)–(31) there obtains the following coarse estimate for the non-zero terms in series (35), (36):

$$\begin{aligned} &\underline{\underline{\theta}}^{(m)}(\rho, \rho', n, h) \sim \sigma^m (k_0 l)^{m-2}, \\ &\underline{\underline{\nu}}^{(m)}(\rho, \rho', n, h) \sim \sigma^m (k_0 l)^{m-2}, \end{aligned} \tag{43}$$

$(m = 2, 4, 6, \ldots)$. As a consequence, the series on the right of Eqs. (35), (36) acquire the status of asymptotic expansions in powers of $\sigma^2 (k_0 l)^2$ and thus can be legitimately truncated for sufficiently small values of said parameter,

$$\sigma^2 (k_0 l)^2 \ll 1. \tag{44}$$

It is essential to note that the requirement (44) is met in case of strong fluctuations ($\sigma \gg 1$) provided their characteristic scale in radial direction is small enough, ($k_0 l \ll \sigma^{-2}$).

5. ILLUSTRATIVE EXAMPLES

We here do two things. First, to further our understanding of the multiple scattering phenomena in a statistically inhomogeneous medium under consideration, we analyze, relying on the bilocal approximation, the macroscopic properties of a spatially dispersive medium with the non-local permittivity $\underline{\underline{\varepsilon}}^{(e)}$. Then the role of multiple scattering effects is given a final emphasis by calculating a shift of propagation constants of the guided modes of the mean field. The respective solution refers to a general model of a lossy inhomogeneous medium with strong permittivity fluctuations and allows for complex values of the discrete spectrum. In this regard, it extends the multiple scattering solution of [16] for an optical fiber with the weak fluctuations, and the classical coupled mode theory combined with the small perturbation approach in [17].

On applying expressions (38), (39) and the relationship [12] $\underline{\underline{G}}(x, x') = \underline{\underline{G}}^{\mathrm{T}}(x', x)$ one can readily confirm the equalities

$$\underline{\underline{\varepsilon}}^{(e)}(\rho, \rho', \varphi - \varphi', z - z') = \underline{\underline{\varepsilon}}^{(e)\mathrm{T}}(\rho, \rho', \varphi - \varphi', z - z'), \tag{45}$$

$$\underline{\underline{\varepsilon}}^{(e)}(\rho, \rho', n, h) = \underline{\underline{\varepsilon}}^{(e)\mathrm{T}}(\rho', \rho, -n, -h), \tag{46}$$

which rest on similar relations for the effective perturbation operator and point to a reciprocal character of the spatially dispersive medium [12–18]. Here T denotes the matrix transposition operation. On consulting Eqs. (B14)–(C4), one can readily show that if a waveform (9), with $\underline{E}(\rho) = col[E_1, E_2, E_3]$, $\underline{H}(\rho) = col[H_1, H_2, H_3]$ and given n, h satisfies the source-free ($\underline{J} = 0$) Maxwell's equations (5), (6) then these equations also allow for a solution of the form

$$\begin{aligned} \underline{E}^-(x) &= \underline{E}^-(\rho) \exp[i(n\varphi - hz)], \\ \underline{H}^-(x) &= \underline{H}^-(\rho) \exp[i(n\varphi - hz)], \end{aligned} \tag{47}$$

whence

$$\begin{aligned} \underline{E}^-(\rho) &= col[E_1, E_2, -E_3], \\ \underline{H}^-(\rho) &= col[-H_1, -H_2, H_3], \end{aligned} \tag{48}$$

or, alternatively,

$$\begin{aligned} \tilde{\underline{E}}(x) &= \tilde{\underline{E}}(\rho) \exp[i(-n\varphi + hz)], \\ \tilde{\underline{H}}(x) &= \tilde{\underline{H}}(\rho) \exp[i(-n\varphi + hz)], \end{aligned} \tag{49}$$

with

$$\begin{aligned} \tilde{\underline{E}}(\rho) &= col[E_1, -E_2, E_3], \\ \tilde{\underline{H}}(\rho) &= col[-H_1, H_2, -H_3], \end{aligned} \tag{50}$$

provided all of the correlation functions in Eq. (41) are even functions of variable $z - z'$ or $\varphi - \varphi'$, respectively. Eqs. (47), (49) represent the mean field harmonics which travel in the opposite direction or possess a complex-conjugate azimuthal dependence compared to an original weaveform (9).

In the next instance, we shall examine the dissipative properties of the effective medium which are described by the hermitian operator $\underline{\underline{\varepsilon}}^{(e)\prime\prime}$,

$$\underline{\underline{\varepsilon}}^{(e)\prime\prime} = \left(\underline{\underline{\varepsilon}}^{(e)} - \underline{\underline{\varepsilon}}^{(e)\dagger}\right)/2i. \tag{51}$$

Here the dagger symbolizes the hermitian conjugate of an x-acting operator. The significance of $\underline{\underline{\varepsilon}}^{(e)\prime\prime}$ is uncovered, e.g., by an expression for the time-average power Q dissipated by a passive medium ($Q < 0$) or generated by an active medium ($Q > 0$) with the nonlocal permittivity $\underline{\underline{\varepsilon}}^{(e)} : Q = (\omega/8\pi)\int \underline{E}^{*\mathrm{T}}(x) \circ \underline{\underline{\varepsilon}}^{(e)\prime\prime} \circ \underline{E}(x)dV$, the asterisk denoting a complex conjugate of the function to which it is appended. When the permittivity operator has the form (7) and the EM field is given by Eq. (9), the divergence of an integral over z in the appertaining expression for $Q = (\omega/8\pi)\int_{-\infty}^{+\infty} dz \int_0^{2\pi} d\varphi q$ signals a preferable status of the density quantity

$$q = \int_0^{+\infty} \rho d\rho \int_0^{+\infty} \rho' d\rho' \underline{E}^{*\mathrm{T}}(\rho) \circ \underline{\underline{\varepsilon}}^{(e)\prime\prime}(\rho, \rho', n, h) \circ \underline{E}(\rho') \tag{52}$$

