

Electromagnetic Theory with Discrete Exterior Calculus

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Abstract—A self-contained electromagnetic theory is developed on a simplicial lattice. Instead of dealing with vectorial field, discrete exterior calculus (DEC) studies the discrete differential forms of electric and magnetic fields, and circumcenter dual is adopted to achieve diagonal Hodge star operators. In this paper, Gauss’ theorem and Stokes’ theorem are shown to be satisfied inherently within DEC. Many other electromagnetic theorems, such as Huygens’ principle, reciprocity theorem, and Poynting’s theorem, can also be derived on this simplicial lattice consistently with an appropriate definition of wedge product between cochains. The preservation of these theorems guarantees that this treatment of Maxwell’s equations will not lead to spurious solutions.

1. INTRODUCTION

Finite difference (FD) and finite element methods (FEM) have been widely applied in electromagnetics [1, 2]. And they are both based on the vectorial version of Maxwell’s equations. As is well known, differential forms can be used to recast Maxwell’s equations in a more succinct fashion, which completely separates metric-free and metric-dependent components [4–9]. Discrete exterior calculus (DEC), a discrete version of differential geometry, provides a numerical treatment of differential forms directly [10–12]. Compared to the popular finite difference method, DEC is based on unstructured simplicial mesh as in FEM, e.g., triangular mesh for 2-D space and tetrahedral mesh for 3-D space. Therefore, this method is adaptable to arbitrary complex structures. Moreover, as will be shown, this method also exactly preserves important structural features of Maxwell’s equations as in Yee grid applied to FD, e.g., Gauss’s law $\nabla \cdot \mathbf{D} = \rho$ and $\mathbf{E} = -\nabla\phi$. Meanwhile, unlike finite integration technique (FIT) [15, 16], the dual mesh is constructed by connecting circumcenters instead of barycenters. In this way, the Hodge star operators can be built as diagonal matrices. Effective dual volume can be applied to incorporate the material information to include inhomogeneity as shown in [14].

Shown in [14], we have been able to solve eigenvalue problems, excitation problems, and scattering problems with DEC method. It should be noted that the only limitation of our approach is that without Delaunay triangulation, the positivity of Laplacian cannot be guaranteed [13, 14], which will lead to instability of time domain analysis. However it is still of interest to investigate if a self-consistent and self-contained discrete electromagnetic theory can be developed on simplicial mesh using DEC, which has been accomplished based on Yee grid [3, 4]. In this paper, we show that many electromagnetic theorems still exist in this discrete electromagnetic theory with DEC, such as Huygens’ principle, reciprocity theorem, Poynting’s theorem and uniqueness theorem. Therefore, all electromagnetic phenomena can be predicted and explained with this discrete electromagnetic theory with a appropriate discretization length scale. Besides, due to the charge conservation property, as will be shown, the numerical simulation of Maxwell’s equations with DEC will not lead to spurious solutions [3, 17].

To develop this discrete electromagnetic theory with DEC, an appropriate discrete treatment of wedge product [11, 18] is crucial. The wedge product operator \wedge is used to construct higher degree forms

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from lower degree ones. Correspondingly, a discrete wedge product operator should be able to obtain higher degree cochains from lower degree ones. For example, with a known primal 1-cochain \mathbf{E} and dual 2-cochain \mathbf{D} , an integral of the wedge product of these two cochains should represent stored electric energy density $\mathbf{E} \cdot \mathbf{D}$. It should be noted that a wedge product between two primal or two dual cochains is discussed in [11]. In this paper, a proper treatment of wedge product between a primal k -cochain and a dual $(n - k)$ -cochain is given. With this discrete wedge product, the discrete electric energy density $\mathbf{E} \cdot \mathbf{D}$, magnetic energy density $\mathbf{H} \cdot \mathbf{B}$, and Poynting's vector $\mathbf{E} \times \mathbf{H}$ can all be constructed. Moreover, the convolution of Green's function with current or charge fields can also be understood as a wedge product. Therefore, Huygens' principle for both scalar and electromagnetic waves can be derived in this discrete world. And it is straightforward to show that many electromagnetic theorems, like Poynting's theorem and reciprocity theorem are also preserved.

In Section 2, discrete calculus based on DEC is introduced, including discrete gradient, curl, and divergence for both primal and dual cochains. In Section 3, we define the discrete wedge product between primal k -cochain and dual $(n - k)$ -cochain in n -D space. In the next section, the discrete Green's functions and Huygens' principle for scalar waves and vector waves are formulated and derived. In addition, discrete Poynting's theorem and reciprocity theorem are also derived.

2. DISCRETE CALCULUS WITH DEC

In DEC, continuous scalar or vector field are represented with *cochains*, which are column vectors. Then the derivative operators acting on them are replaced by *co-boundary* or discrete exterior derivative operators, which are highly sparse matrices. As mentioned, DEC is based on arbitrary simplicial mesh, which is called the primal mesh, and the dual mesh is constructed by connecting nearest circumcenters. Cochains defined on the primal and dual mesh is called primal cochains and dual cochains, respectively. Due to the existence of boundaries, the dual mesh is truncated. Therefore, discrete calculus for dual cochains will give rise to boundary terms.

2.1. Discrete Gradient, Curl and Divergence for Primal Cochains

2.1.1. Discrete Gradient

With a scalar function $\phi(\mathbf{r})$, a gradient can be defined. With DEC, a scalar function corresponds to a 0-cochain ϕ , which lives on N_0 vertices $\{p_1, \dots, p_{N_0}\}$, and it can be written as a $N_0 \times 1$ column vector:

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{N_0} \end{pmatrix}, \quad \phi_i \triangleq \phi(\mathbf{r}_{p_i}) = \phi(\mathbf{r}_i). \quad (1)$$

If we view gradient of function $\phi(\mathbf{r})$, $\nabla\phi(\mathbf{r})$, as a 1-cochain \mathbf{e} , then the value of this 1-cochain along the i -th primal edge l_i is defined as:

$$e_i = \int_{l_i} \nabla\phi(\mathbf{r}) \cdot d\mathbf{l} = \phi(\mathbf{r}_{j1}) - \phi(\mathbf{r}_{j2}) = \phi_{j1} - \phi_{j2} = \sum_j d_{i,j}^{(0)} \phi_j \quad (2)$$

where $d_{i,j}^{(0)}$ is the (i, j) element of discrete exterior derivative operator $\bar{\mathbf{d}}^{(0)}$. More specifically,

$$[\bar{\mathbf{d}}^{(0)}]_{i,j} = d_{i,j}^{(0)} = \begin{cases} +1 & \text{if } p_j \text{ is the ending vertex of } l_i, \\ -1 & \text{if } p_j \text{ is the starting vertex of } l_i, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where the superscript "(0)" means that it only operates on 0-cochains. In fact, the rightmost of Equation (2) is just the i -th element of 1-cochain $\bar{\mathbf{d}}^{(0)}\phi$, which means that the gradient relation $\mathbf{e} = \bar{\mathbf{d}}^{(0)}\phi$ is naturally and exactly preserved.

2.1.2. Discrete Curl and Stokes' Theorem

A curl of a vector field, i.e., $\mathbf{E}(\mathbf{r})$, can measure its circulation or rotation about a point. In both 2-D and 3-D spaces, we view this vector field as a 1-cochain \mathbf{E} , because only a path integral of it gives rise to a sensible physical meaning. Therefore, its curl $\nabla \times \mathbf{E}(\mathbf{r})$ corresponds to a 2-cochain \mathbf{b} in both 2-D and 3-D spaces.

In a 2-D or 3-D simplicial mesh, if there are N_1 edges in total, 1-cochain \mathbf{E} can be defined as:

$$\mathbf{E} = \begin{pmatrix} E_1 \\ \vdots \\ E_{N_1} \end{pmatrix}, \quad E_i \triangleq \int_{l_i} [\mathbf{E}(\mathbf{r}) \cdot \hat{l}_i] dl. \quad (4)$$

Here, \hat{l}_i is the unit vector along the i -th primal edge l_i .

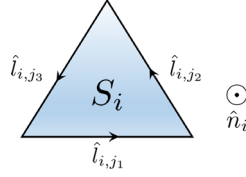


Figure 1. A primal face S_i with oriented edges. Here \hat{n}_i is assumed as the positive direction of S_i .

As shown in Figure 1, the value of 2-cochain \mathbf{b} on i -th primal face S_i can be derived from Stokes' theorem as:

$$b_i = \int_{S_i} [\nabla \times \mathbf{E}(\mathbf{r})] \cdot \hat{n}_i dS = \oint_{\partial S_i} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} = \sum_{l_j \in \partial S_i} \int_{l_j} [\mathbf{E}(\mathbf{r}) \cdot \hat{l}_{i,j}] dl = \sum_j d_{i,j}^{(1)} E_j. \quad (5)$$

where ∂S_i refers to the three primal edges around S_i ; $\hat{l}_{i,j}$ is a unit vector surrounding S_i clockwise and along l_j ; $d_{i,j}^{(1)}$ is the (i, j) element of discrete exterior derivative operator $\bar{\mathbf{d}}_{i,j}^{(1)}$.

$$[\bar{\mathbf{d}}^{(1)}]_{i,j} = d_{i,j}^{(1)} = \begin{cases} +1 & \text{if } \hat{l}_j = \hat{l}_{i,j}, \\ -1 & \text{if } \hat{l}_j = -\hat{l}_{i,j}, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

It can be seen that the rightmost of Equation (5) is the i -th element of $\bar{\mathbf{d}}^{(1)} \mathbf{E}$. Then the relation $\mathbf{b} = \bar{\mathbf{d}}^{(1)} \mathbf{E}$ holds exactly in both 2-D and 3-D.

2.1.3. Discrete Divergence and Gauss' Theorem

With a vector field $\mathbf{D}(\mathbf{r})$, its divergence can be defined in both 2-D and 3-D space. In 2-D, $\mathbf{D}(\mathbf{r})$ is viewed as a primal 1-cochain[†] and its divergence $\nabla_s \cdot \mathbf{D}(\mathbf{r})$ corresponds to a primal 2-cochain σ . While in 3-D, $\mathbf{D}(\mathbf{r})$ is viewed as a primal 2-cochain with divergence $\nabla \cdot \mathbf{D}(\mathbf{r})$ corresponding to a primal 3-cochain ρ .

