

Classical and Quantum Electromagnetic Interferences: What Is the Difference?

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Abstract—The zeroing of second order correlation functions between output fields after interferences in a 50/50 beam splitter has been accepted decades-long in the quantum optics community as an indicator of the quantum nature of lights. But, a recent work [1] presented some notable discussions and experiments that classical electromagnetic fields can still exhibit the zero correlation under specific conditions. Here, we examine analytically classical and quantum electromagnetic field interferences in a 50/50 beam splitter in the context of the second order correlation function for various input conditions. Adopting the Heisenberg picture in quantum electromagnetics, we examine components of four-term interference terms in the numerator of second order correlation functions and elucidate their physical significance. As such, we reveal the fundamental difference between the classical and quantum interference as illustrated by the Hong-Ou-Mandel (HOM) effect. The quantum HOM effect is strongly associated with: (1) the commutator relation that does not have a classical analogue; (2) the property of Fock states needed to stipulate the one-photon quantum state of the system; and (3) a destructive wave interference effect. Here, (1) and (2) imply the indivisibility of a photon. On the contrary, the classical HOM effect requires the presence of two destructive wave interferences without the need to stipulate a quantum state.

1. INTRODUCTION

The fact that light is a wave has been well accepted since Newton's time [2] in the seventeenth century. Even though Newton himself was an advocate of the corpuscular (particle) nature of light, it was not accepted until recently or the twentieth century. In the electromagnetics community, the concept of *coherence* is somewhat trivial since electromagnetic (EM) fields generated from macroscopic electric current sources on antennas are mostly coherent due to their long wavelengths and low frequencies. Namely, their field amplitudes, phases, and frequencies remain roughly constants (i.e., time-independent) in the timescale of the measurement, and in the absence of any artificial modulations. The phase-locked loop can be used to stabilize the phase of many electromagnetic sources [3] or electronic systems. On the contrary, in the optics community, optical sources often produce *chaotic* (incoherent) lights coming from microscopic phenomena such as atomic collisional or Doppler broadening effects [4, 5]. Therefore, to get coherent light, which is important for many practical optical engineering, well-designed lasers or optical masers are needed. Furthermore, the operating wavelength λ is usually much smaller than the size of objects analyzed or equipment settings; consequently, fast oscillations during propagation cause optical incoherence due to the short wavelengths and the interaction with the surrounding EM environment.

Optical coherence is typically assessed by using various orders of *correlation* of fields [5–8] widely used in statistical mechanics. More specifically, they are a measure of the similarity of various physical

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quantities, on the average sense, across space and time with respect to a variable of interest. For example, a first order correlation function, called field fluctuation, is defined by

$$g^{(1)}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \frac{\langle E^{(-)}(\mathbf{r}_2, t_2) E^{(+)}(\mathbf{r}_1, t_1) \rangle}{\sqrt{\langle |E^{(+)}(\mathbf{r}_1, t_1)|^2 \rangle \langle |E^{(-)}(\mathbf{r}_2, t_2)|^2 \rangle}} \quad (1)$$

where $E^{(+)}$ and $E^{(-)}$ are positive and negative frequency components of an electric field,[†] respectively, and \mathbf{r}_ν and t_ν for $\nu = 1, 2$ are observation positions and time instants. Here, angular brackets denote an ensemble average.[‡] It measures the extent of similarity or correlation of two field values probed at different positions and time instants. It has a range of $0 \leq g^{(1)} \leq 1$ due to Schwartz inequality. For instance, the degree of coherent light sources are often tested by evaluating $g^{(1)}$ in Michelson, Mach-Zehnder, or Sagnac interferometers.

Meanwhile, a classical second order correlation function is used to measure the extent of intensity fluctuations and correlations, defined by

$$g^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \frac{\langle I(\mathbf{r}_2, t_2) I(\mathbf{r}_1, t_1) \rangle}{\langle I(\mathbf{r}_1, t_1) \rangle \langle I(\mathbf{r}_2, t_2) \rangle} \quad (2)$$

where an instantaneous intensity $I(\mathbf{r}_\nu, t_\nu) = E^{(-)}(\mathbf{r}_\nu, t_\nu) E^{(+)}(\mathbf{r}_\nu, t_\nu)$ for $\nu = 1, 2$.[§] Originally, it has been exploited to improve the accuracy of stellar intensity interferometers in astronomy, known as *Hanbury Brown and Twiss* (HBT) effect [9]. The experiment is also vividly described in Fox [8]. Unlike $g^{(1)}$ for field fluctuations, $g^{(2)} \geq 1$ for classical lights, and as indicated in [8], it is usually less than 2.^{||}

More importantly, the concept of $g^{(2)}$ is significantly useful to identify particle nature of lights. Thus, the use of $g^{(2)}$ (for *coincidence count*) has been a standard measurement protocol in quantum optics experiments. For example, two photons interfering inside a 50/50 beam splitter always exit through one of the output ports while being bunched, known as *Hong-Ou-Mandel* (HOM) effect [10, 11]. This is understood as a fully-quantum-mechanical phenomenon. Since the two-photon destructive interference always produces zero coincidence counts, the zeroing of $g^{(2)}$ has been implicitly believed as the direct evidence of the quantum nature of lights [12]. This effect has spawned a lot of interest in the quantum optics community, and it has been avidly studied. Subsequent theoretical explanation of HOM was given in [11, 13–16]. HOM has also been studied in plasmons [17], numerically [18], in microwave [19], in atoms [20], in frequency domain [21, 22], in Gaussian wave packets [23, 24], with and without beam splitters [25, 26], as well as in many particle systems [27]. Of interest is a paper demonstrating this effect at astronomical length scale [28].

Recently, however, it has been experimentally shown that one can mimic $g^{(2)} = 0$ in a classical HOM given careful adjustment in the relative phase difference between two classical lights [1]. Here, we examine classical and quantum electromagnetic field interferences in a 50/50 beam splitter in the context of the second order correlation function for various input conditions. Adopting the Heisenberg picture, we examine components of the interference terms in the numerator of second order correlation functions to find their physical significance. As such, we reveal the fundamental difference between the classical and quantum HOM effects. The latter turns out to be strongly associated with: (1) the commutator relation that does not have a classical analogue; (2) the property of pure Fock states to represent the single-photon states of the quantum system; and (3) a wave interference effect. Moreover, (1) and (2) imply the indivisibility of a photon. On the contrary, the classical HOM effect requires the presence of two destructive wave interferences; whereas the quantum HOM effect requires only one of these cancellations.