and a kernel $\underline{\underline{\varepsilon}}^{(e)\prime\prime}(\rho, \rho', n, h)$ of the spectral-domain counterpart of $\underline{\underline{\varepsilon}}^{(e)\prime\prime}$,

$$\underline{\underline{\varepsilon}}^{(e)\prime\prime}(\rho, \rho', n, h) = \left[\underline{\underline{\varepsilon}}^{(e)}(\rho, \rho', n, h) - \underline{\underline{\varepsilon}}^{(e)*\mathrm{T}}(\rho, \rho', n, h)\right]/2i, \tag{53}$$

as the energetic characteristics. Unless otherwise stated, parameter h assumes here the real values. We focus our attention on the case where the random medium is free of dissipative losses and is characterized with a positive permittivity function $\varepsilon^{(r)}$. Also, to afford a relative simplicity in calculations without incurring any significant loss of generality it is further assumed that the random medium's fluctuations are confined to a region $0 < \rho < \rho_0$, $(\rho_0 = const)$, and the deterministic region $\rho_0 < \rho < +\infty$ is homogeneous with permittivity ε. On this basis, it follows from Eq. (37) that the background medium is characterized by real positive functions $\varepsilon_{\|}(\rho), \varepsilon_{\perp}(\rho)$ which assume constant positive value ε if $\rho > \rho_0$. Turning to Appendix A, one can consequently deduce that the continuous spectrum Γ_c of Eqs. (A2)–(A5) consists of the imaginary semiaxis from $\kappa = +i\infty$ to $\kappa = 0$ and an interval on the real axis from $\kappa = 0$ to $\kappa = k$, $(k = k_0\sqrt{\varepsilon})$. The latter interval together with the discrete spectrum points on the real axis [which may occur to the right from the point $\kappa = k$] constitute the real-valued part $\Gamma_p(n)$ of spectrum that corresponds to the propagating eigenmodes (of given n) in the background medium.

Eqs. (38), (39) make plain that in the present circumstances $\underline{\underline{\varepsilon}}^{(e)\prime\prime}$ coincides with $\underline{\underline{\xi}}^{(e)\prime\prime}$, the latter operator defined in full analogy with Eq. (51). It suffices accordingly to calculate $\underline{\underline{\xi}}^{(e)}(\rho, \rho', n, h)$ and then find the desired quantity $\underline{\underline{\varepsilon}}^{(e)\prime\prime}(\rho, \rho', n, h)$ via formula similar to Eq. (53). To effect the passage to a lossless background medium in the real-axis integral over h' which figures on the right of Eq. (38), it is expedient to express $\underline{\underline{G}}'(\rho, \rho', n', h')$ in terms of $\underline{\underline{G}}(\rho, \rho', n', h')$ via Eqs. (15), (A11), and displace the contour of integration by an infinitesimal shift from the real axis into the second and fourth quadrants in the complex h'-plane to avoid integration through the singularities of $\underline{\underline{G}}(\rho, \rho', n', h')$ contributed by $\Gamma_p(n')$. If one now lets the losses vanish, it follows that the new path of integration should be indented around the poles of $\underline{\underline{G}}(\rho, \rho', n', h')$ on the real axis into the second and fourth quadrants and pass along the right side of the parts $-k < h' < 0,\ 0 < h' < k$ of the respective branch cuts Γ_{sc}, Γ_c. As regards the computation of the new contour integral, the portion over a semicircular arc of the circle of an infinitesimal small radius about each pole on the real axis is expressed through the respective residues given by Eq. (A12), and the limiting value of $\underline{\underline{G}}(\rho, \rho', n', h')$ on the right side of the branch cuts are determined by Eq. (A13). On using the resultant representation of $\underline{\underline{\xi}}^{(e)}(\rho, \rho', n, h)$ for the calculation of $\underline{\underline{\xi}}^{(e)\prime\prime}(\rho, \rho', n, h)$ we obtain

$$\begin{aligned}\underline{\underline{\varepsilon}}^{(e)\prime\prime}(\rho, \rho', n, h) &= \underline{\underline{\xi}}^{(e)\prime\prime}(\rho, \rho', n, h) \\ &= \sum_{n'=-\infty}^{n'=+\infty} \Sigma^{p}_{\kappa'} \left[\underline{\underline{\sigma}}^{+}(\rho, \rho', n, h|n', \kappa') + \underline{\underline{\sigma}}^{-}(\rho, \rho', n, h|n', \kappa')\right],\end{aligned} \tag{54}$$

with

$$\begin{aligned}&\underline{\underline{\sigma}}^{\pm}(\rho, \rho', n, h|n', \kappa') \\ &= \frac{k_0}{8\pi P(n', \kappa')} \int_{-\infty}^{+\infty} dz \int_0^{2\pi} d\varphi \exp\{-i[(n-n')\varphi + (h \mp \kappa')z]\} \\ &\quad \Big\langle \underline{\underline{\xi}}(\rho, \varphi, z) \circ [\underline{E}_t(\rho, n', \kappa') \pm \underline{E}_l(\rho, n', \kappa')] \\ &\quad \circ [\underline{E}_t(\rho', n', \kappa') \pm \underline{E}_l(\rho', n', \kappa')]^{*\mathrm{T}} \circ \underline{\underline{\xi}}(\rho', 0, 0) \Big\rangle .\end{aligned} \tag{55}$$

Here κ' stands for an eigenvalue of a problem obtainable from Eqs. (A2)–(A5) after replacing n with n', $\Sigma^{p}_{\kappa'}$, denotes "summation" over spectral points in $\Gamma_p(n')$, i.e., the standard summation over the real-vaIued points of discrete spectrum $\Gamma_d(n')$ and the operation $\sum_{j=a,b} \int_0^k d\kappa' \ldots$ referring to a (degenerate) part $0 < \kappa' < k$ of the continuous spectrum, $\underline{E}_t$ and $\underline{E}_l$ determine the transverse and the longitudinal constituents of an eigenmode in the background medium, P is the normalization constant (see Appendix A for more details). In the derivation of Eqs. (54), (55) we are aided by taking note of the fact that in the case under consideration the eigensolutions to Eqs. (A2)–(A5) belonging to real (and consequently positive) eigevalues can always be chosen in such a way that the components E_1, H_2, H_3 are real, E_2, E_3, H_1 are purely imaginary, and the quantity P is real and positive. Accounting for representations (54), (55) in Eq. (52) shows that $q \geq 0$, as should be for a passive medium.