For 2-D triangular mesh, if there are N_1 edges, the primal 1-cochain \mathbf{D} is defined as:

$$\mathbf{D} = \begin{pmatrix} D_1 \\ \vdots \\ D_{N_1} \end{pmatrix}, \quad D_i \triangleq \int_{l_i} [\mathbf{D}(\mathbf{r}) \cdot (\hat{z} \times \hat{l}_i)] dl. \quad (7)$$

Here, $\hat{z} \times \hat{l}_i$ is a unit vector perpendicular to the i -th primal edge l_i .

[†] Generally, the electric displacement field $\mathbf{D}(\mathbf{r})$ is treated as a dual cochain. Here we only use it to illustrate the idea since the divergence of magnetic field $\mathbf{B}(\mathbf{r})$ is always zero.

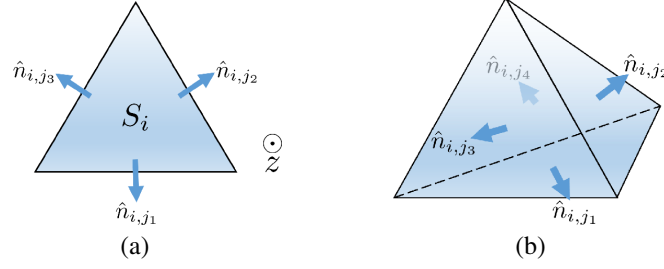


Figure 2. (a) Primal face S_i in a 2-D mesh. (b) Primal volume V_i in a 3-D mesh.

Therefore, as shown in Figure 2(a), the value of 2-cochain σ on i -th primal triangle S_i can be derived by 2-D Gauss' theorem:

$$\sigma_i = \int_{S_i} [\nabla_s \cdot \mathbf{D}(\mathbf{r})] dS = \oint_{\partial S_i} [\mathbf{D}(\mathbf{r}) \cdot \hat{\mathbf{n}}] dl = \sum_{l_j \in \partial S_i} \int_{l_j} [\mathbf{D}(\mathbf{r}) \cdot \hat{\mathbf{n}}_{i,j}] dl = - \sum_j d_{i,j}^{(1)} D_j, \quad (8)$$

where $\hat{\mathbf{n}}_{i,j} = -\hat{\mathbf{z}} \times \hat{\mathbf{l}}_{i,j}$ is a unit vector pointing outside S_i and perpendicular to $\hat{\mathbf{l}}_j$, and ∂S_i , $d_{i,j}^{(1)}$ are introduced earlier.

Therefore, the rightmost of Equation (8) is the i -th element of $-\bar{\mathbf{d}}^{(1)} \mathbf{D}$. Then relation $\sigma = -\bar{\mathbf{d}}^{(1)} \mathbf{D}$ is exact. It should be pointed out that this *negative* sign is introduced by the definition of cochain \mathbf{D} , shown in Equation (7).

Similarly, in a 3-D tetrahedral mesh, if there are N_2 triangular faces. The 2-cochain \mathbf{D} is defined as:

$$\mathbf{D} = \begin{pmatrix} D_1 \\ \vdots \\ D_{N_2} \end{pmatrix}, \quad D_i \triangleq \int_{S_i} [\mathbf{D}(\mathbf{r}) \cdot \hat{\mathbf{n}}_i] dS \quad (9)$$

where $\hat{\mathbf{n}}_i$ is the unit vector normal to the i -th primal face S_i .

Therefore, as shown in Figure 2(b), the value of 3-cochain ρ on i -th tet V_i can be derived by 3-D Gauss' theorem:

$$\rho_i = \int_{V_i} [\nabla \cdot \mathbf{D}(\mathbf{r})] dV = \oint_{\partial V_i} [\mathbf{D}(\mathbf{r}) \cdot \hat{\mathbf{n}}] dS = \sum_{S_j \in \partial V_i} \int_{S_j} [\mathbf{D}(\mathbf{r}) \cdot \hat{\mathbf{n}}_{i,j}] dS = \sum_j d_{i,j}^{(2)} D_j. \quad (10)$$

Here, ∂V_i is the closed surface of V_i , which includes four triangles; $\hat{\mathbf{n}}_{i,j}$ is a unit vector pointing outside V_i and normal to primal face S_j ; $d_{i,j}^{(2)}$ is the (i, j) element of discrete exterior derivative operator $\bar{\mathbf{d}}_{i,j}^{(2)}$, which is defined as

$$[\bar{\mathbf{d}}^{(2)}]_{i,j} = d_{i,j}^{(2)} = \begin{cases} +1 & \text{if } \hat{\mathbf{n}}_{i,j} = \hat{\mathbf{n}}_j, \\ -1 & \text{if } \hat{\mathbf{n}}_{i,j} = -\hat{\mathbf{n}}_j, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

Notice that the rightmost of Equation (10) is the i -th element of $\bar{\mathbf{d}}^{(2)} \mathbf{D}$. Then relation $\rho = \bar{\mathbf{d}}^{(2)} \mathbf{D}$ is also exact.

2.2. Discrete Gradient, Curl and Divergence for Dual Cochains

In the previous section, we demonstrate the discrete gradient, divergence, and curl for cochains on the primal mesh. Indeed, for dual cochains, similar relations can be derived. Here, dual cochains are defined on dual *cells*,[‡] which are constructed by connecting circumcenters, as shown in Figure 3. And notice that, Gauss' theorem and Stokes' theorem can also be applied for complex structures, such as polygons or polyhedrons. Therefore, in the same way, exact relations can also be constructed for gradient, divergence and curl for dual cochains.

[‡] Usually, dual faces or volumes are not simplices, as in Figure 3. So we use dual *cells* to denote them.

However, since the dual mesh is deduced from the primal mesh by connecting nearest circumcenters, the mesh is incomplete at the boundary. For example, in Figure 3, the dual edges $\overline{d_1 e_1}$ and $\overline{e_2 d_2}$ are in fact “truncated” by the primal mesh boundary. And these two dual edges are not line segment between two dual vertices, because there is no well defined dual vertices outside the primal mesh boundary. Instead, dual edges on the boundary are constructed by connecting circumcenters (d_1 and d_2) and centers of corresponding primal edges (e_1 for $\overline{p_1 p_2}$, and e_2 for $\overline{p_2 p_3}$). Moreover, the dual faces and volumes surrounded by these boundary dual edges and faces are also incomplete. For example, the dual face \mathcal{A}_{p_2} [§] in Figure 3 is only enclosed by three dual edges (solid lines): $\overline{d_1 e_1}$, $\overline{e_2 d_2}$, and $\overline{d_2 d_1}$. The other two edges (dashed lines) $\overline{e_1 p_2}$ and $\overline{p_2 e_2}$ are half of primal edges. Due to this incompleteness of dual mesh on the boundary, a certain kind of boundary condition needs to be imposed for discrete gradient, curl, divergence of dual cochains.

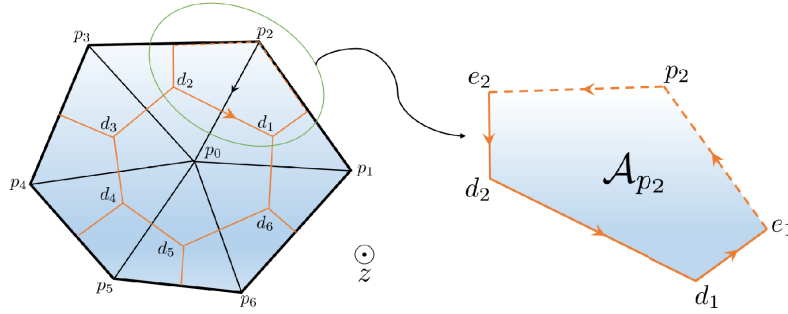


Figure 3. Dual cells at the boundary (bold edges) in a 2-D mesh. Here, d_1, \dots, d_6 are circumcenters, and e_1, e_2 are edge centers.

2.2.1. Discrete Gradient for Dual Cochains

In a 2-D mesh, a scalar function $\psi(\mathbf{r})$ can also be treated as a dual 0-cochain ψ with length N_2 on the dual mesh:

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{N_2} \end{pmatrix}, \quad \psi_i \triangleq \psi(\mathbf{r}_{d_i}) \tag{12}$$

where d_i is the circumcenter of i -th primal face, also the i -th dual vertex. In 3-D meshes, the length of this 0-cochain becomes N_3 , the number of tetrahedrons.

Then the gradient of $\psi(\mathbf{r})$, $\nabla\psi(\mathbf{r})$, corresponds to a dual 1-cochain \mathbf{h} . The value of this dual 1-cochain along dual edges strictly inside, which are not at the boundary, can be derived in the same manner as we did for primal cochains in the previous section. In a 2-D mesh, along i -th dual edge \mathcal{L}_i :

$$h_i = \int_{\mathcal{L}_i} \left[\nabla\psi(\mathbf{r}) \cdot (\hat{z} \times \hat{l}_i) \right] dl = \psi(\mathbf{r}_{d_{j_1}}) - \psi(\mathbf{r}_{d_{j_2}}) = \sum_j d_{\text{dual},i,j}^{(0)} \psi_j = \sum_j d_{j,i}^{(1)} \psi_j \tag{13}$$

Here, the unit vector along dual edge \mathcal{L}_i is obtained by rotating unit vector along corresponding primal edge l_i by 90 degrees anticlockwise,

$$\hat{\mathcal{L}}_i = \hat{z} \times \hat{l}_i$$

as illustrated in Figure 3 by the direction of dual edge $\overline{d_2 d_1}$ and primal edge $\overline{p_2 p_0}$. The last equality in Equation (13) is because that in a 2-D mesh below relations hold true [12].

$$\bar{\mathbf{d}}_{\text{dual}}^{(0)} = \left(\bar{\mathbf{d}}^{(1)} \right)^T \quad \text{and} \quad \bar{\mathbf{d}}_{\text{dual}}^{(1)} = - \left(\bar{\mathbf{d}}^{(0)} \right)^T. \tag{14}$$

[§] In this paper, we use \mathcal{A} to denote dual faces, which are usually polygons.

Therefore, for cochain values defined on edges strictly inside, $\mathbf{h} = (\bar{\mathbf{d}}^{(1)})^T \boldsymbol{\psi}$ is exactly satisfied. While in 3-D, since Equation (14) becomes [12]

$$\bar{\mathbf{d}}_{\text{dual}}^{(0)} = (\bar{\mathbf{d}}^{(2)})^T, \quad \bar{\mathbf{d}}_{\text{dual}}^{(1)} = (\bar{\mathbf{d}}^{(1)})^T, \quad \bar{\mathbf{d}}_{\text{dual}}^{(2)} = -(\bar{\mathbf{d}}^{(0)})^T, \quad (15)$$

Then the equation becomes $\mathbf{h} = (\bar{\mathbf{d}}^{(2)})^T \boldsymbol{\psi}$.

