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A NEW METHOD FOR THE SYNTHESIS OF NON-UNIFORM LINEAR ARRAYS WITH SHAPED POWER PATTERNS

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Abstract—Antenna arrays with shaped power patterns have many applications in communications and radars. Many antenna array synthesis techniques for shaped patterns have been developed in the past years, and most of them deal only with uniformly spaced arrays. In this paper, a new method is proposed for the synthesis of nonuniform linear antenna arrays with shaped power patterns. The proposed synthesis method consists of three steps. First, we find a satisfactory power pattern for the required radiation characteristics by solving a constrained least-squares problem which is obtained with the help of non-redundant representation of squared magnitude of a linear array factor. Then, we factorize the polynomial associated with the power pattern by using polynomial rooting, and consequently obtain the corresponding field patterns. Finally, the forward-backward matrix pencil method is used to obtain a nonuniform linear array with optimized excitation magnitudes, phases and locations for a specific choice of field patterns. The synthesized array has a smaller number of elements than the one with uniformly spaced elements for the same

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pattern performance. Several synthesis experiments are conducted to validate the effectiveness and advantages of the proposed synthesis method.

1. INTRODUCTION

Antenna arrays with shaped power patterns have many applications in communications and radars. Many antenna array synthesis techniques for the synthesis of shaped patterns have been developed in the past years [1–11], and most of them deal only with uniformly spaced arrays, such as in [1–9]. As is well known, the synthesis of uniformly spaced arrays sometimes requires a large number of antenna elements to radiate a desired pattern shape. Naturally, if utilizing nonuniform element spacings can reduce the total number of elements, it will be very useful in some applications, for example, in satellite communications where the weight of antenna systems is extremely limited. Hence, several methods for synthesizing nonuniform antenna arrays have been developed [9–16]. However, nonuniform array synthesis is a highly nonlinear inverse problem that involves finding the solution of many unknowns (element positions, excitation magnitudes and phases). Many iterative synthesis techniques cannot guarantee a global optimum for all the variables. Some stochastic optimization algorithms capable of finding the global optimal solutions such as genetic algorithms [12] may be appropriate, but they can be time-consuming. In addition, to our knowledge, most of existing nonuniform array geometry synthesis techniques are proposed for the case of pencil beam patterns, and it is not clear whether they can be directly applicable to the synthesis of shaped-beam patterns.

Here, we present a new method for the synthesis of nonuniform linear antenna arrays with shaped power patterns. The idea of the proposed method is to factorize the whole synthesis process into three steps. First, we find a satisfactory power pattern for the required radiation characteristics by solving a constrained least-squares problem which is obtained with the help of non-redundant representation of squared magnitude of a linear array factor [7]. Then, we factorize the polynomial associated with the power pattern by using polynomial root-finding [13], and consequently obtain the corresponding field patterns. Finally, the forward-backward matrix pencil method (FBMPM) [14, 15] is used to obtain a nonuniform linear array with optimized excitation magnitudes, phases and locations for any choice of field patterns. In general, the synthesized nonuniform array has a smaller number of elements than the one with uniformly spaced elements for the same pattern characteristics. Note that we can choose the 'best' one by scanning all the synthesis results for different field patterns. Several synthesis experiments have been conducted, and the results have shown the effectiveness of the proposed method and the robustness for different radiation pattern requirements.

It should be noted that, a similar three-step synthesis idea was used in the last synthesis example of our previous paper [15] for providing a satisfactory field pattern for validating the performance of the FBMPM. However, that paper focuses on how to reconstruct the desired field pattern with fewer elements by the FBMPM, and the current work instead deals with the problem of power pattern synthesis which is more useful in real applications. To do so, the three-step synthesis strategy will be formulated in detail, and be validated with more synthesis examples.

2. FORMULATION AND ALGORITHM

2.1. Power Pattern Synthesis for Uniformly Spaced Arrays

First, we will derive the non-redundant representation of the squared magnitude of a linear array factor. Consider a linear array with M uniformly spaced identical antenna elements. The far field array factor is given by

$$F(u) = \sum_{p=0}^{M-1} R_p e^{jpu} \tag{1}$$

where $j = \sqrt{-1}$, $u = \beta d \cos\theta$ and $\beta = 2\pi/\lambda$. The parameter d is the element spacing, R_p is the complex excitation coefficient of the pth element, λ is the wavelength, and θ is the angle between the direction of observation and the linear array geometry. The squared magnitude of the above expression can be written as

$$P(u) = F(u)F^*(u) = \sum_{p=0}^{M-1}\sum_{q=0}^{M-1} R_p R_q^* e^{j(p-q)u} \tag{2}$$

where the subscript $*$ indicates the complex conjugate of a variable. The above equation can be reformulated into the following form

$$P(u) = \sum_{p=-M+1}^{M-1} D_p e^{jpu} \tag{3}$$

where $D_p = D_{-p}^*$. Sampling the above equation with $u_l = 2\pi l/(2M - 1)$, we obtain