Eqs. (54), (55) serve in a useful capacity to highlight a physical origin of "dissipative" losses in the effective medium. According to these equations, the quantity $\underline{\underline{\varepsilon}}^{(e)\prime\prime}(\rho, \rho', n, h)$ which is responsible for said losses is formed as "sum" of contributions $\underline{\underline{\sigma}}^{\pm}(\rho, \rho', n, h|n', \kappa')$ due to scattering into the eigenwaves in the background medium with different n', κ' which propagate in the positive (+) and negative (−) z-directions. The absence of contributions due to evanescent modes in Eq. (54) is in keeping with common sense, for these waves do not transfer energy in the longitudinal direction and consequently cause no attenuation of the mean field in the process of scattering.

We will now discuss another feature of wave propagation through a statistically inhomogeneous medium, namely the distortion of the guided, or discrete spectrum modes. For brevity, we shall concentrate on only one aspect of said distortion concerned with the propagation constants of the guided modes of the mean field. For this, we return to a general model of a lossy random medium and consider a sourceless waveform (9) of the mean field with fixed n and arbitrary complex h. It will determine a guided mode of the mean field if its spectral amplitudes $\underline{E}(\rho), \underline{H}(\rho)$ multiplied by factor $\rho^{1/2}$ tend to zero sufficiently fast as $\rho \to +\infty$. Following the substitution of Eq. (9) into source-free Maxwell's equations (5), (6) and the introduction of column matrix $\underline{s}(\rho)$,

$$\underline{s}(\rho) \equiv \int_0^{+\infty} \underline{\underline{\nu}}(\rho, \rho', n, h) \circ \underline{E}(\rho')\rho' d\rho', \tag{56}$$

where $\underline{\underline{\nu}}(\rho, \rho', n, h)$ is defined by Eq. (36), the arising equations for the components of $\underline{E}(\rho), \underline{H}(\rho)$ can be recast in the form which is obtainable from Eqs. (A2)–(A5) after replacing there κ with h and adding in their right-hand sides the terms $ik_0 s_2, -ik_0 s_1,\ -(\partial/\partial\rho)(s_3/\varepsilon_\perp),\ -ins_3/\rho\varepsilon_\perp$ and $s_3/\varepsilon_\perp$ respectively. We now have everything to derive a standard formula relating the discrete spectrum propagation constants h, κ of two arbitrarily chosen guided modes (with the same azimuthal number n) in the effective and background media. Taking as a specimen the calculations in [16], we find a rigorous formula

$$h = \pm(\kappa + k_0 q'/2p'), \tag{57}$$

where

$$q' = \int_0^{+\infty} \rho d\rho \int_0^{+\infty} \rho' d\rho' \left[\underline{\tilde{E}}_t(\rho, n, \kappa) \mp \underline{E}_l(\rho, n, \kappa)\right]^{\mathrm{T}} \circ \underline{\underline{\nu}}(\rho, \rho', n, h) \circ \underline{E}(\rho'), \tag{58}$$

$$2p' = \int_0^{+\infty} \rho d\rho [E_1(\rho) H_2(\rho, n, \kappa) + E_2(\rho) H_1(\rho, n, \kappa) \\ \pm E_1(\rho, n, \kappa) H_2(\rho) \pm E_2(\rho, n, \kappa) H_1(\rho)]. \tag{59}$$

The quantity $\underline{\tilde{E}}_t(\rho, n, \kappa)$ which figures in Eq. (58) is defined after Eq. (A11). The convergence of the integral in Eq. (59) on the upper limit is secured by vanishing of the integrand at infinity. This implies that neither of the guided modes involved is in the cut-off regime. We recall that, according to a convention in Appendix A, there should be $0 \le arg\kappa < \pi$. Said convention, however, does not apply to the quantity h.

Within the framework of the renormalization approach the first-order effect of strong permittivity fluctuations is absorbed in the permittivity of the background medium defined by Eq. (37). Treating the quantity $\underline{\underline{\nu}}$ as small parameter, let choose the mean-field guided mode (9) in such a way that $\underline{E}(x) \to \underline{E}^{\pm}(x|n, \kappa)$, $\underline{H}(x) \to \underline{H}^{\pm}(x|n, \kappa)$, and $h \to \pm\kappa$ when $\underline{\underline{\nu}} \to 0$. Here the quantities $\underline{E}^{\pm}(x|n, \kappa)$, $\underline{H}^{\pm}(x|n, \kappa)$ determine a guided mode in the background medium which travels in the $+z(-z)$ direction — see Eq. (A8) for explicit characterization; the upper or lower signs should be selected simultaneously. In establishing the aforementioned relationship we utilize the fact that the discrete spectrum eigenvalues of the unperturbed problem (A2)–(A5) are non-degenerate. The implementation of perturbation approach in Eqs. (58), (59), as well as the employment of Eq. (39) leads to a second, approximate formula

$$\delta h^{\pm} \equiv h \mp \kappa \simeq \pm k_0 q'' / 2P(n, \kappa), \tag{60}$$

for a shift of propagation constant of a guided mode progressing along the z-axis (δh^+) or in the opposite direction (δh^-). In this formula,

$$q'' = \int_0^{+\infty} \rho d\rho \int_0^{+\infty} \rho' d\rho' \left[\underline{\tilde{E}}_t(\rho, n, \kappa) \mp \underline{E}_l(\rho, n, \kappa)\right]^{\mathrm{T}} \\ \circ \underline{\underline{\xi}}^{(e)}(\rho, \rho', n, \pm\kappa) \circ [\underline{E}_t(\rho, n, \kappa) \pm \underline{E}_l(\rho, n, \kappa)], \tag{61}$$

$$P(n, \kappa) = \int_0^{+\infty} [E_1(\rho, n, \kappa) H_2(\rho, n, \kappa) + E_2(\rho, n, \kappa) H_1(\rho, n, \kappa)] \rho d\rho. \tag{62}$$