However, along dual edges at the boundary, such as $\overline{d_1 e_1}$ in Figure 3, Dirichlet or Neumann boundary conditions need to be imposed. For these two kinds of boundary conditions, the value of dual 1-cochain \mathbf{h} on the boundary dual edges can be defined as:

$$h_{\overline{d_1 e_1}} = \int_{\mathbf{r}_{d_1}}^{\mathbf{r}_{e_1}} \nabla \psi(\mathbf{r}) \cdot d\mathbf{l} = c - \psi_1, \quad \psi = c \text{ on the boundary}, \quad (16)$$

$$h_{\overline{d_1 e_1}} = \int_{\mathbf{r}_{d_1}}^{\mathbf{r}_{e_1}} \nabla \psi(\mathbf{r}) \cdot d\mathbf{l} \approx c |\overline{d_1 e_1}|, \quad \frac{\partial \psi}{\partial n} = c \text{ on the boundary}. \quad (17)$$

Here, $|\dots|$ means the length of a line segment.

2.2.2. Discrete Curl for Dual Cochains

For a vectorial case, if we have a vector field $\mathbf{H}(\mathbf{r})$, then it can be treated as a dual 1-cochain \mathbf{H} on the dual mesh as:

$$\mathbf{H} = \begin{pmatrix} H_1 \\ \vdots \\ H_{N_1} \end{pmatrix}, \quad H_i \triangleq \begin{cases} \int_{\mathcal{L}_i} [\mathbf{H}(\mathbf{r}) \cdot (\hat{z} \times \hat{l}_i)] dl, & \text{2-D mesh} \\ \int_{\mathcal{L}_i} [\mathbf{H}(\mathbf{r}) \cdot \hat{n}_i] dl, & \text{3-D mesh} \end{cases} \quad (18)$$

Then the curl of this vector field $\nabla \times \mathbf{H}(\mathbf{r})$ is represented by a dual 2-cochain \mathbf{d} , and its value on dual faces strictly inside can be derived the same as before, which means that

$$\mathbf{d} = \bar{\mathbf{d}}_{\text{dual}}^{(1)} \mathbf{H} = -(\bar{\mathbf{d}}^{(0)})^T \mathbf{H}$$

for a 2-D mesh, and

$$\mathbf{d} = \bar{\mathbf{d}}_{\text{dual}}^{(1)} \mathbf{H} = (\bar{\mathbf{d}}^{(1)})^T \mathbf{H}$$

for a 3-D mesh.

However, as mentioned, these relations are incomplete at the boundary. For example, in a 2-D mesh as shown in Figure 3, the value of \mathbf{d} on \mathcal{A}_{p_2} can be derived by Stokes' theorem as:

$$\begin{aligned} d_{\mathcal{A}_{p_2}} &= \int_{\mathcal{A}_{p_2}} [\nabla \times \mathbf{H}(\mathbf{r})] \cdot d\mathbf{S} = \oint_{\partial \mathcal{A}_{p_2}} \mathbf{H}(\mathbf{r}) \cdot d\mathbf{l} = \sum_{\mathcal{L}_j \in \partial \mathcal{A}_{p_2}} \int_{\mathcal{L}_j} \mathbf{H}(\mathbf{r}) \cdot d\mathbf{l} \\ &= H_{\overline{d_1 e_1}} + H_{\overline{e_2 d_2}} + H_{\overline{d_2 d_1}} + H_{\overline{e_1 p_2}} + H_{\overline{p_2 e_2}}. \end{aligned} \quad (19)$$

Here, the first three terms are the value of dual 2-cochain $\bar{\mathbf{d}}_{\text{dual}}^{(1)} \mathbf{H}$ or $-(\bar{\mathbf{d}}^{(0)})^T \mathbf{H}$, associated with dual face \mathcal{A}_{p_2} , while the last two terms $H_{\overline{e_1 p_2}}$ and $H_{\overline{p_2 e_2}}$ need to be determined by a certain boundary conditions. For example, with perfect magnetic conductor (PMC) boundary condition, they are both zero. For a general boundary condition, they have nonzero values which need to be left. Therefore, we should have a corrected relation as below:

$$\mathbf{d} = -(\bar{\mathbf{d}}^{(0)})^T \mathbf{H} + \mathbf{J}_s. \quad (20)$$

Here, \mathbf{J}_s is a dual 2-cochain in this 2-D mesh, and represents the tangential component of \mathbf{H} on the boundary. In fact, if we restrict just on the boundary, which is a 1-D discrete manifold, this \mathbf{J}_s cochain

is viewed as a dual 1-cochain. This cochain is only defined on the boundary dual faces. For example, its value on \mathcal{A}_{p_2} is

$$\begin{aligned} J_{s, \mathcal{A}_{p_2}} &= H_{\overline{e_1 p_2}} + H_{\overline{p_2 e_2}} = \int_{\overline{e_1 p_2, p_2 e_2}} \mathbf{H}(\mathbf{r}) \cdot d\mathbf{l} \\ &= H_t (|\overline{e_1 p_2}| + |\overline{p_2 e_2}|) = H_t |\mathcal{L}_{p_2}^\partial|. \end{aligned} \quad (21)$$

Here, H_t is the average tangential value of $\mathbf{H}(\mathbf{r})$ field, and $|\mathcal{L}_{p_2}^\partial| = |\overline{e_1 p_2}| + |\overline{p_2 e_2}|$. In practice, with different boundary conditions, \mathbf{J}_s may have different values [14]. When there is no boundary, which means that the space we consider is closed, although not possible in computational electromagnetics, \mathbf{J}_s also disappears.

It should be noted that although a 2-D example is used to illustrate the curl operation on vector field $\mathbf{H}(\mathbf{r})$, in 3-D space, the derivation would be exactly the same except that the dual face at the boundary is no longer just associated with a vertex, e.g., \mathcal{A}_{p_2} and p_2 . Instead, the dual face at the boundary is associated with a primal edge, and below relation holds

$$\mathbf{d} = \left(\bar{\mathbf{d}}^{(1)}\right)^T \mathbf{H} + \mathbf{J}_s. \quad (22)$$

Here, \mathbf{J}_s is also a dual 2-cochain, and can be viewed as a dual 1-cochain on the boundary, which is a 2-D discrete manifold now.

2.2.3. Discrete Divergence for Dual Cochains

The discrete divergence for dual cochains can also be induced similarly. Certain boundary conditions are also needed to complete this relation. To be clear, we first consider the divergence of a vector field $\mathbf{D}(\mathbf{r})$ in 2-D space. This field can be viewed as a dual 1-cochain \mathbf{D} as:

$$\mathbf{D} = \begin{pmatrix} D_1 \\ \vdots \\ D_{N_1} \end{pmatrix}, \quad D_i \triangleq \int_{\mathcal{L}_i} [\mathbf{D}(\mathbf{r}) \cdot \hat{l}_i] dl. \quad (23)$$

Here, \hat{l}_i , the unit vector along primal edge l_i , is normal to dual edge \mathcal{L}_i , shown in Figure 4. Then, by Gauss' theorem, the discrete divergence of this vector field $\nabla_s \cdot \mathbf{D}$ can be represented as a dual 2-cochain σ . Strictly inside the boundary, following relation holds

$$\sigma = \bar{\mathbf{d}}_{\text{dual}}^{(1)} \mathbf{D} = -\left(\bar{\mathbf{d}}^{(0)}\right)^T \mathbf{D}.$$

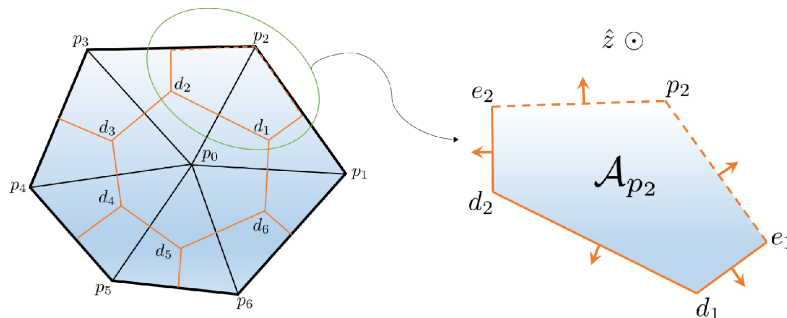


Figure 4. Dual cells at the boundary (bold edges) in 2-D space. \hat{z} direction is out of the paper.

At the boundary, the value of σ on dual face \mathcal{A}_{p_2} in Figure 4 is written as:

$$\begin{aligned} \sigma_{\mathcal{A}_{p_2}} &= \int_{\mathcal{A}_{p_2}} [\nabla_s \cdot \mathbf{D}(\mathbf{r})] dS = \oint_{\partial \mathcal{A}_{p_2}} [\mathbf{D}(\mathbf{r}) \cdot \hat{n}] dl = \sum_{\mathcal{L}_j \in \partial \mathcal{A}_{p_2}} \int [\mathbf{D}(\mathbf{r}) \cdot \hat{n}] dl \\ &= D_{\overline{d_1 e_1}} + D_{\overline{e_2 d_2}} + D_{\overline{d_2 d_1}} + D_{\overline{e_1 p_2}} + D_{\overline{p_2 e_2}} \end{aligned} \quad (24)$$

where \hat{n} is a unit vector normal to the boundary and pointing outside \mathcal{A}_{p_2} . Similarly, the first three terms in the second line of Equation (24) are the value of dual 2-cochain $\bar{\mathbf{d}}_{\text{dual}}^{(1)} \mathbf{D}$ or $-(\bar{\mathbf{d}}^{(0)})^T \mathbf{D}$ on this dual face \mathcal{A}_{p_2} , but the last two terms $D_{\underline{e_1 p_2}}$ and $D_{\underline{p_2 e_2}}$ are not included. Therefore, we should have relation:

$$\boldsymbol{\sigma} = -(\bar{\mathbf{d}}^{(0)})^T \mathbf{D} + \boldsymbol{\lambda}_s. \quad (25)$$

Here, $\boldsymbol{\lambda}_s$ is a dual 2-cochain representing the normal component of \mathbf{D} on the boundary. If only considering the boundary, which is a discrete 1-D manifold, $\boldsymbol{\lambda}_s$ is a dual 1-cochain. And it only contains elements defined on the boundary dual faces. For example, on dual face \mathcal{A}_{p_2} , the value of $\boldsymbol{\lambda}_s$ is

$$\lambda_{s, \mathcal{A}_{p_2}} = D_{\underline{e_1 p_2}} + D_{\underline{p_2 e_2}} = \int_{\underline{e_1 p_2, p_2 e_2}} [\mathbf{D}(\mathbf{r}) \cdot \hat{n}] dl \approx D_n |\mathcal{L}_{p_2}^\partial| \quad (26)$$

where $D_n = \mathbf{D}(\mathbf{r}) \cdot \hat{n}$ is the average normal component of \mathbf{D} field. Therefore, with the value of D_n given as a boundary condition, we can determine $\boldsymbol{\lambda}_s$.