[†] The positive frequency component of a real-valued electric field is the analytic signal of the electric field, i.e., $\mathcal{E}(\mathbf{r}, t) = E^{(+)}(\mathbf{r}, t) + E^{(-)}(\mathbf{r}, t)$ where $E^{(-)}(\mathbf{r}, t)$ is the complex conjugate of $E^{(+)}(\mathbf{r}, t)$.

[‡] An ensemble average is the statistical average of many identical events.

[§] In fact, this intensity expression is the result of the short-time averaged intensity with the assumption that fields are quasi-monochromatic [6, p.100 and p.162]. In other words, in the short-time average of the rigorous instantaneous intensity given by $I(\mathbf{r}, t) = \mathcal{E}(\mathbf{r}, t)^2 = [E^{(+)}(\mathbf{r}, t) + E^{(-)}(\mathbf{r}, t)]^2$, diagonal terms such as $[E^{(+)}(\mathbf{r}, t)]^2$ and $[E^{(-)}(\mathbf{r}, t)]^2$ have no contribution.

^{||} Due to the quirk of history, the radio astronomy community did not fully utilize Schwarz inequality in its definition. Otherwise, it should have been bounded from above by 1 just as $g^{(1)}$.

2. CLASSICAL INTERFERENCE IN A BEAM SPLITTER

Consider a two dimensional lossless beam splitter, as illustrated in Fig. 1. For simplicity, we assume **monochromatic** electromagnetic (EM) waves (TM_z polarized) impinge on the beam splitter. Input and output field amplitudes can be related in such a way that [13, 29]

$$\begin{bmatrix} E_3^{(+)} \\ E_4^{(+)} \end{bmatrix} = \begin{bmatrix} \mathcal{T}' & \mathcal{R} \\ \mathcal{R}' & \mathcal{T} \end{bmatrix} \cdot \begin{bmatrix} E_1^{(+)} \\ E_2^{(+)} \end{bmatrix} \quad (3)$$

where $E_\nu^{(+)}$ denotes a complex-valued amplitude of the positive frequency component of an electric field at ν -th port and \mathcal{R} (or \mathcal{R}') and \mathcal{T} (or \mathcal{T}') are reflection and transmission coefficients, respectively. For monochromatic signals, $E^{(+)}$, also termed complex signal, is equivalent to a phasor in electrical engineering parlance. Its real part is a time-harmonic signal [30].

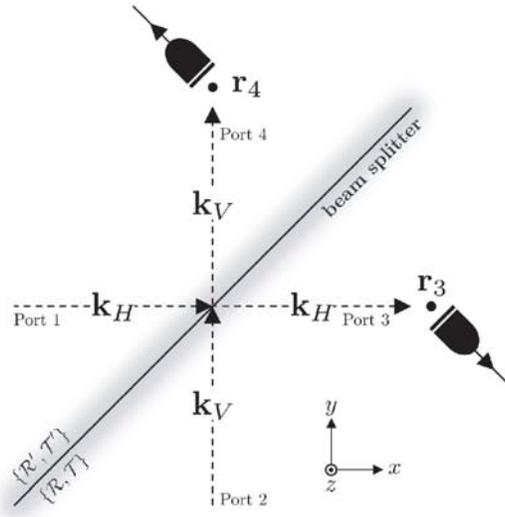


Figure 1. Illustration of a two dimensional beam splitter. Incident lights enter the beam splitter via input ports 1 and 2 and exit through output ports 3 and 4. The second order correlation function is evaluated for output fields measured at \mathbf{r}_3 and \mathbf{r}_4 .

Then, we can decompose spatial normal modes of this system into two modes:

$$\phi_1(\mathbf{r}) = \begin{cases} e^{i\mathbf{k}_H \cdot \mathbf{r}}: & \text{incident from port 1} \\ \mathcal{R}' e^{i\mathbf{k}_V \cdot \mathbf{r}}: & \text{reflected to port 4} \\ \mathcal{T}' e^{i\mathbf{k}_H \cdot \mathbf{r}}: & \text{transmitted toward port 3} \end{cases}, \quad (4)$$

$$\phi_2(\mathbf{r}) = \begin{cases} e^{i\mathbf{k}_V \cdot \mathbf{r}}: & \text{incident from port 2} \\ \mathcal{R} e^{i\mathbf{k}_H \cdot \mathbf{r}}: & \text{reflected to port 3} \\ \mathcal{T} e^{i\mathbf{k}_V \cdot \mathbf{r}}: & \text{transmitted toward port 4} \end{cases}, \quad (5)$$

where \mathbf{k}_H and \mathbf{k}_V are wavevectors along horizontal and vertical axes, respectively, and subscripts on the left-hand side denote modal indices. It is to be noted that these two modes are orthogonal to each other. They can also be thought of the limiting case of the Bloch-Floquet modes which can be found numerically [31, 32]. When the period of the Bloch-Floquet modes tends to infinity, above analytic solutions ensue.

The second order correlation function for classical lights is defined by [5, 6, 8]

$$g^{(2)}(\tau = 0) = \frac{\langle f^* f \rangle}{\langle g^* g \rangle \langle h^* h \rangle} = \frac{\langle |f|^2 \rangle}{\langle |g|^2 \rangle \langle |h|^2 \rangle} = \frac{\langle E^{(-)}(\mathbf{r}_3) E^{(-)}(\mathbf{r}_4) E^{(+)}(\mathbf{r}_4) E^{(+)}(\mathbf{r}_3) \rangle}{\langle E^{(-)}(\mathbf{r}_3) E^{(+)}(\mathbf{r}_3) \rangle \langle E^{(-)}(\mathbf{r}_4) E^{(+)}(\mathbf{r}_4) \rangle} \quad (6)$$

where τ is a temporal delay for two input fields; the asterisk “*” implies complex conjugate; and an angular bracket denotes an ensemble average. Since the temporal delay is just a phase delay, it is redundant when dealing with monochromatic fields, and we discard the temporal delay from our analysis.