The quantities $\Re e \delta h^{\pm}$ define the displacement of the phase velocity, and $\Im m e \delta h^{\pm}$ that of attenuation of the eigenmode. Reference to expressions (B14)–(C4) shows that, in the situation in which Eqs. (47), (48) hold, the following relation takes place: $\delta h^- = -\delta h^+$, i.e., the shifts $\delta h^{\pm}$ are different as regards the sense of propagation only. For a more general situation however $\delta h^- \neq -\delta h^+$. In the case of a lossless random medium which is described by Eqs. (54), (55), for real κ one can replace in Eq. (61) $\left[\underline{\tilde{E}}_t(\rho,n,\kappa) \mp \underline{E}(\rho,n,\kappa)\right]^{\mathrm{T}}$ with $[\underline{E}_t(\rho,n,\kappa) \pm \underline{E}_l(\rho,n,\kappa)]^{*\mathrm{T}}$, and the quantity $P(n,\kappa)$ altered by a factor $c/8\pi$ can be identified with the total energy flux of the guided mode through the transverse cross section of a background medium in the direction of propagation which is obviously a positive value. Thus one gets from Eq. (61) $\Im me\delta h^{\pm} = \pm k_0 \Im m q''/2P(n,\kappa)$, where

$$\Im m q'' = \int_0^{+\infty} \rho' d\rho' \, [\underline{E}_t(\rho,n,\kappa) \pm \underline{E}_l(\rho,n,\kappa)]^{*\mathrm{T}} \circ \underline{\underline{\xi}}^{(e)\prime\prime}(\rho,\rho',n,\pm\kappa) \circ [\underline{E}_t(\rho,n,\kappa) \pm \underline{E}_l(\rho,n,\kappa)] > 0. \quad (63)$$

This in conjunction with expressions (54), (55) leads to inequalities $\Im m\delta h^+ > 0$, $\Im m\delta h^- < 0$ indicative of attenuation of guided modes of the mean field in a lossless random medium due to scattering losses. It is interesting to note that in the weak fluctuation case when one may disregard the difference between the permittivity of the background medium and the original medium in absence of fluctuations i.e., the difference between $1/\langle\varepsilon^{(r)}\rangle$ and $\langle 1/\varepsilon^{(r)}\rangle$, and the difference between the random quantities $\underline{\underline{\xi}}$ and $(\varepsilon^{(r)} - \langle\varepsilon^{(r)}\rangle)\underline{\underline{I}}$ the resulting expressions for $\Im m\delta h^{\pm}$ can be arrived at via a coupled mode approach of [17]. This clearly extends the applicability of the coupled mode calculations of the attenuation in lossless weakly fluctuating media to the case where the multiple scattering effects play an essential role. We conclude this Section with a remark that the presence of dissipative losses in a random medium should complicate the manifestation of multiple scattering effects, especially in the strong fluctuation case, and might lead, under certain circumstances, to a decrease in attenuation ($\Im m\delta h^+$ or $-\Im m\delta h^- < 0$). However, this point needs separate consideration.

6. CONCLUSIONS

In the paper, a strong fluctuation approach for the mean EM field is developed referring to a lossy electrically isotropic random medium whose physical and probabilistic properties possess rotational and translational symmetry with respect to a fixed axis, and vary in arbitrary manner with the distance from this axis. The main emphasis of the approach is devoted to the elimination of secular terms in the spectral representation of the EPO in the basis set of waves associated with the aforementioned type of random media which is completely different from a plane-wave basis set relevant to statistically homogeneous media. The aforesaid problem is resolved by invoking a renormalized integral equation of scattering whose kernel has been obtained by extracting a delta function singularity from the spectral Green's function in the background medium. As a consequence of such renormalization, the arising perturbation series solution for the EPO acquires, in the spectral domain, the status of an asymptotic expansion in powers of a classical small parameter of the strong fluctuation theory which does not presuppose the weakness of permittivity fluctuations. The background medium appears to be spatially inhomogeneous and electrically anisotropic, and its permittivity matrix is found in the explicit form in contrast to the existing strong fluctuation theories for statistically homogeneous media. Within the framework of a bilocal approximation, the basic physical properties of the effective medium are interpreted, and a distortion of the guided mode spectrum of the mean field is analyzed.

ACKNOWLEDGMENT

This research was supported in part by the Alexander von Humboldt Foundation through a research fellowship to N. P. Zhuk.

APPENDIX A.

A spectral expansion for $\underline{\underline{G}}(\rho, \rho', n, h)$ over the eigenfunctions pertaining to a background medium is presented here.

First, following the well known approach toward spectral problems in open regions [12–14] we introduce suitable transverse eigenfunctions

$$\begin{aligned} \underline{E}_t(\rho, n, \kappa) &= col[E_1(\rho, n, \kappa), E_2(\rho, n, \kappa), 0], \\ \underline{H}_t(\rho, n, \kappa) &= col[H_1(\rho, n, \kappa), H_2(\rho, n, \kappa), 0] \end{aligned} \tag{A1}$$

for the simultaneous equations

$$-\frac{\partial}{\partial\rho}\frac{n}{k_0\rho}E_1 + \left[\frac{\partial}{\partial\rho}\frac{1}{ik_0\rho}\frac{\partial}{\partial\rho}\rho - ik_0\varepsilon_\perp(\rho)\right]E_2 = i\kappa H_1, \tag{A2}$$