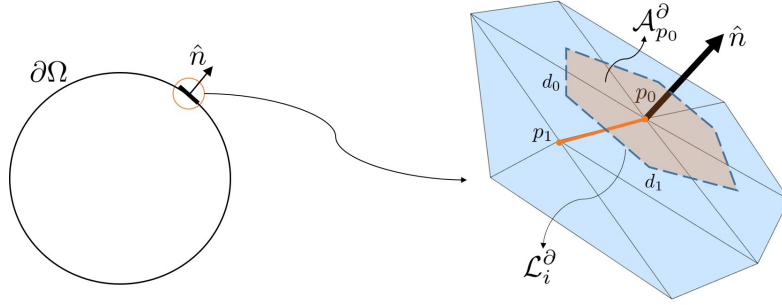


Figure 5. Dual face at the boundary (surrounded by dashed edges) in 3-D space. \hat{n} is a unit vector normal to the surface, and $\mathcal{A}_{p_0}^\partial$ (orange region) is the dual 2-cell on the surface boundary associated with p_0 .

In a 3-D mesh, the relation is quite similar. But now the vector field $\mathbf{D}(\mathbf{r})$ should be viewed as a dual 2-cochain, and its divergence is a dual 3-cochain $\boldsymbol{\rho}$, which represents the volume charge density including a strictly inside component and a boundary (surface) component:

$$\boldsymbol{\rho} = -(\bar{\mathbf{d}}^{(0)})^T \mathbf{D} + \boldsymbol{\sigma}_s. \quad (27)$$

Here, $\boldsymbol{\sigma}_s$ is a dual 3-cochain, and it also only contains elements on the boundary. As shown in Figure 5, the value of $\boldsymbol{\sigma}_s$ on $\mathcal{A}_{p_0}^\partial$ is

$$\sigma_{s, \mathcal{A}_{p_0}^\partial} = \int_{\mathcal{A}_{p_0}^\partial} (\mathbf{D}(\mathbf{r}) \cdot \hat{n}) dS \approx D_n |\mathcal{A}_{p_0}^\partial| \quad (28)$$

where $|\mathcal{A}_{p_0}^\partial|$ refers to the area of $\mathcal{A}_{p_0}^\partial$. Since the boundary surface $\partial\Omega$ can be viewed as a discrete 2-D manifold, cochain $\boldsymbol{\sigma}_s$ is a dual 2-cochain on this manifold. If the normal component of $\mathbf{D}(\mathbf{r})$ field D_n at the surface is given as a boundary condition, $\boldsymbol{\sigma}_s$ is determined.

2.3. Discrete Laplacian Operator

With discrete divergence and gradient defined for both primal and dual cochains, the Laplacian operator $\nabla \cdot \nabla$ can then be well defined for a scalar field $\phi(\mathbf{r})$. For simplicity, we treat this scalar field as a primal 0-cochain ϕ . Then the first gradient operator take the primal 0-cochain to primal 1-cochain $\bar{\mathbf{d}}^{(0)} \phi$. Then, in 3-D space, for next divergence operation to function properly, a Hodge star operator is needed to map this primal 1-cochain to a dual 2-cochain:

$$\bar{\mathbf{d}}^{(0)} \phi \rightarrow \star^{(1)} \bar{\mathbf{d}}^{(0)} \phi$$

where the superscript of Hodge star operator means the degree of cochains it acts on. Then as discussed earlier in Equation (27), the divergence of this dual 2-cochain will introduce a boundary term:

$$\nabla \cdot \nabla \phi(\mathbf{r}) \Rightarrow - \left[\left(\bar{\mathbf{d}}^{(0)} \right)^T \star^{(1)} \bar{\mathbf{d}}^{(0)} \right] \phi + \partial_n \phi. \quad (29)$$

where $\partial_n \phi$ is the boundary term introduced by discrete divergence acting on dual 2-cochain $\star^{(1)} \bar{\mathbf{d}}^{(0)} \phi$, and it is a dual 3-cochain in the whole space and a dual 2-cochain on the boundary surface.

A vectorial version of this, $\nabla \times \nabla \times$ operator^{||}, can be written similarly. We will then examine how this operator applies to a vector field $\mathbf{E}(\mathbf{r})$. First, this vector field $\mathbf{E}(\mathbf{r})$ can be viewed as a primal 1-cochain \mathbf{E} . Then the first curl operator acts as $\bar{\mathbf{d}}^{(1)}$ and will generate a primal 2-cochain:

$$\nabla \times \mathbf{E}(\mathbf{r}) \Rightarrow \bar{\mathbf{d}}^{(1)} \mathbf{E}. \quad (30)$$

Then a hodge star operator is needed for the second curl operator to function appropriately, which leads to a dual 1-cochain $\star^{(2)} \bar{\mathbf{d}}^{(1)} \mathbf{E}$. Then the second curl operator will introduce an extra boundary term as in Equation (20):

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \Rightarrow \left[\left(\bar{\mathbf{d}}^{(1)} \right)^T \star^{(2)} \bar{\mathbf{d}}^{(1)} \right] \mathbf{E} + i\omega\mu \mathbf{J}_s. \quad (31)$$

Here, coefficient $i\omega\mu$ is added to be consistent with Maxwell's equations, and \mathbf{J}_s represents the tangential component of $\nabla \times \mathbf{E}(\mathbf{r})$ or $\mathbf{H}(\mathbf{r})$ on the boundary.

3. WEDGE PRODUCT

In exterior calculus, wedge product is used to construct higher degree forms from lower degree ones. For example, with two 1-forms e and h in 3-D space, their wedge product $e \wedge h$ is a 2-form. And this corresponds to the fact that the cross product of two vector fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ gives rise to a new vector field $\mathbf{S}(\mathbf{r}) = \mathbf{E} \times \mathbf{H}$. Similarly, a wedge product between a 1-form e and a 2-form d in 3-D space gives rise to a 3-form $e \wedge d$, which refers to a scalar field generated by dot product between two vector fields. In fact, the dot product between electric field $\mathbf{E}(\mathbf{r})$ and displacement field $\mathbf{D}(\mathbf{r})$ is proportional to electric energy density, which can be integrated over a volume to obtain stored electric energy.

In this paper, we only need to introduce a discrete treatment of wedge product between a primal cochain and its corresponding dual cochain, which was described briefly in [12]. With a primal k -cochain \mathbf{b} and a dual $(n-k)$ -cochain \mathbf{h} , their wedge product should give rise to a n -cochain, or a volume cochain. In practice, we are more interested in the integration of a volume form or a volume cochain, so it is not crucial whether this n -cochain should be a primal one or dual one.

Since cochains are column vectors, we use an inner product between two column vectors to represent the volume integration of their wedge product. For example, with a primal 1-cochain \mathbf{E} representing the electric field and dual 2-cochain \mathbf{D} for the electric displacement field in a given mesh, the stored electric energy (up to a constant difference) can be represented as:

$$\int [\mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r})] dV \Rightarrow \mathbf{E}^T \cdot \mathbf{D} \quad (32)$$

Here, the dot product is valid since a dual 2-cochain have the same length with a primal 1-cochain, which is N_1 , the number of primal edges.

Similarly, we can also represent the magnetic energy in a discrete fashion, which is an inner product between a primal 2-cochain \mathbf{B} and a dual 1-cochain \mathbf{H} both with length N_2

$$\int [\mathbf{B}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r})] dV \Rightarrow \mathbf{B}^T \cdot \mathbf{H} \quad (33)$$

Moreover, we can also use this to represent the integration of two scalar functions, $\phi(\mathbf{r})$ and $\rho(\mathbf{r})$. We think $\phi(\mathbf{r})$ and $\rho(\mathbf{r})$ as a primal 0-cochain ϕ and dual 3-cochain ρ . Then the integration of their product over a volume will be:

$$\int [\phi(\mathbf{r}) \rho(\mathbf{r})] dV \Rightarrow \phi^T \cdot \rho. \quad (34)$$

^{||} Strictly speaking, this is not the Laplacian operator for a vector field, $-\nabla \nabla \cdot$ needs to be added.

Next we would like to show this treatment of discrete wedge product is rigorous under local constant field assumption on arbitrary 2-D surfaces. Since the relation in Equation (34) is trivial, we will focus on the wedge product between two vector fields. To show this, a mathematical identity needs to be introduced.

3.1. A Mathematical Identity

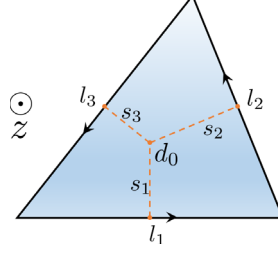


Figure 6. A triangle with primal edges l_1, l_2, l_3 and dual edges s_1, s_2, s_3 . Here, d_0 is the circumcenter.

In Figure 6, we use $\hat{l}_1, \hat{l}_2, \hat{l}_3$ to represent three unit vectors along three primal edges, and l_i, s_i to denote the length of each primal and dual edge for simplicity. Then, below relation holds true:

$$l_1 s_1 \hat{l}_1 \hat{l}_1^T + l_2 s_2 \hat{l}_2 \hat{l}_2^T + l_3 s_3 \hat{l}_3 \hat{l}_3^T = S \bar{\mathbf{I}} \quad (35)$$

where S is the area of this triangle, and $\bar{\mathbf{I}}$ is a 2×2 identity matrix. We will prove this in the appendix.