Classical monochromatic electric fields at \mathbf{r}_3 and \mathbf{r}_4 , from Eqs. (4) and (5) are

$$E^{(+)}(\mathbf{r}_3) = (\mathcal{T}'A_1 + \mathcal{R}A_2)e^{i\theta_0} \quad (7)$$

$$E^{(+)}(\mathbf{r}_4) = (\mathcal{R}'A_1 + \mathcal{T}A_2)e^{i\theta_0}, \quad (8)$$

$$E^{(-)}(\mathbf{r}_3) = (\mathcal{T}'^*A_1^* + \mathcal{R}^*A_2^*)e^{-i\theta_0} \quad (9)$$

$$E^{(-)}(\mathbf{r}_4) = (\mathcal{R}'^*A_1^* + \mathcal{T}^*A_2^*)e^{-i\theta_0}, \quad (10)$$

where A_ν is a field amplitude for $\nu = 1, 2$ and $\theta_0 = \mathbf{k}_H \cdot \mathbf{r}_3 = \mathbf{k}_V \cdot \mathbf{r}_4$. In what follows, without loss of generality, we assume that $\theta_0 = 0$. It should be mentioned that the above is similar to the phasor technique in electrical engineering where for a time-harmonic signal, it can be represented by a complex signal [30]. Also, if a broadband time-domain real-valued signal is written in terms of Fourier expansion in time, the real frequency part is analogous to $E^{(+)}(\mathbf{r}_3)$ which is complex. The negative frequency part is analogous to $E^{(-)}(\mathbf{r}_3)$ to ensure the realness of the field.

Substituting Eqs. (7) to (10) into Eq. (6) yields

$$f = (\mathcal{R}'A_1 + \mathcal{T}A_2)(\mathcal{T}'A_1 + \mathcal{R}A_2), \quad (11)$$

$$\langle |f|^2 \rangle = \langle (\mathcal{T}'A_1 + \mathcal{R}A_2)^*(\mathcal{R}'A_1 + \mathcal{T}A_2)^*(\mathcal{R}'A_1 + \mathcal{T}A_2)(\mathcal{T}'A_1 + \mathcal{R}A_2) \rangle, \quad (12)$$

$$\langle |g|^2 \rangle = \langle (\mathcal{T}'A_1 + \mathcal{R}A_2)^*(\mathcal{T}'A_1 + \mathcal{R}A_2) \rangle, \quad (13)$$

$$\langle |h|^2 \rangle = \langle (\mathcal{R}'A_1 + \mathcal{T}A_2)^*(\mathcal{R}'A_1 + \mathcal{T}A_2) \rangle. \quad (14)$$

The physical meaning of the numerator of $g^{(2)}$ is the ensemble average or cross correlation of two field intensities at both outputs. When expanding Eq. (12), there are a total 16 terms, each of which represents one of possible combinations for the intensity correlation in terms of two input fields; hence each can be given a physical meaning. For example, $\langle (\mathcal{R}'A_1\mathcal{T}'A_1)^*\mathcal{R}'A_1\mathcal{T}'A_1 \rangle$ results from the input field 1, i.e., A_1 , that is reflected and transmitted at the same time.

To find more physical significance in the interferences, we examine an expanded version of f , viz.,

$$f = \left(\underbrace{\mathcal{R}'A_1\mathcal{T}'A_1}_{\text{self-divided by input 1}} + \underbrace{\mathcal{T}A_2\mathcal{R}A_2}_{\text{self-divided by input 2}} \right) + \left(\underbrace{\mathcal{R}'A_1\mathcal{R}A_2}_{\text{both reflected}} + \underbrace{\mathcal{T}'A_1\mathcal{T}A_2}_{\text{both transmitted}} \right). \quad (15)$$

The first two terms inside the first parenthesis in the second equation above is associated with two self-divided components, as illustrated in Figs. 2(a) and 2(b). On the other hand, the next two terms in the second parenthesis comes from both reflected and transmitted components, as depicted in Figs. 2(c) and 2(d). Hence, to make $g^{(2)}$ zero, one requires two distinct conditions: (1) introducing a quadrature relative phase between input fields, i.e., $A_2 = e^{\pm i\pi/2}A_1$, causes the destructive interference for self-divided components; and (2) using a 50/50 beam splitter with $\mathcal{R}' = \mathcal{R} = i/\sqrt{2}$, $\mathcal{T}' = \mathcal{T} = 1/\sqrt{2}$ causes the destructive interference for both reflected and transmitted components in the second parenthesis above. In other words, in the classical regime, the zeroing of $g^{(2)}$ can be achieved by having simultaneously, the occurrence of such two destructive interferences.

2.1. Coherent Lights

If classical waves are coherent, the ensemble average in Eq. (6) can be nulled since phasors of waves do not contain any random variables but are always constant; hence, $\langle |f|^2 \rangle = |f|^2$, $\langle |g|^2 \rangle = |g|^2$, and $\langle |h|^2 \rangle = |h|^2$. As a consequence, $g^{(2)}$ becomes always unity due to the cancellation of numerator and denominators in Eq. (6) for arbitrary reflection and transmission coefficients and field amplitudes. When both numerator and denominators go to zero, one can still show that $g^{(2)} = 1$ by using L'Hospital's rule to deal with the zero over zero limit.

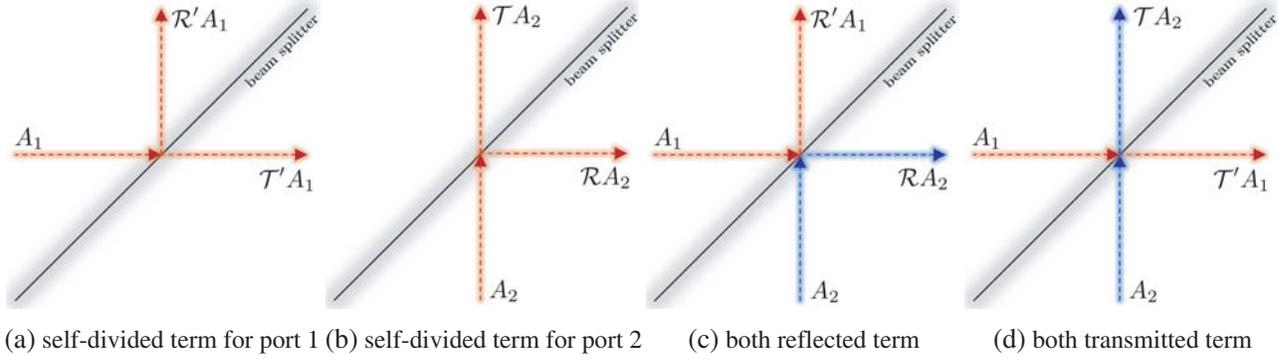


Figure 2. Four possible contributions of the simultaneous detection of lights at output ports.