$$i\left[k_0\varepsilon_\|(\rho) - \frac{n^2}{k_0\rho^2}\right]E_1 + \frac{n}{k_0\rho^2}\frac{\partial}{\partial\rho}\rho E_2 = i\kappa H_2, \tag{A3}$$

$$-\frac{\partial}{\partial\rho}\frac{n}{k_0\rho\varepsilon_\perp(\rho)}H_1 + \left[\frac{\partial}{\partial\rho}\frac{1}{ik_0\rho\varepsilon_\perp(\rho)}\frac{\partial}{\partial\rho}\rho - ik_0\right]H_2 = -i\kappa E_1, \tag{A4}$$

$$i\left[k_0 - \frac{n^2}{k_0\rho^2\varepsilon_\perp(\rho)}\right]E_1 + \frac{n}{k_0\rho^2\varepsilon_\perp(\rho)}\frac{\partial}{\partial\rho}\rho H_2 = -i\kappa E_2, \tag{A5}$$

that satisfy the condition of continuity of E_2, H_2 and of the quantities

$$E_3(\rho, n, \kappa) \equiv \frac{1}{ik_0\rho\varepsilon_\perp(\rho)}\left[inH_1(\rho, n, \kappa) - \frac{\partial}{\partial\rho}\rho H_2(\rho, n, \kappa)\right] \quad \text{(A6)}$$

$$H_3(\rho, n, \kappa) \equiv \frac{1}{ik_0\rho}\left[\frac{\partial}{\partial\rho}\rho E_2(\rho, n, \kappa) - inE_1(\rho, n, \kappa)\right], \quad \text{(A7)}$$

are bounded at $\rho = 0$ and when multiplied by $\rho^{1/2}$ remain finite as $\rho \to +\infty$. Here κ is an eigenvalue, and n is an arbitrary integer which remains fixed throughout this Appendix. In subsequent considerations the eigenvalues of Eqs. (A2)–(A5) will be identified with the respective points in the complex h-plane. It can be easily seen that if $\underline{E}_t$, $\underline{H}_t$ is a solution to the above problem related to the eigenvalue κ then $\underline{E}_t$, $-\underline{H}_t$ will form a solution associated with an eigenvalue $-\kappa$. For our purposes it suffices accordingly to restrict the admissible set of eigenvalues by the condition that κ belong to a complex set $D = \{h : 0 \le \arg h < \pi\}$. The physical meaning of the eigenfunctions (A1) is revealed after noting that

$$\begin{aligned} \underline{E}^{\pm}(x|n, \kappa) &= [\underline{E}_t(\rho, n, \kappa) \pm \underline{E}_l(\rho, n, \kappa)] \exp[i(n\varphi + \kappa z)], \\ \underline{H}^{\pm}(x|n, \kappa) &= [\pm\underline{H}_t(\rho, n, \kappa) + \underline{H}_l(\rho, n, \kappa)] \exp[i(n\varphi + \kappa z)] \end{aligned} \quad \text{(A8)}$$

satisfy source-free Maxwell's equations for the background medium and thus represent the fields of eigenmodes progressing (or evanescent) in the $+z$ and $-z$ directions with the propagation constant κ. Here $\underline{A}_l = A_3\underline{e}_l$, $(A = E, H)$, and $\underline{e}_l$ stands for a matrix $col[0, 0, 1]$.

The spectrum we seek, Γ, has the discrete part $\Gamma_d(c)$ which consists of finite number of points $\kappa = \kappa_q(n)$, $q = 1, 2, ...Q(n)$, and gives rise to guided modes, and the continuous part Γ_c associated with the radiation modes [12–14]. It is now assumed for the sake of simplicity that $\varepsilon_{||}(\rho), \varepsilon_\perp(\rho)$ take constant value ε if ρ is sufficiently large say $\rho > \rho_0$. This requirement suggests, via Eq. (37), that the random medium fluctuations are constrained to the region $0 \le \rho < \rho_0$ whilst the outer space $\rho > \rho_0$ is deterministic and homogeneous. In this circumstance Γ_c comprises the roots of equation $\Im m(k_0^2\varepsilon - h^2)^{1/2} = 0$ in D. Anticipating further needs it is convenient to view Γ_c as oriented contour drawn from infinity to the origin. In what follows the discrete spectrum points κ_q [for a fixed n] are treated as nondegenerate, since otherwise the degeneracy can be escaped by introducing a proper infinitesimal perturbation into the background medium. In contrast to this, each κ from the continuum Γ_c is associated with two linearly independent eigensolutions which we denote as $\underline{E}_t^{(a)}, \underline{H}_t^{(a)}$ and $\underline{E}_t^{(b)}, \underline{H}_t^{(b)}$. By invoking an orthogonalization procedure, it is always possible to render them mutually orthogonal, i.e., secure the property

$$\ll \underline{E}_t^{(a)}(\rho, n, \kappa), \underline{H}_t^{(b)}(\rho, n, \kappa') \gg \tag{A9}$$
$$\equiv \int_0^{+\infty} \Big[E_1^{(a)}(\rho, n, \kappa) H_2^{(b)}(\rho, n, \kappa') + E_2^{(a)}(\rho, n, \kappa) H_1^{(b)}(\rho, n, \kappa') \Big] \rho d\rho = 0,$$

for all κ, $\kappa' \in \Gamma_c$. Then it is easy to verify, by utilizing standard techniques [12–14] that the eigenfunctions (A1) satisfy the orthogonality conditions

$$\begin{aligned} \ll \underline{E}_t(\rho, n, \kappa), \underline{H}_t(\rho, n, \kappa') \gg \; &= \; P(n, \kappa)\delta_{\kappa\kappa'}, (\kappa, \kappa' \in \Gamma_d), \\ \ll \underline{E}_t(\rho, n, \kappa), \underline{H}_t(\rho, n, \kappa') \gg \; &= \; 0, \quad (\kappa \in \Gamma_d, \; \kappa' \in \Gamma_c), \\ \ll \underline{E}_t^{(j)}(\rho, n, \kappa), \underline{H}_t^{(j)}(\rho, n, \kappa') \gg \; &= \; P^{(j)}(n, \kappa)\delta(\kappa - \kappa')_{|\Gamma_c}, \\ &\qquad (j = a, b; \kappa, \kappa' \in \Gamma_c). \end{aligned} \tag{A10}$$

Here $\delta_{\kappa\kappa'}$, is Kronecker's delta, $\delta_{|\Gamma_c}$ is complex Dirac delta function [15] associated with contour Γ_c, and $P, P^{(j)}$ are the normalization constants.