Then with two local constant two-dimensional vector field $\mathbf{E}(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$, the integration of their dot product over the triangle in Figure 6 is

$$\begin{aligned} \int [\mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r})] dS &\approx [\mathbf{E} \cdot \mathbf{D}] S = \mathbf{E} \cdot S \bar{\mathbf{I}} \cdot \mathbf{D} \\ &= \mathbf{E} \cdot \left(\sum_{i=1}^3 l_i s_i \hat{l}_i \hat{l}_i^T \right) \cdot \mathbf{D} \\ &= \sum_{i=1}^3 [(\mathbf{E} \cdot \hat{l}_i) l_i] [(\hat{l}_i^T \cdot \mathbf{D}) s_i] \\ &= \sum_{i=1}^3 E_i D_i \end{aligned} \quad (36)$$

where $E_i = (\mathbf{E} \cdot \hat{l}_i) l_i$ is the value of cochain \mathbf{E} on primal edge l_i , and $D_i = (\mathbf{D} \cdot \hat{l}_i) s_i$ is the value of cochain \mathbf{D} on dual edge s_i . In a 2-D mesh with multiple triangles, the value of dual cochain \mathbf{D} on each dual edge normally comes from two triangles, which can be combined. Then the relation becomes

$$\int [\mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r})] dS = \sum_{i=1}^{N_2} \int_{A_i} [\mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r})] dS \approx \sum_{j=1}^{N_1} E_j D_j = \mathbf{E}^T \cdot \mathbf{D} \quad (37)$$

where A_i refers to i -th primal face. Therefore, Equation (37) shows that the dot product between a primal 1-cochain and a dual 1-cochain is a good approximation of the integration of $\mathbf{E} \cdot \mathbf{D}$.

The relation in Equation (35) is also true with \hat{l}_i changing to $\hat{n}_i = \hat{z} \times \hat{l}_i$

$$l_1 s_1 \hat{n}_1 \hat{n}_1^T + l_2 s_2 \hat{n}_2 \hat{n}_2^T + l_3 s_3 \hat{n}_3 \hat{n}_3^T = S \bar{\mathbf{I}} \quad (38)$$

With this relation, we can prove

$$\int [\mathbf{B}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r})] dS \approx \sum_{j=1}^{N_1} B_j H_j = \mathbf{B}^T \cdot \mathbf{H} \quad (39)$$

in a 2-D mesh in the same way.

Moreover, we can also consider the integration of $\mathbf{E} \times \mathbf{H}$ on a 2-D surface, which is not necessarily a plane, as a wedge product

$$\begin{aligned} \int [\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})] \cdot \hat{\mathbf{n}} \, dS &= - \int [\mathbf{E}(\mathbf{r}) \cdot (\hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r}))] \, dS \\ &\approx - \sum_{j=1}^{N_1} E_j \left[\tilde{l}_j^T \cdot (\hat{\mathbf{n}} \times \mathbf{H}) s_j \right] \\ &= \mathbf{E}^T \cdot \mathbf{H} \end{aligned} \quad (40)$$

where $\hat{\mathbf{n}}$ is the normal unit vector of this surface.

It should be pointed out that in a general 3-D mesh, although the counterpart relation of Equation (35) does not hold exactly, we still adopt inner product in Equations (37) and (39) as the volume integration of discrete primal-dual wedge product. With this discrete treatment of wedge product, we can derive the discrete version of Huygens' principle, Poynting's theorem, and reciprocity theorem.

4. DISCRETE ELECTROMAGNETIC THEORY

With the discrete calculus introduced in Section 2, in 3-D space without a boundary, discrete Maxwell's equations can be written as

$$\bar{\mathbf{d}}^{(1)} \mathbf{E} = i\omega \mathbf{B}, \quad (41a)$$

$$\left(\bar{\mathbf{d}}^{(1)} \right)^T \mathbf{H} = -i\omega \mathbf{D} + \mathbf{J}, \quad (41b)$$

$$\bar{\mathbf{d}}^{(2)} \mathbf{B} = 0, \quad (41c)$$

$$- \left(\bar{\mathbf{d}}^{(0)} \right)^T \mathbf{D} = \rho, \quad (41d)$$

where relations in Equation (15) have been applied.

For any k -cochain ω_k , it is easy to verify that:

$$\gamma_{k+2} = \bar{\mathbf{d}}^{(k+1)} \bar{\mathbf{d}}^{(k)} \omega_k = 0, \quad k \leq 1 \text{ in a 3-D mesh, and } k = 0 \text{ in a 2-D mesh.}$$

In fact, this is because, in the boundary of the boundary of a k -simplex, a $(k-2)$ -simplex always appears twice with different signs [12]. In exterior calculus language, this means that any exact form is automatically closed. The opposite way is also guaranteed by Poincaré lemma. Therefore, in a 3-D mesh, due to relations $\bar{\mathbf{d}}^{(1)} \cdot \bar{\mathbf{d}}^{(0)} = 0$ and $\bar{\mathbf{d}}^{(2)} \cdot \bar{\mathbf{d}}^{(1)} = 0$, $\nabla \times \nabla = 0$ and $\nabla \cdot \nabla \times = 0$ are exactly preserved. In fact, the second relation gives rise to charge continuity equation, also derived in [14].

$$\left(\bar{\mathbf{d}}^{(0)} \right)^T \cdot \left(\bar{\mathbf{d}}^{(1)} \right)^T \mathbf{H} = 0 \quad \Rightarrow \quad \left(\bar{\mathbf{d}}^{(0)} \right)^T (-i\omega \mathbf{D} + \mathbf{J}) = 0 \quad \Rightarrow \quad i\omega \rho + \left(\bar{\mathbf{d}}^{(0)} \right)^T \mathbf{J} = 0 \quad (42)$$

This means that Maxwell's equations in DEC are consistent with charge conservation, which is the same with continuous theory. Therefore, numerical simulations based on DEC will not generate spurious charges.

Moreover, two constitutive relations are needed to complete Maxwell's equations

$$\mathbf{D} = \star_{\epsilon}^{(1)} \mathbf{E} \quad \text{and} \quad \mathbf{H} = \star_{\mu^{-1}}^{(2)} \mathbf{B}. \quad (43)$$

Here, two Hodge star operators $\star_{\epsilon}^{(1)}$ and $\star_{\mu^{-1}}^{(2)}$ map primal cochains \mathbf{E} and \mathbf{B} to dual cochains \mathbf{D} and \mathbf{H} , superscripts refer to the degree of cochains they are acting on, and subscripts ϵ and μ^{-1} means that they are characterized by these two material information, which is described in detail in [14]. For homogeneous medium, $\star_{\epsilon}^{(1)} = \epsilon \star^{(1)}$ and $\star_{\mu^{-1}}^{(2)} = \mu^{-1} \star^{(2)}$.

Noted that the choice of \mathbf{E} and \mathbf{B} as primal cochains and \mathbf{H} , \mathbf{D} and \mathbf{J} as dual cochains is only by convention, not a requirement. We have shown that the other choice is beneficial for PEC boundary condition implementation [14].

4.1. Green's Function

4.1.1. Scalar Waves

In continuous theory, a Green's function of a wave equation is the solution of the wave equation with a point source. When this point source solution is known, we can obtain the solution to a general source through superposition. For example, the scalar wave equation in 3-D can be written as:

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = s(\mathbf{r}). \quad (44)$$

Then by treating $\psi(\mathbf{r})$, $s(\mathbf{r})$ as a primal 0-cochain $\boldsymbol{\psi}$ and a primal 3-cochain \mathbf{s} (both are $N_0 \times 1$ column vectors), we can rewrite this equation in a discrete fashion as:

$$\left[\left(\bar{\mathbf{d}}^{(0)} \right)^T \star^{(1)} \bar{\mathbf{d}}^{(0)} - k^2 \star^{(0)} \right] \cdot \boldsymbol{\psi} = -\mathbf{s}. \quad (45)$$

Noted that the space considered is assumed to be infinite, so that the boundary term is dropped.

The scalar Green's function $g(\mathbf{r}, \mathbf{r}')$ is the solution to following equation:

$$(\nabla^2 + k^2)g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'). \quad (46)$$

Here, for each given \mathbf{r}' , $g(\mathbf{r}, \mathbf{r}')$ should be treated as a $N_0 \times 1$ column vector (0-cochain). Then with N_0 possible \mathbf{r}' , Green's function $g(\mathbf{r}, \mathbf{r}')$ should be considered as a $N_0 \times N_0$ matrix $\bar{\mathbf{g}}$. Similarly, for any given \mathbf{r}' , $\delta(\mathbf{r} - \mathbf{r}')$ should be considered as a $N_0 \times 1$ column vector (3-cochain) with only one nonzero entry with value 1, where $\mathbf{r} = \mathbf{r}'$. Therefore, a discrete $\delta(\mathbf{r} - \mathbf{r}')$ can be replaced by a $N_0 \times N_0$ identity matrix.

$$\left[\left(\bar{\mathbf{d}}^{(0)} \right)^T \star^{(1)} \bar{\mathbf{d}}^{(0)} - k^2 \star^{(0)} \right] \cdot \bar{\mathbf{g}} = \bar{\mathbf{I}}. \quad (47)$$

Here the boundary term is also dropped. Then,

$$\bar{\mathbf{g}} = - \left[\left(\bar{\mathbf{d}}^{(0)} \right)^T \star^{(1)} \bar{\mathbf{d}}^{(0)} + k^2 \star^{(0)} \right]^{-1} \quad (48)$$

is the discrete solution of Green's function in an infinite homogeneous space. Notice that since Hodge operators are diagonal, this discrete Green's function is a symmetric matrix. This coincides with the fact that for the continuous solution, $g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}', \mathbf{r})$.

Then we left multiply Equation (45) with $\bar{\mathbf{g}}^T$, and right multiply Equation (47)'s transpose with $\boldsymbol{\psi}$:

$$\bar{\mathbf{g}}^T \cdot (45) \Rightarrow \bar{\mathbf{g}}^T \cdot \left[\left(\bar{\mathbf{d}}^{(0)} \right)^T \star^{(1)} \bar{\mathbf{d}}^{(0)} - k^2 \star^{(0)} \right] \cdot \boldsymbol{\psi} = -\bar{\mathbf{g}}^T \cdot \mathbf{s} \quad (49a)$$

$$(47)^T \cdot \boldsymbol{\psi} \Rightarrow \bar{\mathbf{g}}^T \cdot \left[\left(\bar{\mathbf{d}}^{(0)} \right)^T \star^{(1)} \bar{\mathbf{d}}^{(0)} - k^2 \star^{(0)} \right] \cdot \boldsymbol{\psi} = \boldsymbol{\psi} \quad (49b)$$

Therefore, we can represent the solution $\boldsymbol{\psi}$ due to an arbitrary source \mathbf{s} with Green's function as:

$$\boldsymbol{\psi} = -\bar{\mathbf{g}}^T \cdot \mathbf{s}. \quad (50)$$

Obviously, this is a discrete analog of the continuous version

$$\psi(\mathbf{r}) = - \int_V d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') s(\mathbf{r}'). \quad (51)$$

In fact, this can also be understood from wedge product between a primal 0-cochain and a dual 3-cochain. More specifically, for any given \mathbf{r} , scalar field $g(\mathbf{r}, \mathbf{r}')$, with respect to \mathbf{r}' , corresponds to a primal 0-cochain, and $s(\mathbf{r}')$ is viewed as a dual 3-cochain. In Section 3, we have introduced the volume integration of a wedge product between these two cochains can be represented by an inner product. Therefore, with N_0 possible \mathbf{r} , this integration can be represented by a matrix vector product.