2.2. Randomized Relative Phase Difference

In order for $g^{(2)}$ to be zero classically, one can also make the numerator zero while keeping denominators non-zero. One can achieve this by introducing a random variable for the relative phase difference, denoted by φ , between input classical fields and performing the ensemble average with respect to φ . Suppose that $A_2 = e^{i\varphi} A_1$ and φ is to be randomized from 0 to 2π based on a certain probability density function, denoted by $P(\varphi)$. Then, the numerator of $g^{(2)}$ can be explicitly represented by

$$\begin{aligned} \langle |f|^2 \rangle &= \int_0^{2\pi} d\varphi |f|^2 P(\varphi) \\ &= \int_0^{2\pi} d\varphi (T'A_1 + \mathcal{R}A_1 e^{i\varphi})^* (\mathcal{R}'A_1 + \mathcal{T}A_1 e^{i\varphi})^* (\mathcal{R}'A_1 + \mathcal{T}A_1 e^{i\varphi}) (T'A_1 + \mathcal{R}A_1 e^{i\varphi}) P(\varphi). \end{aligned} \quad (16)$$

One can evaluate $\langle |g|^2 \rangle$ and $\langle |h|^2 \rangle$ in the similar fashion. For the 50/50 beam splitter, one can derive that

$$\langle |f|^2 \rangle = \int_0^{2\pi} d\varphi |A_1|^4 (1 - \sin^2 \varphi) P(\varphi), \quad (17)$$

$$\langle |g|^2 \rangle = \int_0^{2\pi} d\varphi |A_1|^2 (1 - \sin \varphi) P(\varphi), \quad (18)$$

$$\langle |h|^2 \rangle = \int_0^{2\pi} d\varphi |A_1|^2 (1 + \sin \varphi) P(\varphi). \quad (19)$$

If $P(\varphi)$ is the discrete distribution for φ to be either $\pi/2$ or $3\pi/2$, i.e., $P(\varphi) = 0.5\delta(\varphi - \pi/2) + 0.5\delta(\varphi - 3\pi/2)$, by the delta function sifting property

$$\langle |f|^2 \rangle = 0, \quad \langle |g|^2 \rangle = |A_1|^2, \quad \langle |h|^2 \rangle = |A_1|^2. \quad (20)$$

Hence, it always guarantees $g^{(2)} = 0$ even in the classical regime. Again, it should be emphasized that the above results from two cancellations: one is between two self-divided terms and the other is between both reflected and transmitted terms shown in (15). Note that [1] assumed that $\mathcal{R}' = -1/\sqrt{2}$ and $\mathcal{R} = \mathcal{T} = \mathcal{T}' = 1/\sqrt{2}$, such that, for $g^{(2)} = 0$, φ should be picked in $\{0, \pi\}$.[¶] On the other hand, $P(\varphi) = 0.5\delta(\varphi) + 0.5\delta(\varphi - \pi)$ results in

$$\langle |f|^2 \rangle = |A_1|^4, \quad \langle |g|^2 \rangle = |A_1|^2, \quad \langle |h|^2 \rangle = |A_1|^2, \quad (21)$$

thus, $g^{(2)} = 1$ which means that the self-divided terms in Eq. (15) are not extinguished. Likewise, when $P(\varphi) = 0.25\delta(\varphi) + 0.25\delta(\varphi - \pi/2) + 0.25\delta(\varphi - \pi) + 0.25\delta(\varphi - 3\pi/2)$, gives $g^{(2)} = 0.5$.

[¶] By the similar procedure shown here, when $\mathcal{R}' = -1/\sqrt{2}$ and $\mathcal{R} = \mathcal{T} = \mathcal{T}' = 1/\sqrt{2}$, one can check that $\langle |f|^2 \rangle = \langle |A_1|^4 (1 - \cos^2 \varphi) \rangle$, $\langle |g|^2 \rangle = \langle |A_1|^2 (1 + \cos \varphi) \rangle$, $\langle |h|^2 \rangle = \langle |A_1|^2 (1 - \cos \varphi) \rangle$.

Again, the classical second order $g^{(2)}$ correlation function strongly depends on the extent of the two destructive interferences. But when the two destructive interferences occur simultaneously, classical HOM effects appear.

3. BRIEF REVIEW ON QUANTIZATION OF EM FIELDS

It has been shown previously that quantum Maxwell's equations can be derived to be [33]

$$\nabla \times \hat{\mathbf{H}}(\mathbf{r}, t) - \partial_t \hat{\mathbf{D}}(\mathbf{r}, t) = \hat{\mathbf{J}}_{\text{ext}}(\mathbf{r}, t), \quad (22)$$

$$\nabla \times \hat{\mathbf{E}}(\mathbf{r}, t) + \partial_t \hat{\mathbf{B}}(\mathbf{r}, t) = 0, \quad (23)$$

$$\nabla \cdot \hat{\mathbf{D}}(\mathbf{r}, t) = \hat{\rho}_{\text{ext}}(\mathbf{r}, t), \quad \nabla \cdot \hat{\mathbf{B}}(\mathbf{r}, t) = 0. \quad (24)$$

In the above, all the field quantities in the classical Maxwell's equations are represented by field operators (analogous to infinite dimensional matrix operators).⁺ For example, when a classical field \mathbf{E} is elevated to be represented by a field operator $\hat{\mathbf{E}}$, each of the field components, E_x , E_y , or E_z is now a random variable with a mean and a variance. Since the fields are now operators, the above quantum Maxwell's equations make sense only if they operate on a state vector $|\Psi\rangle$ that describes the state of the quantum system. To endow the field components further with properties of random variables, they are represented by operators that operate on a state vector $|\Psi\rangle$ that represents the state of a quantum system. These operators are called quantum observables analogous to their classical observables. Since the classical variable such as the field component is a random variable, its means and variance can be "observed" in the laboratory. The mean and variance of E_x , for instance, are given by

$$\bar{E}_x = \langle \Psi | \hat{E}_x | \Psi \rangle, \quad \sigma_{E_x} = \langle \Psi | (\hat{E}_x - \bar{E}_x)^2 | \Psi \rangle. \quad (25)$$