If we now Fourier transform Eq. (13) and eliminate $\underline{e}_l \circ \underline{\underline{G}}(\rho, \rho', n, h)$ in favour of $(\underline{\underline{I}} - \underline{e}_l \circ \underline{e}_l^{\mathrm{T}} \circ \underline{\underline{G}}(\rho, \rho', n, h)$ the resultant equation lends itself pretty easy to analytic solution via eigenfunction expansion technique involving functional set (A1). Subsequent reconstruction of the spectral GF yields the representation that we are seeking:

$$\underline{\underline{G}}(\rho, \rho', n, h) = -\underline{\underline{e}}_l \circ \underline{\underline{e}}_l^{\mathrm{T}} \frac{\delta(\rho - \rho')}{k_0^2 \rho' \varepsilon_\perp(\rho')}$$
$$+ \sum_k \frac{1}{k_0 P(\kappa, n)(h^2 - \kappa^2)} \Big\{ \underline{E}_t(\rho, n, \kappa) \circ \Big[\kappa \underline{\tilde{E}}_t(\rho', n, \kappa) - h \underline{E}_l(\rho', n, \kappa) \Big]^{\mathrm{T}}$$
$$+ \underline{E}_l(\rho, n, \kappa) \circ \Big[h \underline{\tilde{E}}_t(\rho', n, \kappa) - \kappa \underline{E}_l(\rho', n, \kappa) \Big] \Big\}. \tag{A11}$$

Here $\underline{\tilde{E}}_t(\rho, n, \kappa)$ is obtainable from $\underline{E}_t(\rho, n, \kappa)$ by replacing in Eq. (A1) E_2 with $-E_2$, T denotes the matrix transposition operation, Σ_k indicates the standard summation over the discrete spectrum points κ_q and integration along contour Γ (with contribution of both linear independent eigenfunctions included), the respective superscripts $j = a, b$ are dropped to save on notation. It should be remarked for the sake of clarity that expansion (A11) rests upon a premise that neither h not $-h$ belong to the spectral set Γ. It appears on taking note of Eq. (15) and the results in Appendix B that the "sum" on the right of Eq. (A11) represents a generalized function of variables ρ, ρ' which consists of an integrable function $\underline{\underline{G}}'(\rho, \rho', n, h)$ and a singular part $\delta(\rho - \rho')[\underline{e}_l \circ \underline{e}_l^{\mathrm{T}} \varepsilon_\perp^{-1}(\rho') - \underline{\underline{I}}_{\|} \varepsilon_{\|}^{-1}(\rho')]/k_0^2 \rho'$.

The display (A11) makes clear that a discrete spectrum eigenvalue κ_q of Eqs. (A2)–(A5) bears responsibility for the occurrence of simple poles $h = \pm\kappa_q$ of spectral GF in the complex h-plane, with the concomitant residues

$$res\underline{\underline{G}}(\rho, \rho', n, h) = \pm \frac{1}{2k_0 P(n, \kappa_q)} [\underline{E}_t(\rho, n, \kappa_q) \pm \underline{E}_l(\rho, n, \kappa_q)] \circ [\underline{\tilde{E}}_t(\rho', n, \kappa_q) \mp \underline{E}_l(\rho', n, \kappa_q)]^{\mathrm{T}}. \quad \text{(A12)}$$

Also, from the theory of the integrals of Cauchy type [19] one can infer that the continuous spectrum contribution endows $\underline{\underline{G}}(\rho, \rho', n, h)$ with two branch points $h = \pm k_0\sqrt{\varepsilon}$ and the branch cuts along the line $\Im m(k_0^2 \varepsilon - h^2)^{1/2} = 0$. The latter consists of contour Γ_c and another contour Γ_{sc} which is symmetric to Γ_c with respect to the point $h = 0$. When the Plemelj-Sokhotskii theorem [19] is applied to calculate a limiting value of spectral GF as h approaches an arbitrary point h' on Γ_c (Γ_{sc}) it turns out that

$$\underline{\underline{G}}(\rho, \rho', n, h') = \underline{\underline{G}}^{(pv)}(\rho, \rho', n, h') + \sum_{j=a,b} \frac{\pi i}{2k_0 P^{(j)}(n, \pm h')} \left[\underline{E}_t^{(j)}(\rho, n, \pm h') + \underline{E}_l^{(j)}(\rho, n, \pm h')\right] \circ \left[\underline{\tilde{E}}_t^{(j)}(\rho', n, \pm h') \mp \underline{E}_l^{(j)}(\rho', n, \pm h')\right]^{\mathrm{T}}, \quad \text{(A13)}$$

if the passage $h \to h'$ is made along a path lying to the right of the oriented contour Γ_c (Γ_{sc}). The notation $\underline{\underline{G}}^{(pv)}(\rho, \rho', n, h')$ denotes the quantity which follows from Eq. (A8) if we adopt the Cauchy principal value of the integral along Γ_c, in which the integrand has a pole on the path of integration contributed by the denominator $h'^2 - \kappa^2$.

APPENDIX B.

Here we derive a regularized representation for the spectral Green's matrix $\underline{\underline{G}}(\rho, \rho', n, h)$ as generalized function of ρ, ρ'.