4.1.2. Vector Waves

For vector waves, a dyadic Green's function is introduced to describe the solution to a Hertzian dipole source. First, the equation for the vector wave in a homogeneous, isotropic medium is

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2 \mathbf{E}(\mathbf{r}) = i\omega\mu \mathbf{J}(\mathbf{r}). \quad (52)$$

where $k = \omega\sqrt{\mu\epsilon}$ is the wave number.

Then, similarly, by treating $\mathbf{E}(\mathbf{r})$ and $\mathbf{J}(\mathbf{r})$ as a primal 1-cochain \mathbf{E} and a dual 2-cochain \mathbf{J} (both are $N_1 \times 1$ column vectors), this vector wave equation can be written in a discrete fashion as:

$$\left[\left(\bar{\mathbf{d}}^{(1)} \right)^T \star^{(2)} \bar{\mathbf{d}}^{(1)} - k^2 \star^{(1)} \right] \cdot \mathbf{E} = i\omega\mu \mathbf{J}. \quad (53)$$

Here, the equation is assumed for the whole space, so boundary term is also dropped.

The dyadic Green's function $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ is solution to following equation:

$$\nabla \times \nabla \times \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - k^2 \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \bar{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}'). \quad (54)$$

Here, slightly different from the scalar case, for each given \mathbf{r}' , $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ is a tensor field, which is a 3×3 matrix at any \mathbf{r} . In fact, for each given Hertzian dipole current source placed at primal edge around \mathbf{r}' , if the unit vector along this primal edge is \hat{l}' , then $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \hat{l}'$ is a vector field and can be treated as a primal 1-cochain, which is a $N_1 \times 1$ (N_1 is the number of primal edges) column vector. Then with N_1 possible \mathbf{r}' , $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ can be viewed as an $N_1 \times N_1$ matrix, and $\bar{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}')$ on the right hand side should be treated as an $N_1 \times N_1$ identity matrix. Then, with DEC, the equation for dyadic Green's function can be written as:

$$\left[\left(\bar{\mathbf{d}}^{(1)} \right)^T \star^{(2)} \bar{\mathbf{d}}^{(1)} - k^2 \star^{(1)} \right] \cdot \bar{\mathbf{G}} = \bar{\mathbf{I}}. \quad (55)$$

Therefore,

$$\bar{\mathbf{G}} = \left[\left(\bar{\mathbf{d}}^{(1)} \right)^T \star^{(2)} \bar{\mathbf{d}}^{(1)} - k^2 \star^{(1)} \right]^{-1} \quad (56)$$

is the discrete dyadic Green's function in homogeneous, isotropic medium. Because Hodge star matrices are diagonal, hence symmetric, this $\bar{\mathbf{G}}$ is also a symmetric matrix. Combining Equations (53) and (55), we can obtain

$$\mathbf{E} = i\omega\mu \bar{\mathbf{G}}^T \cdot \mathbf{J}. \quad (57)$$

Clearly, this is the discrete version of

$$\mathbf{E}(\mathbf{r}) = i\omega\mu \int_V \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') \, d\mathbf{r}'. \quad (58)$$

We can also view Equation (57) as a discrete wedge product. To see this, we can write the value of primal 1-cochain \mathbf{E} on a primal edge l_i around \mathbf{r} by using Equation (58):

$$E_i = \int_{l_i} \left[\hat{l}_i^T \cdot \mathbf{E}(\mathbf{r}) \right] dl = i\omega\mu \int_V \left[\int_{l_i} \hat{l}_i^T \cdot \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') dl \right] \cdot \mathbf{J}(\mathbf{r}') \, d\mathbf{r}' \quad (59)$$

For a given primal edge l_i , $\int_{l_i} \hat{l}_i^T \cdot \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') dl$ gives rise to a vector field, and can be viewed as a primal 1-cochain. In fact, this primal 1-cochain is just the i -th column of $\bar{\mathbf{G}}$. Therefore, the integration in Equation (58) can be viewed as a wedge product between this primal 1-cochain and dual 2-cochain \mathbf{J} . From the discussion in Section 3, the integration of this wedge product can be carried out through a vector inner product as in Equation (57).

So far, we have introduced the Green's function for both scalar waves and vector waves. Although we assumed homogeneous medium for simplicity, inhomogeneous medium can be considered similarly with characterized Hodge star operators, which we discussed in [14]. Next we will include the boundary effects and obtain Huygens' principle for both scalar waves and vector waves.

4.2. Huygens' Principle

Huygens' principle states that the wave field on a closed surface S can determine the field elsewhere. This principle is valid for both scalar and vector waves. We will derive the discrete version of this principle in the following. The scalar wave case is considered first.

4.2.1. Scalar Waves

When deriving Equations (45) and (47), we dropped the boundary term since the whole space is considered. However, generally, the interested space is a finite region. Hence the boundary term cannot be ignored.

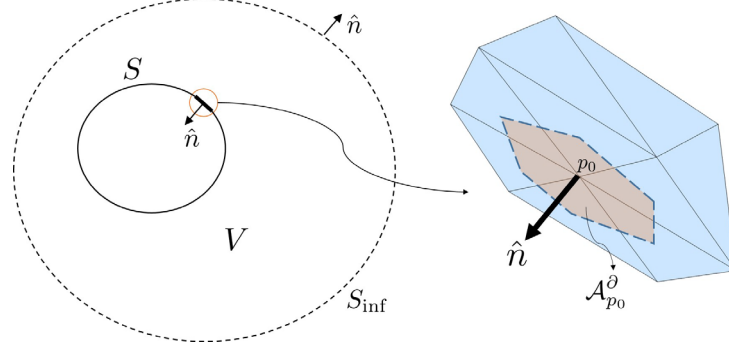


Figure 7. The geometry for Huygens' principle derivation. p_0 is a vertex on the surface S , \hat{n} is the unit normal vector. The orange shaded region $\mathcal{A}_{p_0}^\partial$ is obtained by connecting circumcenters of triangles around p_0 on S .

As shown in Figure 7, the considered region is bounded by surface S and S_{inf} , and the source only exists within S . Then, by using the conclusion in Equation (29), the discrete scalar wave equation with a boundary term can be written as:

$$\left[\left(\bar{\mathbf{d}}^{(0)} \right)^T \star^{(1)} \bar{\mathbf{d}}^{(0)} - k^2 \star^{(0)} \right] \cdot \psi - \partial_n \psi = 0. \quad (60)$$

Here, since S_{inf} is located at infinity, its contribution is assumed to be zero. So $\partial_n \psi$ has nonzero value only on S . More specifically, $\partial_n \psi$ is a dual 3-cochain with nonzero values only on the dual 3-cells associated with vertices on S . In fact, as mentioned, $\partial_n \psi$ can also be viewed as a dual 2-cochain on S , which is a 2-D manifold. The value of $\partial_n \psi$ on the dual 2-cell \mathcal{A}_{p_0} on S in Figure 7 is

$$[\partial_n \psi]_{\mathcal{A}_{p_0}^\partial} = \int_{\mathcal{A}_{p_0}^\partial} [\nabla \psi(\mathbf{r}) \cdot \hat{n}] dS = \int_{\mathcal{A}_{p_0}^\partial} \frac{\partial \psi}{\partial n} dS \approx \frac{\partial \psi(\mathbf{r}_{p_0})}{\partial n} |\mathcal{A}_{p_0}^\partial|. \quad (61)$$

Here, the last approximation is given because this form is easier to implement boundary conditions. The equation for scalar Green's function is written as:

$$\left[\left(\bar{\mathbf{d}}^{(0)} \right)^T \star^{(1)} \bar{\mathbf{d}}^{(0)} - k^2 \star^{(0)} \right] \cdot \bar{\mathbf{g}} - \partial_n \bar{\mathbf{g}} = \bar{\mathbf{I}}. \quad (62)$$

Here, $\partial_n \bar{\mathbf{g}}$ is a $N_0 \times N_0$ matrix denoting the boundary term, and its (i, j) element is defined as:

$$[\partial_n \bar{\mathbf{g}}]_{i,j} = \int_{\mathcal{A}_i^\partial} [\nabla g(\mathbf{r}, \mathbf{r}_j) \cdot \hat{n}] dS = \int_{\mathcal{A}_i^\partial} \frac{\partial g(\mathbf{r}, \mathbf{r}_j)}{\partial n} dS \approx \frac{\partial g(\mathbf{r}_i, \mathbf{r}_j)}{\partial n} |\mathcal{A}_i^\partial| \quad (63)$$

where \mathbf{r}_j is the j -th primal vertex, and \mathcal{A}_i^∂ is dual face on the boundary associated with i -th primal vertex \mathbf{r}_i , such as $\mathcal{A}_{p_0}^\partial$ in Figure 7. With N_0^∂ vertices on S , $\partial_n \bar{\mathbf{g}}$ will only have N_0^∂ nonzero rows. It is noted that each column of $\partial_n \bar{\mathbf{g}}$ can be viewed as a dual 3-cochain, or a dual 2-cochain on S , just as $\partial_n \psi$.

Combining Equations (60) and (62), we can derive

$$\boldsymbol{\psi} = \bar{\mathbf{g}}^T \cdot \partial_n \boldsymbol{\psi} - (\partial_n \bar{\mathbf{g}})^T \cdot \boldsymbol{\psi} \quad (64)$$

Here, the two matrix-vector production can be understood as a series of wedge product between primal 0-cochains and dual 3-cochains as defined in Equation (34), or a wedge product between primal 0-cochains and dual 2-cochains on 2-D manifold S , which refers to a surface integral. Clearly, this is a discrete analog to the continuous theory:

$$\boldsymbol{\psi}(\mathbf{r}') = \oint_S [g(\mathbf{r}, \mathbf{r}') \hat{\mathbf{n}} \cdot \nabla \boldsymbol{\psi}(\mathbf{r}) - \boldsymbol{\psi}(\mathbf{r}) \hat{\mathbf{n}} \cdot \nabla g(\mathbf{r}, \mathbf{r}')] dS. \quad (65)$$

Moreover, if we set $g(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in S$, then the first term on the right hand side of Equation (64) will vanish and become

$$\boldsymbol{\psi} = -(\partial_n \bar{\mathbf{g}})^T \cdot \boldsymbol{\psi}. \quad (66)$$

On the other hand, if we require $\hat{\mathbf{n}} \cdot \nabla g(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in S$, then instead, the second term of (64) will disappear and

$$\boldsymbol{\psi} = \bar{\mathbf{g}}^T \cdot \partial_n \boldsymbol{\psi}. \quad (67)$$

Equations (64), (66) and (67) are various forms of discrete Huygens' principle for scalar field depending on the definitions of $g(\mathbf{r}, \mathbf{r}')$. This principle states that only the value of $\boldsymbol{\psi}$ or $\partial_n \boldsymbol{\psi}$ on surface S are needed to obtain the field value at any point between S and S_{inf} .