In order to have real value observables \bar{E}_x and σ_{E_x} , the operator representations of observables have to be Hermitian. And for normalization purpose, $\langle \Psi | \Psi \rangle = 1$ to give it probabilistic interpretation. The Dirac notation is entirely analogous to the linear algebra notation where " $|$ " implies an inner product, and $|\Psi\rangle$ is a vector, with $\langle \Psi |$ as its conjugate transpose. In general, the variance, σ_{E_x} , is non-zero, and this is quantum "noise" which cannot be eliminated, but part of the nature of quantum theory.

In addition to the above, there is a Hamiltonian \hat{H} associated with an eigenstate $|\Psi\rangle$, and with the above quantum Maxwell's equations. Moreover, the corresponding quantum state equation has to be satisfied:^{*}

$$\hat{H}|\Psi\rangle = i\hbar\partial_t|\Psi\rangle \quad (26)$$

It turns out that if the classical Maxwell's equations can be derived from a classical Hamiltonian [33], then the quantum Maxwell's equations can be derived from a quantum Hamiltonian [36]. But one way of deriving quantum Maxwell's equations is to use the mode decomposition approach [5, 6, 37–39].[‡] In this approach, the classical fields are first decomposed in terms of the modes of the system. Then a homomorphism is established between a mode and a quantum harmonic oscillator or quantum pendulum. In this manner, the quantum fields are represented by quantum operators. Here, we will use a modal view of the quantum fields. Therefore, we illustrate the quantization of EM fields in an inhomogeneous medium represented by a slab [31–33].

Using Lorenz gauge with $\Phi = 0$, one can represent a classical vector potential in mode space via eigenmode decomposition as

$$A(\mathbf{r}, t) = \sum_k \phi_k(\mathbf{r}) a_k e^{-i\omega_k t} + \text{h.c.} \quad (27)$$

where subscript k is the modal index; h.c. denotes Hermitian conjugate; $\phi_k(\mathbf{r})$, ω_k , a_k are the k -th (traveling-wave) eigenmode, eigenfrequency, complex-valued modal amplitudes, respectively. It is to be noted that if one were to fix \mathbf{r} and observe the field, it has a simple harmonic motion just like that of

⁺ The infinite dimensional linear vector spaces associated with such operators are generally known as Hilbert spaces.

^{*} Again, this equation is like the state equation in modern control theory using the state-variable approach [34]. It was first proposed by Schrödinger [35] for the hydrogen atom.

[‡] One can also refer to quite recent textbooks [7, 40–45] that also discussed the mode decomposition approach.

a simple pendulum. Again, each term of the above is very similar to the phasor technique in electrical engineering [30], a poor-man's Fourier transform technique.

Then, one can evaluate the total energy of the system which is equivalent to the Hamiltonian, denoted by H . And by using the orthonormal property of eigenmodes, we have [33]

$$H = \sum_k H_k = \sum_k a_k^* a_k = \frac{1}{2} \sum_k (p_k^2 + \omega_k^2 q_k^2) \quad (28)$$

where

$$a_k = \frac{\omega_k q_k + i p_k}{\sqrt{2}}, \quad a_k^* = \frac{\omega_k q_k - i p_k}{\sqrt{2}}, \quad (29)$$

q_k and p_k are (real-valued) canonical "position" and "momentum" of k -th eigenmode. Here, the reference to position and momentum is entirely by analogy or mathematical homomorphism to a pendulum. It implies that an electromagnetic field oscillates like a pendulum. The above physical picture is that the time variation of the field of the system can be decomposed into sum of oscillations of each individual mode H_k . Moreover, the total energy of the system, denoted by H , is now decomposed into the sum of the energy of each individual mode. Each mode is analogous to a pendulum or a simple harmonic oscillator. Therefore, the above Hamiltonian is mathematically homomorphic to that of uncoupled harmonic oscillators: each mode is uncoupled to the other modes in this picture. Thus, one can employ canonical quantization for a classical pendulum to quantize each mode of the field. Then, by the correspondence principle, the canonical variables (also called conjugate variables) are represented by quantum operators, i.e., $q_k \rightarrow \hat{q}_k$ and $p_k \rightarrow \hat{p}_k$. In this manner, these classical variables p_k and q_k become random variables with means and variances.

Furthermore, these operators need not commute, and hence, \hat{p}_k and \hat{q}_k satisfy the fundamental commutator relations

$$[\hat{q}_k, \hat{p}_{k'}] = i\hbar\delta_{k,k'}\hat{I}, \quad [\hat{q}_k, \hat{q}_{k'}] = 0 = [\hat{p}_k, \hat{p}_{k'}], \quad (30)$$

where the commutator $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$, and \hbar is Planck constant, and \hat{I} is an identity operator. In the coordinate space representation, $\hat{p} = -i\hbar\partial/\partial q\hat{I}$ and $\hat{q} = q\hat{I}$ [33].

When two operators do not share the same eigenstates, their commutator is not zero [33, 46, 47]. It also implies that the order of measurements associated with canonical position and momentum operators does matter because the two Hermitian operators do not commute. Also, if we can prepare a pure eigenstate for one of the operators, it cannot be a pure eigenstate for the second operator. One can easily show that the variance, in accordance to (25), is zero if the quantum state is a pure eigenstate. This implies that if a pure eigenstate is prepared for one quantum operator, its eigenvalue can be determined precisely. However, this eigenstate cannot be a pure eigenstate for the second operator, if the second operator does not commute with the first one. This is the gist of the Heisenberg uncertainty principle regarding two non-commuting operators representing two observables. (Note that the fundamental commutator relations can be derived from Heisenberg equations of motion and quantum state equation under the energy conservation [36].)