It may be directly verified that the expression obtainable via scalarization technique [12]

$$\underline{\underline{G}}(\rho, \rho', n, h) = -\underline{\underline{A}}(\rho') \frac{\delta(\rho - \rho')}{\rho'} + k_0^{-1} \left[\underline{v} \circ \left(\underline{\tilde{v}}^{\mathrm{T}} G_{ee} - \underline{\tilde{w}}^{\mathrm{T}} G_{mm}\right) + \underline{w} \circ \left(\underline{\tilde{w}}^{\mathrm{T}} G_{mm} - \underline{\tilde{v}}^{\mathrm{T}} G_{me}\right)\right] \quad \text{(B1)}$$

meets all the stipulations laid down for the spectral-domain GF, including the Fourier-transformed version of Eq. (13), provided the scalar GFs $G_{\alpha\beta} \equiv G_{\alpha\beta}(\rho, \rho', n, h)$, $(\alpha, \beta = e, m)$, satisfy a coupled system of equations

$$\begin{bmatrix} D_{11} & -D_{12} \\ D_{21} & D_{22} \end{bmatrix} \circ \begin{bmatrix} G_{ee} & G_{me} \\ G_{em} & G_{mm} \end{bmatrix} = \frac{\delta(\rho - \rho')}{\rho} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{B2}$$

remain bounded as $\rho \to +\infty$, preserve continuity together with the quantities

$$\begin{aligned} &\frac{1}{\chi_1(\rho,h)}\left[\frac{nh}{\rho}G_{m\beta} - ik_0\varepsilon_{\|}(\rho)\frac{\partial G_{e\beta}}{\partial \rho}\right], \\ &\frac{1}{\chi_2(\rho,h)}\left(\frac{nh}{\rho}G_{e\beta} + ik_0\frac{\partial G_{m\beta}}{\partial \rho}\right), \quad (\beta = e, m), \end{aligned} \tag{B3}$$

at the interfaces of the background medium, and demonstrate an outgoing wave behaviour with $\rho \to +\infty$. In these relations,

$$\underline{\underline{A}}(\rho') = diag\left[\frac{1}{\chi_2(\rho',h)}, \frac{1}{\chi_1(\rho',h)}, 0\right] \tag{B4}$$

$$\chi_1(\rho,h) = k_0^2\varepsilon_{\|}(\rho) - h^2, \quad \chi_2(\rho,h) = k_0^2\varepsilon_{\perp}(\rho) - h^2, \tag{B5}$$

$\underline{v}, \underline{w}$ and $\underline{\tilde{v}}, \underline{\tilde{w}}$ are matrix differential operators which act on variables ρ and ρ', respectively:

$$\underline{v} = col\left[\frac{ih}{\chi_1(\rho,h)}\frac{\partial}{\partial\rho}, -\frac{nh}{\rho\chi_2(\rho,h)}, 1\right], \tag{B6}$$

$$\underline{w} = col\left[\frac{nk_0}{\rho\chi_1(\rho,h)}, -\frac{ik_0}{\chi_2(\rho,h)}\frac{\partial}{\partial\rho}, 0\right], \tag{B7}$$

$$\underline{\tilde{v}} = col\left[\frac{ih}{\chi_1(\rho',h)}\frac{\partial}{\partial\rho'}, \frac{nh}{\rho'\chi_2(\rho',h)}, -1\right], \tag{B8}$$

$$\underline{\tilde{w}} = col\left[-\frac{nk_0}{\rho'\chi_1(\rho',h)}, \frac{ik_0}{\chi_2(\rho',h)}\frac{\partial}{\partial\rho'}, 0\right], \tag{B9}$$

(B10)

and D_{jk}, $(j, k = 1, 2)$, are scalar differential operators acting on ρ:

$$k_0^{-1}D_{11} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\frac{\rho\varepsilon_{\|}(\rho)}{\chi_1(\rho,h)}\frac{\partial}{\partial\rho} - \frac{n^2\varepsilon_{\perp}(\rho)}{\rho^2\chi_2(\rho,h)} + \varepsilon_{\perp}(\rho), \tag{B11}$$

$$D_{12} = \frac{inh}{\rho}\left[\frac{1}{\chi_2(\rho,h)}\frac{\partial}{\partial\rho} - \frac{\partial}{\partial\rho}\frac{1}{\chi_1(\rho,h)}\right], \tag{B12}$$

$$D_{11} \to D_{22},\ D_{12} \to D_{21},\ (\varepsilon_{\|,\perp} \to 1,\ \chi_1 \leftrightarrow \chi_2). \tag{B13}$$

Physically, $-iG_{e\beta}$ and $-iG_{m\beta}$ give, respectively, the z components of the spectral electric and magnetic field amplitudes created in the background medium by spatially harmonic impressed sources of the electric ($\beta = e$) or magnetic ($\beta = m$) type with zero transversal components and the z component equal to $\exp[i(n\varphi + hz)]\delta(\rho - \rho')/\rho$.

Due to presence of a Dirac delta function in the right-hand side of Eq. (B2) the quantities $G_{\alpha\beta}$, $\underline{\underline{G}}$ as well as the differential operations which figure on the right of Eq. (B1) should be properly interpreted in the spirit of the generalized function theory [15]. The latter point is only essential with regard to the second mixed derivatives $\partial^2 G_{ee}/\partial\rho\partial\rho'$ and $\partial^2 G_{mm}/\partial\rho\partial\rho'$. Each of them contains the usual derivative signified by proper index and the Dirac delta function, e.g.,

$$\frac{\partial^2 G_{ee}(\rho, \rho', n, h)}{\partial\rho\partial\rho'}_{|gen} = \frac{\partial^2 G_{ee}(\rho, \rho', n, h)}{\partial\rho\partial\rho'}_{|usu} - \delta(\rho - \rho')\frac{\chi_1(\rho', h)}{k_0\rho'\varepsilon_{\|}(\rho')} \tag{B14}$$

(similar formula for G_{mm} which follows from Eq. (B13) after replacing $\varepsilon_{\|} \to 1$, $\chi_1 \to \chi_2$, is omitted). This claim is verified by applying a well-known method of differentiating a function with a step-like discontinuity [15] to $\partial G_{ee}/\partial\rho$, $\partial G_{mm}/\partial\rho$ characterized by such a discontinuity at $\rho' = \rho$. Said discontinuity can be found explicitly by integrating Eq. (B2) over ρ within an infinitely small interval containing ρ'. On collecting delta functions in Eq. (B1), one arrives at the regularized representation (15). The regular constituent $\underline{\underline{G}}(\rho, \rho', n, h)$ is formally obtainable from the expression (B1) after letting out the first term with the Dirac delta and replacing, in the remaining terms, the generalized derivatives with usual ones. Evidently, the result will be a finite-valued integrable function of each variable ρ, ρ' which has a discontinuity at $\rho = \rho'$. This should be contrasted to the case of spatial GF where the regular, or principal-value constituent appears to be a non-integrable function with a polar irregularity, and is defined non-uniquely — see, e.g., [3–6]

APPENDIX C.