4.2.2. Electromagnetic Waves

For electromagnetic waves, we consider the same geometry, as shown in Figure 8. It is noted that the red solid line $\mathcal{L}_{d_0 d_1}^\partial$, connecting d_0 and d_1 , is from d_0 to middle point of primal edge $\overline{p_0 p_1}$ then to d_1 , and not necessarily straight. And dual face $\mathcal{A}_{\overline{p_0 p_1}}$ is perpendicular to $\overline{p_0 p_1}$, and is inside the mesh.

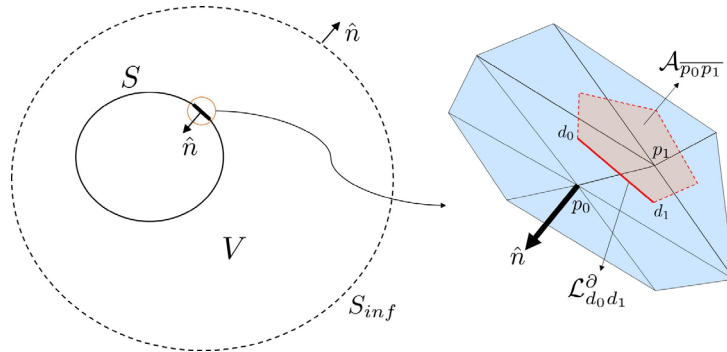


Figure 8. The geometry for vector Huygens' principle derivation. Here, $\overline{p_0 p_1}$ is a primal edge on surface S , $\mathcal{A}_{\overline{p_0 p_1}}$ is the dual face associated with $\overline{p_0 p_1}$, d_0 and d_1 are the circumcenters of two triangles containing $\overline{p_0 p_1}$, and $\mathcal{L}_{d_0 d_1}^\partial$ is the red solid line connecting d_0 and d_1 .

Including a boundary term induced from the Laplacian operator, shown in Equation (31) in Section 2, in a source-free region bounded by surface S and S_{inf} , Equation (53) is adjusted as

$$\left[\left(\bar{\mathbf{d}}^{(1)} \right)^T \star^{(2)} \bar{\mathbf{d}}^{(1)} - k^2 \star^{(1)} \right] \cdot \mathbf{E} + i\omega\mu \mathbf{J}_s = 0. \quad (68)$$

The contribution of S_{inf} is also assumed to be zero, then \mathbf{J}_s has nonzero value only on S . More specifically, \mathbf{J}_s is a dual 2-cochain with nonzero entities only on the incomplete dual faces at the boundary, e.g., $\mathcal{A}_{\overline{p_0 p_1}}$ in Figure 8. Due to the one to one relation between dual faces and primal edges,

with N_1^∂ primal edges on S , \mathbf{J}_s would have N_1^∂ nonzero entries. For example, the value of \mathbf{J}_s defined on the dual face $\mathcal{A}_{\overline{p_0 p_1}}$ is

$$\begin{aligned} J_{s, \mathcal{A}_{\overline{p_0 p_1}}} &\triangleq \frac{1}{i\omega\mu} \int_{d_0}^{d_1} \nabla \times \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} = \int_{d_0}^{d_1} \mathbf{H}(\mathbf{r}) \cdot d\mathbf{l} \\ &= \int_{d_0}^{d_1} [\hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r})] \cdot (\hat{\mathbf{n}} \times d\mathbf{l}) \\ &\approx \frac{|\mathcal{L}_{d_0 d_1}^\partial|}{|\overline{p_0 p_1}|} \int_{p_0}^{p_1} [\hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r})] \cdot d\mathbf{l} \end{aligned} \quad (69)$$

It should be noted that \mathbf{J}_s is a dual 1-cochain on 2-D manifold S , since it is defined on dual edges on S such as $\mathcal{L}_{d_0 d_1}^\partial$. By writing as the second line of Equation (69), we can easily construct the wedge product between \mathbf{J}_s and a primal 1-cochain on S , as we derived in Equation (40) in Section 3.

The third line of Equation (69) is listed because this form is easier to implement certain kind of boundary condition, such as absorption boundary conditions (ABCs) and impedance boundary condition (IBC). For example, the first-order ABC reads as

$$\hat{\mathbf{n}} \times (\nabla \times \mathbf{E}) = i\omega\mu_0(\hat{\mathbf{n}} \times \mathbf{H}) \approx -ik_0\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}).$$

Then, the line integral of $\hat{\mathbf{n}} \times \mathbf{H}$ in Equation (69) can be replace by the value of primal cochain \mathbf{E} on $\overline{p_0 p_1}$ [14].

Similarly, with a boundary term, the equation for vector Green's function becomes

$$\left[\left(\bar{\mathbf{d}}^{(1)} \right)^T \star^{(2)} \bar{\mathbf{d}}^{(1)} - k^2 \star^{(1)} \right] \cdot \bar{\mathbf{G}} + \bar{\mathbf{K}} = \bar{\mathbf{I}}. \quad (70)$$

Here, $\bar{\mathbf{K}}$ is an $N_1 \times N_1$ matrix with only $N_{1,\partial}$ nonzero rows, similar with $\partial_n \bar{\mathbf{g}}$ introduced for scalar wave. The (i, j) element $[\bar{\mathbf{K}}]_{i,j}$ associated with j -th primal edge and i -th dual face at surface, which only has one edge \mathcal{L}_i^∂ on the surface, is defined as:

$$\begin{aligned} [\bar{\mathbf{K}}]_{i,j} &\triangleq \int_{\mathcal{L}_i^\partial} \left[\int_{l_j} \nabla \times \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot d\mathbf{l}' \right] \cdot d\mathbf{l} \\ &= \int_{\mathcal{L}_i^\partial} \left[\hat{\mathbf{n}} \times \int_{l_j} \nabla \times \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot d\mathbf{l}' \right] \cdot (\hat{\mathbf{n}} \times d\mathbf{l}) \\ &\approx \frac{|\mathcal{L}_i^\partial|}{|l_i|} \int_{l_i} \left[\hat{\mathbf{n}} \times \int_{l_j} \nabla \times \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot d\mathbf{l}' \right] \cdot d\mathbf{l} \end{aligned} \quad (71)$$

It is noted that with the j -th primal edge l_j , $\int_{l_j} \nabla \times \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot d\mathbf{l}'$ gives rise to a vector field such as $\mathbf{H}(\mathbf{r})$. The tangential component of this vector field on the boundary, as defined in Equation (71), should be viewed as a dual 2-cochain in a 3-D mesh, or a dual 1-cochain on S , which is an $N_1 \times 1$ column vector with N_1^∂ nonzero entries, just as \mathbf{J}_s . Therefore, with N_1 possible l_j , these cochains form an $N_1 \times N_1$ matrix, which is $\bar{\mathbf{K}}$.

Combining Equations (68) and (70), we can obtain the vectorial Huygens' principle:

$$\mathbf{E} = \bar{\mathbf{K}}^T \cdot \mathbf{E} - i\omega\mu \bar{\mathbf{G}}^T \cdot \mathbf{H}_\parallel \quad (72)$$

This formula can also be understood as a discrete wedge product between dual 1-cochains and primal 1-cochains on surface S . More specifically, \mathbf{E} and each column of $\bar{\mathbf{G}}$ are primal 1-cochains, while \mathbf{J}_s and each column of $\bar{\mathbf{G}}_\parallel$ are dual 1-cochains on S . Therefore, an inner product between them refers to a surface integral. Equation (72) is a discrete analog of the continuous theory [1]

$$\mathbf{E}(\mathbf{r}) = \int_S dS \{ i\omega\mu \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r}') - [\nabla \times \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}')] \cdot \hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r}') \}. \quad (73)$$

The extra negative sign is due to the definition of normal unit vector $\hat{\mathbf{n}}$.

4.3. Poynting's Theorem

The complex Poynting's theorem is written as [1]

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = i\omega \mathbf{H}^* \cdot \mathbf{B} - i\omega \mathbf{E} \cdot \mathbf{D}^* - \mathbf{E} \cdot \mathbf{J}^*, \quad (74a)$$

$$\oint_{\partial\Omega} [(\mathbf{E} \times \mathbf{H}^*) \cdot \hat{n}] dS = \int_{\Omega} (i\omega \mathbf{H}^* \cdot \mathbf{B} - i\omega \mathbf{E} \cdot \mathbf{D}^* - \mathbf{E} \cdot \mathbf{J}^*) dV. \quad (74b)$$

where Equation (74b) is the integral form of Poynting's theorem on domain Ω and its boundary $\partial\Omega$.