At this junction, it is customary to introduce the so-called annihilation and creation operators similar to Eq. (29) given by

$$\hat{a}_k = \frac{\omega_k \hat{q}_k + i \hat{p}_k}{\sqrt{2\hbar\omega_k}}, \quad \hat{a}_k^\dagger = \frac{\omega_k \hat{q}_k - i \hat{p}_k}{\sqrt{2\hbar\omega_k}}, \quad (31)$$

satisfying the commutator relations

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{k,k'}\hat{I}, \quad [\hat{a}_k, \hat{a}_{k'}] = 0 = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger]. \quad (32)$$

Correspondingly, the quantum Hamiltonian operator becomes

$$\hat{H} = \sum_k \hat{H}_k = \frac{1}{2} \sum_k (\hat{p}_k^2 + \omega_k^2 \hat{q}_k^2) = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right). \quad (33)$$

It is noted here that the classical Hamiltonian represented by Eq. (28) is diagonalized in the coordinate space by mode decomposition. When the classical conjugate variables are elevated to be represented by

quantum operators, these operators now operate on a quantum Hilbert space, and they are in general non-diagonal in this new Hilbert space. To diagonalize these operators, we need to solve the Schrödinger equation^{††} or the quantum state equation associated with each of these modes. To this end, one can find an eigenstate of k -th quantum harmonic oscillator by using the k -th Hamiltonian from Eq. (33), viz.,

$$\hat{H}_k|\Psi\rangle_k = \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right) |\Psi\rangle_k = \hbar\omega_k \left(n_k + \frac{1}{2} \right) |\Psi\rangle_k = E_{n_k} |\Psi\rangle_k \quad (34)$$

where eigenstates are Fock (number) states denoted by $|\Psi\rangle_k = |n_k\rangle$, and $E_{n_k} = \left(n_k + \frac{1}{2} \right) \hbar\omega_k$ denotes the eigenenergy contained in k -th eigenmode. In the above, n_k is related to the number of photons since the energy levels are equally spaced $\hbar\omega_k$ apart, which is equivalent to the energy of a single photon with frequency ω_k . These are also known as non-classical states since they do not have classical equivalence. Also, notice that in $|\Psi\rangle_k = |n_k\rangle$, n_k denotes the number of photons in the state as well as used as an index. The action of the annihilation and creation operators on a number state yields [46]

$$\hat{a}_k |n_k\rangle = \sqrt{n_k} |n_k - 1\rangle, \quad (35)$$

$$\hat{a}_k^\dagger |n_k\rangle = \sqrt{n_k + 1} |n_k + 1\rangle, \quad (36)$$

$$\hat{a}_k^\dagger \hat{a}_k |n_k\rangle = n_k |n_k\rangle. \quad (37)$$

The multimode quantum state can be denoted by the (tensor) outer product of different eigenmodes or eigenstates, each of which is expressed in terms of a number state, or

$$|\Psi\rangle = |n_1\rangle |n_2\rangle \dots |n_k\rangle \dots \quad (38)$$

and these number states are orthogonal according to

$$\langle n_1, \dots, n_k, \dots | n'_1, \dots, n'_k, \dots \rangle = \delta_{n_1, n'_1} \dots \delta_{n_k, n'_k} \dots \quad (39)$$

where $|n'_1, \dots, n'_k, \dots\rangle$ is the shorthand notation of $|n_1\rangle \dots |n_k\rangle \dots$. Annihilation and creation operators are called ladder operators since their action on Fock states either increase or decrease the photon number by 1. Consequently, a quantum vector potential operator can be written as

$$\hat{A}(\mathbf{r}, t) = \sum_k \sqrt{\frac{\hbar}{2\omega_k}} \phi_k(\mathbf{r}) \hat{a}_k e^{-i\omega_k t} + \text{h.c.} \quad (40)$$

By taking the time derivative to Eq. (40), one can derive

$$\hat{E}(\mathbf{r}, t) = \underbrace{\sum_k -i \sqrt{\frac{\hbar\omega_k}{2}} \phi_k(\mathbf{r}) \hat{a}_k e^{-i\omega_k t}}_{\hat{E}^{(+)}(\mathbf{r}, t)} + \text{h.c.} \quad (41)$$

In the above, $\phi_k(\mathbf{r})$ can be found numerically using the Bloch-Floquet mode decomposition, and the relevant periodic boundary condition [32]. Again, since we are considering a monochromatic wave, we ignore the time dependence when evaluating quantum interferences. In the subsequent analysis in the next section, we will assume that only two of the modes exist in the quantum system.

4. QUANTUM INTERFERENCE IN A BEAM SPLITTER

4.1. Quantum Electric Field Operators

As shown in the previous section, to describe quantum fields, modal amplitudes of classical fields are represented by operators associated with a quantum state. In the system shown in Fig. 1, we can consider the presence of two modes: one excited by an incident wave in port 1 while the second one is excited by an incident wave in port 2. These modes are orthogonal per the theory outlined in [32]. At

^{††} For those with little background in quantum theory, the introduction given in [30] could be useful.

locations \mathbf{r}_3 and \mathbf{r}_4 , one can deduce the analytical expressions of the modes by using Eqs. (4) and (5). The resulting positive components of quantum electric field operators at \mathbf{r}_3 and \mathbf{r}_4 can be written as

$$\hat{E}^{(+)}(\mathbf{r}_3) = \sqrt{\frac{\hbar\omega}{2}} (\mathcal{T}'\hat{a}_1 + \mathcal{R}\hat{a}_2), \quad (42)$$

$$\hat{E}^{(+)}(\mathbf{r}_4) = \sqrt{\frac{\hbar\omega}{2}} (\mathcal{R}'\hat{a}_1 + \mathcal{T}\hat{a}_2), \quad (43)$$

where subscripts of the ladder operators denote the modal index. Note that here we adopt the Heisenberg picture. One can refer to [13, 48] for the interference of quantum lights in beam splitters as well. It is to be noted that although we have emphasized the non-commutativity of quantum operators, it is easily shown that the above two operators commute. Also, since the field is monochromatic, the above is analogous to the phasor representation of a time-harmonic signal [30].