The elements $\theta^{(2)}_{pq}(\rho, \rho', n, h) \equiv k_0^{-1}\zeta_{pq}$ of matrix $\underline{\underline{\theta}}^{(2)}(\rho, \rho', n, h)$ are determined by the expressions:

$$\begin{aligned}\zeta_{11} = -\sum_{n'}\int \frac{B_{\|\|\|}}{\chi_1'\chi_1''}\Bigg[&h'^2\frac{\partial^2 G_{ee}}{\partial\rho\partial\rho'} + \frac{n'^2 k_0^2}{\rho\rho'}G_{mm}\\ &+ in'h'k_0\left(\frac{1}{\rho}\frac{\partial G_{me}}{\partial\rho'} - \frac{1}{\rho'}\frac{\partial G_{em}}{\partial\rho}\right)\Bigg]\,dh',\end{aligned}$$

$$\begin{aligned}\zeta_{22} &= -\sum_{n'}\int \frac{B_{\perp\perp}}{\chi_2'\chi_2''}\left[k_0^2\frac{\partial^2 G_{mm}}{\partial\rho\partial\rho'} + \frac{n'^2h'^2}{\rho\rho'}G_{ee}\right.\\ &\quad\left. + in'h'k_0\left(\frac{1}{\rho'}\frac{\partial G_{me}}{\partial\rho} - \frac{1}{\rho}\frac{\partial G_{em}}{\partial\rho}\right)\right]dh',\end{aligned} \tag{C1}$$

$$\begin{aligned}\zeta_{12} &= \sum_{n'}\int \frac{B_{\parallel\perp}}{\chi_1'\chi_2''}\left[k_0h'\frac{\partial^2 G_{em}}{\partial\rho\partial\rho'} - \frac{n'^2k_0h'}{\rho\rho'}G_{me}\right.\\ &\quad\left. + in'\left(\frac{h'^2}{\rho'}\frac{\partial G_{ee}}{\partial\rho} + \frac{k_0^2}{\rho'}\frac{\partial G_{mm}}{\partial\rho}\right)\right]dh',\\ \zeta_{21} &= \sum_{n'}\int \frac{B_{\perp\parallel}}{\chi_2'\chi_1''}\left[k_0h'\frac{\partial^2 G_{me}}{\partial\rho\partial\rho'} - \frac{n'^2k_0h'}{\rho\rho'}G_{em}\right.\\ &\quad\left. - in'\left(\frac{h'^2}{\rho}\frac{\partial G_{ee}}{\partial\rho'} + \frac{k_0^2}{\rho'}\frac{\partial G_{mm}}{\partial\rho}\right)\right]dh',\end{aligned} \tag{C2}$$

$$\begin{aligned}\zeta_{13} &= \sum_{n'}\int \frac{B_{\parallel\perp}}{\chi_1'}\left(ih'\frac{\partial G_{ee}}{\partial\rho'} - \frac{n'k_0}{\rho}G_{me}\right)dh',\\ \zeta_{31} &= -\sum_{n'}\int \frac{B_{\perp\parallel}}{\chi_1''}\left(ih'\frac{\partial G_{ee}}{\partial\rho'} + \frac{n'k_0}{\rho}G_{em}\right)dh',\end{aligned} \tag{C3}$$

$$\begin{aligned}\zeta_{23} &= -\sum_{n'}\int \frac{B_{\perp\perp}}{\chi_2'}\left(ik_0\frac{\partial G_{me}}{\partial\rho} + \frac{n'h'}{\rho}G_{ee}\right)dh',\\ \zeta_{32} &= \sum_{n'}\int \frac{B_{\perp\perp}}{\chi_2''}\left(ik_0\frac{\partial G_{em}}{\partial\rho'} - \frac{n'h'}{\rho'}G_{ee}\right)dh',\end{aligned} \tag{C4}$$

$$\zeta_{33} = -\sum_{n'}\int B_{\perp\perp}G_{ee}dh'. \tag{C5}$$

In these equations, the summation is carried out over all integral n' i.e., $n' = 0, \pm1, \pm2, \ldots$, integration over h' is performed within the ranges $-\infty < h' < +\infty$, the quantities $G_{\alpha\beta} \equiv G_{\alpha\beta}(\rho, \rho', n', h'),\ \alpha, \beta = e, m)$, are defined in Appendix B, $\chi_j \equiv \chi_j(\rho, h'),\ \chi_j'' \equiv \chi_j(\rho', h'),\ (j = 1, 2)$, with $\chi_j(\rho, h)$ given by Eq. (B5), differentiation with respect to ρ, ρ' is understood in the classical (and not distributional) sense, and

$$B_{uv} \equiv B_{uv}(\rho, \rho', n - n', h - h'),\ (u, v = \perp, \parallel), \tag{C6}$$

where the spectral densities $B_{uv}(\rho, \rho', n, h)$ are defined by Eq. (40). Also, Eqs. (B14)–(C4) contain a tacit assumption that the poles and the branch cuts of $G_{\alpha\beta}(\rho, \rho', n', h')$ as function of complex variable h'

do not lie on the real-axis path of integration — e.g., due to presence of small dissipative losses in the background medium.

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