In Section 3, we have introduced the way to represent the right hand side of Equation (74b) using discrete wedge product as

$$\text{RHS of (74b)} \Rightarrow i\omega(\mathbf{H}^*)^T \cdot \mathbf{B} - i\omega \mathbf{E}^T \cdot \mathbf{D}^* - \mathbf{E}^T \cdot \mathbf{J}^* \quad (75)$$

Invoking the discrete Ampere's law in Equation (41b) with a boundary,

$$-i\omega \mathbf{D} + \mathbf{J} = \left(\bar{\mathbf{d}}^{(1)}\right)^T \mathbf{H} + \mathbf{J}_s. \quad (76)$$

Equation (75) can be adjusted as

$$i\omega(\mathbf{H}^*)^T \cdot \mathbf{B} - \mathbf{E}^T \cdot \mathbf{J}^* - \mathbf{E}^T \cdot \left[\left(\bar{\mathbf{d}}^{(1)}\right)^T \mathbf{H}^* + \mathbf{J}_s^* - \mathbf{J}^*\right] = -\mathbf{E}^T \cdot \mathbf{J}_s^* \quad (77)$$

Here we have applied $\bar{\mathbf{d}}^{(1)} \mathbf{E} = i\omega \mathbf{B}$. Next we need to show that the left hand side of Equation (74b) is represented by $-\mathbf{E}^T \cdot \mathbf{J}_s^*$. In fact, \mathbf{E} and \mathbf{J}_s are primal 1-cochain and dual 1-cochain on two-dimensional manifold $\partial\Omega$, representing the tangential components of $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ on the surface. In other words, \mathbf{J}_s can also be written as a dual 1-cochain \mathbf{H}_{\parallel} defined on the 2-D surface. For example, the value of \mathbf{H}_{\parallel} on $\mathcal{L}_{d_0 d_1}^{\partial}$ in Figure 8 is defined as

$$H_{\parallel, \mathcal{L}_{d_0 d_1}^{\partial}} \triangleq J_{s, \overline{p_0 p_1}} \quad (78)$$

Then the discrete counterpart of the left hand side of Equation (74b) can be represented by the inner product of \mathbf{E} and \mathbf{J}_s or \mathbf{H}_{\parallel} , as shown in Equation (40). The negative sign is due to the contradiction between the direction of dual edges on boundary, and normal direction of associated dual faces. For example, in Figure 8, if the direction of $\overline{p_0 p_1}$ is assumed as \hat{l} , which is from p_0 to p_1 . Then, the normal direction of dual face $\overline{A_{p_0 p_1}}$ is also \hat{l} , while the direction of dual edge $\mathcal{L}_{d_0 d_1}^{\partial}$ on the surface is $\hat{n} \times \hat{l}$, which is from d_1 to d_0 and contradicts with the normal direction of $\overline{A_{p_0 p_1}}$.

Therefore, we have shown that the discrete Poynting's theorem still holds true and is written as

$$\begin{aligned} -\mathbf{E}^T \cdot \mathbf{J}_s^* &= i\omega \mathbf{B}^T \cdot \mathbf{H}^* - i\omega \mathbf{E}^T \cdot \mathbf{D}^* - \mathbf{E}^T \cdot \mathbf{J}^* \\ &= i\omega \mathbf{B}^T \cdot \star_{\mu^{-1}}^* \cdot \mathbf{B}^* - i\omega \mathbf{E}^T \cdot \star_{\epsilon}^* \cdot \mathbf{E}^* - \mathbf{E}^T \cdot \mathbf{J}^* \end{aligned} \quad (79)$$

Here, the constitutive relations introduced in Equation (43) are applied, and the superscripts of Hodge star operators are dropped for simplicity. With a lossless and source free region, Hodge star operators are real, then

$$\Re \{-\mathbf{E}^T \cdot \mathbf{J}_s^*\} = \omega \Im \{\mathbf{B}^T \cdot \star_{\mu^{-1}} \cdot \mathbf{B}^* - \mathbf{E}^T \cdot \star_{\epsilon} \cdot \mathbf{E}^*\} = 0. \quad (80)$$

This means that no net time-average power-flow out of or into domain Ω .

4.4. Reciprocity Theorem

A medium is called reciprocal medium when its permittivity $\bar{\epsilon}$ and permeability $\bar{\mu}$ are symmetric tensors. In a reciprocal medium, with an electric field \mathbf{E}_1 generated by \mathbf{J}_1 and \mathbf{E}_2 generated by \mathbf{J}_2 , they should satisfy

$$\langle \mathbf{E}_2, \mathbf{J}_1 \rangle = \langle \mathbf{E}_1, \mathbf{J}_2 \rangle. \quad (81)$$

We will prove this theorem within DEC only assuming two Hodge star operators $\star_{\epsilon}^{(1)}$ and $\star_{\mu^{-1}}^{(2)}$ are symmetric. In fact when the medium is isotropic, these two Hodge star operators are diagonal with

DEC. With anisotropic medium, they are still symmetric, but not diagonal any more, which will be shown in our future work.

Since \mathbf{E} field and source \mathbf{J} can be represented by a primal 1-cochain and dual 2-cochain, respectively, the inner product in Equation (81) can be viewed as a wedge product between cochains. In other words, we need to prove

$$\mathbf{E}_2^T \cdot \mathbf{J}_1 = \mathbf{E}_1^T \cdot \mathbf{J}_2. \quad (82)$$

Cochains \mathbf{E}_1 and \mathbf{E}_2 are generated by source cochains \mathbf{J}_1 and \mathbf{J}_2 via equations

$$\left[\left(\bar{\mathbf{d}}^{(1)} \right)^T \star_{\mu^{-1}}^{(2)} \bar{\mathbf{d}}^{(1)} - k^2 \star_{\epsilon}^{(1)} \right] \cdot \mathbf{E}_1 = i\omega \mathbf{J}_1, \quad (83a)$$

$$\left[\left(\bar{\mathbf{d}}^{(1)} \right)^T \star_{\mu^{-1}}^{(2)} \bar{\mathbf{d}}^{(1)} - k^2 \star_{\epsilon}^{(1)} \right] \cdot \mathbf{E}_2 = i\omega \mathbf{J}_2. \quad (83b)$$

Therefore,

$$\begin{aligned} i\omega \mathbf{E}_2^T \cdot \mathbf{J}_1 &= \mathbf{E}_2^T \cdot \left[\left(\bar{\mathbf{d}}^{(1)} \right)^T \star_{\mu^{-1}}^{(2)} \bar{\mathbf{d}}^{(1)} - k^2 \star_{\epsilon}^{(1)} \right] \cdot \mathbf{E}_1 \\ &= \mathbf{E}_2^T \cdot \left[\left(\bar{\mathbf{d}}^{(1)} \right)^T \star_{\mu^{-1}}^{(2)} \bar{\mathbf{d}}^{(1)} - k^2 \star_{\epsilon}^{(1)} \right]^T \cdot \mathbf{E}_1 \\ &= i\omega \mathbf{J}_2^T \cdot \mathbf{E}_1. \end{aligned} \quad (84)$$

Noted that this relationship is exact in this discrete space.

5. CONCLUSIONS

We have shown that by using DEC and a discrete wedge product, discrete electromagnetic theory can be built on an arbitrary simplicial mesh. The fact that theorems of electromagnetic theory are still preserved in this discrete world, shows that discrete electromagnetic theory with DEC is self-consistent and self-contained. Therefore, these discrete theorems can be used to check the correctness of numerical simulations. Moreover, the discrete Green's functions can be used to solve complex scattering problems, which will be explored in our future works.

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APPENDIX A. PROOF OF EQUATION (35)

Since this is a 2×2 matrix equation, we only need to prove that both sides of Equation (35) acting on an arbitrary 2-D vector give rise to the same result. Since any vector on a 2-D plane can be written as a linear supposition of \hat{l}_1 and \hat{l}_2 , we can only use these two vectors to prove our point.

$$\text{RHS of (35)} \cdot \hat{l}_1 = S \hat{l}_1 \quad (A1)$$

$$\text{LHS of (35)} \cdot \hat{l}_1 = l_1 s_1 \hat{l}_1 + l_2 s_2 \left(\hat{l}_2^T \cdot \hat{l}_1 \right) \hat{l}_2 + l_3 s_3 \left(\hat{l}_3^T \cdot \hat{l}_1 \right) \hat{l}_3 \quad (A2)$$

For any triangle, the following relation holds true

$$l_1 \hat{l}_1 + l_2 \hat{l}_2 + l_3 \hat{l}_3 = 0 \quad (A3)$$

Therefore,

$$\text{LHS of (35)} \cdot \hat{l}_1 = l_1 s_1 \hat{l}_1 + l_2 s_2 \left(\hat{l}_2^T \cdot \hat{l}_1 \right) \hat{l}_2 - s_3 \left(\hat{l}_3^T \cdot \hat{l}_1 \right) \left(l_1 \hat{l}_1 + l_2 \hat{l}_2 \right) \quad (A4)$$

$$= l_1 \left[s_1 - s_3 \left(\hat{l}_3^T \cdot \hat{l}_1 \right) \right] \hat{l}_1 + l_2 \left[s_2 \left(\hat{l}_2^T \cdot \hat{l}_1 \right) - s_3 \left(\hat{l}_3^T \cdot \hat{l}_1 \right) \right] \hat{l}_2 \quad (A5)$$

To prove that the right hand side of Equations (A2) and (A5) are identical, we need to prove the following two relations

$$S = l_1 \left[s_1 - s_3 \left(\hat{l}_3^T \cdot \hat{l}_1 \right) \right] \quad (\text{A6})$$

$$0 = l_2 \left[s_2 \left(\hat{l}_2^T \cdot \hat{l}_1 \right) - s_3 \left(\hat{l}_3^T \cdot \hat{l}_1 \right) \right] \quad (\text{A7})$$

As shown in Figure A1, we have the following relations

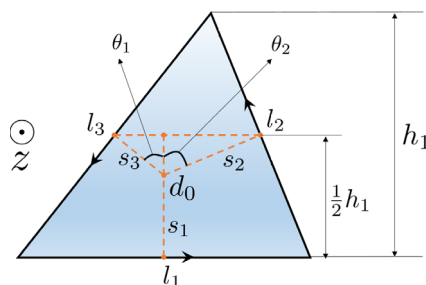


Figure A1. Geometry of an arbitrary triangle. Here, d_0 is the circumcenter, l_1, l_2, l_3 are three primal edges, and d_1, d_2, d_3 are three dual edges.

$$\hat{l}_2^T \cdot \hat{l}_1 = -\cos(\theta_2) \quad (\text{A8})$$

$$\hat{l}_3^T \cdot \hat{l}_1 = -\cos(\theta_1) \quad (\text{A9})$$

$$s_1 - s_3 \left(\hat{l}_3^T \cdot \hat{l}_1 \right) = s_1 + s_3 \cos(\theta_1) = \frac{1}{2} h_1 \quad (\text{A10})$$

$$s_2 \left(\hat{l}_2^T \cdot \hat{l}_1 \right) - s_3 \left(\hat{l}_3^T \cdot \hat{l}_1 \right) = -s_2 \cos(\theta_2) + s_3 \cos(\theta_1) = 0 \quad (\text{A11})$$

Therefore, Equations (A6) and (A7) are proved, which leads to

$$\text{LHS of (35)} \cdot \hat{l}_1 = \text{RHS of (35)} \cdot \hat{l}_1 \quad (\text{A12})$$

Since the same derivation also works for \hat{l}_2 ,

$$\text{LHS of (35)} \cdot \hat{l}_2 = \text{RHS of (35)} \cdot \hat{l}_2 \quad (\text{A13})$$

So Equation (35) is proved.

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