4.2. Quantum Second Order Correlation Function

The quantum second order correlation function was first proposed by Glauber [49], defined by

$$g^{(2)}(\tau = 0) = \frac{\langle f|f \rangle}{\langle g|g \rangle \langle h|h \rangle} \quad (44)$$

where

$$|f\rangle = \hat{E}^{(+)}(\mathbf{r}_4) \hat{E}^{(+)}(\mathbf{r}_3) |\Psi_{in}\rangle, \quad (45)$$

$$|g\rangle = \hat{E}^{(+)}(\mathbf{r}_3) |\Psi_{in}\rangle, \quad (46)$$

$$|h\rangle = \hat{E}^{(+)}(\mathbf{r}_4) |\Psi_{in}\rangle. \quad (47)$$

Note that $|\Psi_{in}\rangle$ is a quantum state vector describing input states of quantum fields in Dirac notation. Moreover, in this quantum $g^{(2)}$, ensemble averages are replaced with expectation values.

In the above, the positive and negative frequency components of electric field operators have their own physical significances. Since the positive frequency component contains a annihilation operator that decreases the photon number by 1, it depicts the photon absorption process. Similarly a creation operator in the negative frequency component describes the photon emission process. In other words, the consecutive action of $\hat{E}^{(+)}(\mathbf{r}_3)$ and $\hat{E}^{(+)}(\mathbf{r}_4)$ on a given quantum state vector is intimately related to two consecutive photodetections at the outputs 3 and 4, respectively [5, 6, 8, 49]. As a result, the quantum second order correlation function becomes the mathematical expression for the underlying physical principle of photodetectors at both outputs. In particular, the numerator of Eq. (44) is the probability of the simultaneous detection of two photons.

The full expression for $\langle f|f \rangle$ can be written as

$$\langle f|f \rangle = \langle \Psi_{in} | \hat{E}^{(-)}(\mathbf{r}_3) \hat{E}^{(-)}(\mathbf{r}_4) \hat{E}^{(+)}(\mathbf{r}_4) \hat{E}^{(+)}(\mathbf{r}_3) | \Psi_{in} \rangle \quad (48)$$

incorporating four-term interferences consisting of various combinations of products of ladder operators. It is important to note that one cannot do the arbitrary change of action orders of the ladder operators on quantum state vectors due to the commutator relation whereas the classical counterpart, i.e., the numerator in Eq. (6), is not affected by the arbitrary change of the product order among field values. This is because, as indicated in Eq. (32), the absorption and emission process of photons are involved in the Heisenberg uncertainty principle. For example, let us compare the expectation value of two operators, viz., $\hat{A} = \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1$ and $\hat{B} = \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1$ with respect to $|\Psi_{in}\rangle = |1_1, 1_2\rangle$. By using Eqs. (35), (36), (39), one can evaluate

$$\langle 1_1, 1_2 | \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 | 1_1, 1_2 \rangle = 0, \quad (49)$$

$$\langle 1_1, 1_2 | \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 | 1_1, 1_2 \rangle = 1. \quad (50)$$

As seen in the above, the change of the action order of operators produces different results. To deal with quantum interference, one should properly account for the commutator relation which has no classical analogue in classical interferences. Next, we investigate the behaviors of $g^{(2)}$ for two specific types of quantum fields. The fact that Eq. (48) is zero also means that a single photon cannot be absorbed twice.

4.3. Fock States

An initial quantum state for two input photons can be modeled by a single-excitation of Fock state in each normal mode as

$$|\Psi_{in}\rangle = |1_2\rangle \otimes |1_1\rangle \equiv |1_2, 1_1\rangle = \hat{a}_2^\dagger \hat{a}_1^\dagger |0\rangle. \quad (51)$$

The above is the two-photon quantum state we have prepared to launch into the beam splitter quantum system from two different ports.

Then, substituting Eqs. (42), (43), and (51) into Eq. (44) yields

$$\begin{aligned} \langle f|f\rangle &= \frac{\hbar^2 \omega^2}{4} \langle 0|\hat{a}_1 \hat{a}_2 \left(\mathcal{R}'^* \mathcal{T}'^* \hat{a}_1^\dagger \hat{a}_1^\dagger + \mathcal{R}'^* \mathcal{R}^* \hat{a}_2^\dagger \hat{a}_1^\dagger + \mathcal{T}^* \mathcal{T}'^* \hat{a}_1^\dagger \hat{a}_2^\dagger + \mathcal{T}^* \mathcal{R}^* \hat{a}_2^\dagger \hat{a}_2^\dagger \right) \\ &\quad \times \left(\mathcal{R}' \mathcal{T}' \hat{a}_1 \hat{a}_1 + \mathcal{R}' \mathcal{R} \hat{a}_1 \hat{a}_2 + \mathcal{T} \mathcal{T}' \hat{a}_2 \hat{a}_1 + \mathcal{T} \mathcal{R} \hat{a}_2 \hat{a}_2 \right) \hat{a}_2^\dagger \hat{a}_1^\dagger |0\rangle. \end{aligned} \quad (52)$$

Thus, it contains multi-term interferences with respect to the vacuum state. Again, due to the commutator relation, one cannot change the action order of ladder operators arbitrarily. To gain more physical insight, when evaluating $|f\rangle$, we expand it into four terms, viz.,

$$\begin{aligned} |f\rangle &= \frac{\hbar\omega}{2} \left(\underbrace{\mathcal{R}' \hat{a}_1 \mathcal{T}' \hat{a}_1}_{\text{self-divided by input 1}} + \underbrace{\mathcal{T} \hat{a}_2 \mathcal{R} \hat{a}_2}_{\text{self-divided by input 2}} + \underbrace{\mathcal{R}' \hat{a}_1 \mathcal{R} \hat{a}_2}_{\text{both reflected}} + \underbrace{\mathcal{T} \hat{a}_2 \mathcal{T}' \hat{a}_1}_{\text{both transmitted}} \right) \hat{a}_2^\dagger \hat{a}_1^\dagger |0\rangle \\ &= \frac{\hbar\omega}{2} \left(\mathcal{R}' \mathcal{T}' \hat{a}_1 \hat{a}_1 \hat{a}_2^\dagger \hat{a}_1^\dagger |0\rangle + \mathcal{T} \mathcal{R} \hat{a}_2 \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1^\dagger |0\rangle + \mathcal{R}' \mathcal{R} \hat{a}_1 \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1^\dagger |0\rangle + \mathcal{T} \mathcal{T}' \hat{a}_2 \hat{a}_1 \hat{a}_2^\dagger \hat{a}_1^\dagger |0\rangle \right) \\ &= \frac{\hbar\omega}{2} \left(\mathcal{R}' \mathcal{T}' \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1 \hat{a}_1^\dagger |0\rangle + \mathcal{T} \mathcal{R} \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 \hat{a}_2^\dagger |0\rangle + \mathcal{R}' \mathcal{R} \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger |0\rangle + \mathcal{T} \mathcal{T}' \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger |0\rangle \right) \\ &= \frac{\hbar\omega}{2} \left(\underbrace{\mathcal{R}' \mathcal{T}' \hat{a}_2^\dagger \hat{a}_1 |0\rangle}_{=0} + \underbrace{\mathcal{T} \mathcal{R} \hat{a}_1^\dagger \hat{a}_2 |0\rangle}_{=0} + \mathcal{R}' \mathcal{R} |0\rangle + \mathcal{T} \mathcal{T}' |0\rangle \right) \\ &= \frac{\hbar\omega}{2} \left(\mathcal{R} \mathcal{R}' + \mathcal{T}' \mathcal{T} \right) |0\rangle \end{aligned} \quad (53)$$

It is noted that \hat{a}_1 and \hat{a}_2 commute since they are from two independent orthogonal modes. Furthermore, the four terms in the first equality describe the four possible scenarios of the simultaneous detection of photons at both outputs, similar to the classical case in Eq. (15). After expanding (in the second equality) and rearranging operators (in the third equality) based on the commutator relation in Eq. (32), we can arrive at the fourth equality by performing raising and lowering process of the photon number from the ground state through Eqs. (35) and (36). One can see that the first two terms in the fourth equality, corresponding to self-divided terms, become zero since annihilation of the ground state is always zero. In other words, a single photon (or a particle) driven from one of two inputs cannot be simultaneously detected at both outputs, or equivalently, cannot be split through the beam splitter. We interpret in the context of $g^{(2)}$ that the non-existence of self-divided terms in quantum interferences comes from the particle nature of lights or the indivisibility of a photon. This is because a single photon passing through the beam splitter gets entangled rather than being split, which can be explicitly explained by taking the Schrödinger picture [7]. Consequently, the simultaneous detection of two photons is coming from the interference of both reflected and transmitted components. It becomes zero when the reflection and transmission coefficients have the correct phase. In summary, the disappearing of the self-divided term above implies that a single photon cannot be divided when it passes through the beam splitter.

4.4. Coherent States

If input quantum fields are described by coherent states [7, 46, 47], which are semi-classical states, the corresponding initial quantum state becomes

$$|\Psi_{in}\rangle = |\alpha_1, \beta_2\rangle \equiv |\alpha_1\rangle \otimes |\beta_2\rangle. \quad (54)$$

where α_1 and β_1 are complex numbers whose magnitudes incorporate the average photon number information, and subscript denotes a modal index. The coherent state is the linear superposition of Fock states as

$$|\alpha_\nu\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n_\nu\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \hat{a}_\nu^\dagger} |0\rangle. \quad (55)$$

Hence, the expectation value of field operators w.r.t. coherent states is not zero. Furthermore, eigenstates of annihilation operators are coherent states, viz.,

$$\hat{a}_\nu |\alpha_\nu\rangle = \alpha_\nu |\alpha_\nu\rangle. \quad (56)$$

This makes the simultaneous detection of lights at the two output ports possible while even obeying the commutator relation since self-divided terms are non-zero. To check this, by substituting Eq. (54) into Eq. (44), one obtains

$$\begin{aligned} |f\rangle &= \frac{\hbar\omega}{2} \left(\underbrace{\mathcal{R}'\hat{a}_1\mathcal{T}'\hat{a}_1}_{\text{self-divided from input 1}} + \underbrace{\mathcal{T}\hat{a}_2\mathcal{R}\hat{a}_2}_{\text{self-divided from input 2}} + \underbrace{\mathcal{R}'\hat{a}_1\mathcal{R}\hat{a}_2}_{\text{both reflected}} + \underbrace{\mathcal{T}\hat{a}_2\mathcal{T}'\hat{a}_1}_{\text{both transmitted}} \right) |\alpha_1, \beta_2\rangle \\ &= \frac{\hbar\omega}{2} \left(\mathcal{R}'\mathcal{T}'\alpha_1^2 + \mathcal{R}'\mathcal{R}\alpha_1\beta_2 + \mathcal{T}\mathcal{T}'\beta_2\alpha_1 + \mathcal{T}\mathcal{R}\beta_2^2 \right) |\alpha_1, \beta_2\rangle, \end{aligned} \quad (57)$$

where the second equality can be derived by using Eq. (56). One observes that self-divided terms exist in quantum interferences since coherent states passing through the beam splitter can be divided into two packets of energy since many photons are involved. Again, the use of the 50/50 beam splitter cancels out both reflected and transmitted terms whereas the cancellation of self-divided terms depends on the relative phase difference between two input coherent states. To show this, suppose $\beta_2 = e^{i\varphi}\alpha_1$. When $\varphi = \pi/2$ or $3\pi/2$, $g^{(2)} = 0$ and when $\varphi = 0$ or π , $g^{(2)} = 1$, similar to the classical interference.

5. CONCLUSION

We have examined classical and quantum electromagnetic field interferences in a 50/50 beam splitter in the context of the second order correlation function for various input conditions. Using closed form solution of wave modes passing through a beam splitter, we extract deeper analytic insight into this difference. Adopting the Heisenberg picture in quantum electromagnetics, we decomposed components of four-term interferences in the numerator of second order correlation functions and found their physical significances. As such, we revealed the fundamental difference between the classical and quantum HOM effects. The quantum HOM is strongly associated with: (1) the commutator relation that does not have a classical analogue; (2) the property of pure Fock states needed to describe single-photon states; and (3) a wave interference effect similar to classical waves. In the quantum HOM effect, two of the four terms disappear due to (1) and (2). Together, they imply the indivisibility of a photon, and the other two terms disappear due to destructive wave interference. On the contrary, the classical HOM effect requires the presence of two destructive wave interferences among all the four terms to have them cancel each other.

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