$$P(u_l) = \sum_{p=-M+1}^{M-1} D_p e^{j2\pi pl/(2M-1)} \tag{4}$$

Clearly, the sequence $\{P(u_i)\}$ is the discrete Fourier transform (DFT) ot $\{D_p\}$ [17], where $l = -M+1, M, \ldots, M-1$ and $p = -M+1, M, \ldots, M-1$. Therefore, $\{D_p\}$ can be expressed as the inverse DFT (IDFT) of $\{P(u_i)\}$. That is,

$$D_p = \frac{1}{2M-1} \sum_{l=-M+1}^{M-1} P(u_l) e^{-j2\pi pl/(2M-1)} \tag{5}$$

Substituting (5) into (3) and taking several mathematical operations, we finally obtain [7]

$$P(u) = \sum_{l=-M+1}^{M-1} P_l W(u - u_l) \tag{6}$$

where $P_l = P(u_i)$ and $W(x) = \sin[(2M-1)x/2]/[(2M-1)\sin(x/2)]$. Equation (6) is the non-redundant representation of squared magnitude of array factor for the linear array with M uniformly spaced elements. This means that for M uniformly spaced elements we actually have $(2M-1)$ degrees of freedom to approximate the desired power pattern.

Now consider the problem of synthesizing a linear array with a shaped power pattern. Here, we formulate this problem as the least-squares under a set of linear inequalities. That is,

$$\begin{cases} \underset{P_l}{\text{minimize}} \ \|P(\beta d \cos\theta_m) - T(\theta_m)\|_2^2, \ \theta_m \in \text{shaped region} \\ \text{subject to } L(\theta_m) \le P(\beta d \cos\theta_m) \le U(\theta_m), \ 0 \le \theta_m \le \pi \end{cases} \tag{7}$$

where $T(\theta)$ is the desired power pattern function, $L(\theta)$ and $U(\theta)$ represent the lower and upper bounds, respectively. By choosing suitable $T(\theta)$, $L(\theta)$ and $U(\theta)$, we can describe complicated radiation requirements including beam shaping and accurate sidelobe level (SLL) control. Such a constrained least-squares solution is very robust for different synthesis applications. In addition, the least squares problem described by (7) can be easily solved, and some programs from standard C or Matlab function library are available for this problem.

Note that since M is unknown, we need to try the solution of (7) multiple times under different values of M to find the minimum number of elements required for satisfying the inequality constraints.

2.2. Pattern Factorization Using Polynomial Rooting

It is shown by Fejer-Riesz theorem [13] that, if $P(u)$, the trigonometric polynomial described in (3), is real and nonnegative for all real u, then there must exist $P(u) = |F(u)|^2$ and $F(u)$ is described by (1). This also means that for an arbitrary nonnegative real function $P(u)$ which is obtained from (7), we can always find a set of uniformly spaced elements to radiate it. Clearly, it is possible that there are many field patterns with different phase distributions corresponding to the same power pattern. Therefore, we can choose a suitable field pattern for getting a better excitation distribution. Here we apply the polynomial rooting method [13] to factorize the power pattern function into the field patterns. To do so, let

$$\psi(\alpha) = \sum_{p=-M+1}^{M-1} D_p \alpha^p \tag{8}$$

Obviously, $P(u) = \psi(e^{ju})$. The polynomial $\psi(\alpha)$ has $(2M-2)$ roots. In addition, since $[\psi(1/\alpha^*)]^* = \psi(\alpha)$, all the roots must exist as a pair of $\{\alpha_i, 1/\alpha_i^*\}$. The field pattern can be constructed as follows

$$F(u) = c \prod\nolimits_{i=1}^{I} (e^{ju} - \alpha_i)^{r_i} \tag{9}$$

where r_i is multiplicity of the root α_i, and $\sum_{i=1}^{I} r_i = M - 1$. Note that whether α_i or $1/\alpha_i^*$ is chosen does not change the value of $|F(u)|^2$. Therefore, if there are M_0 pairs of roots lying off the unit circle, we will have 2^{M_0} different field patterns (all are different in the phase but the same in the magnitude) [3, 7].

2.3. Forward-backward Matrix Pencil Method (FBMPM)

In the previous sections, we have described the method for synthesizing a uniformly spaced array with the desired power pattern. However, to further reduce the number of elements, using nonuniform spacings is required, but such a synthesis process is a highly nonlinear inverse problem. Recently, reference [14] presents the matrix pencil method (MPM) based synthesis technique which can approximate accurately the desired field pattern by using a smaller number of nonuniformly spaced elements. This method is very effective for the case of pencil-beam patterns. More recently, [15] extends such a method to the synthesis of shaped-beam patterns by using forward-backward matrix pencil method (FBMPM). In the rest of this section, we will briefly describe this method.

Mathematically, the problem of reducing the number of elements can be described as follows

$$F(\beta d\eta) = \sum_{i=1}^{Q} R'_i e^{j\beta d'_i \eta} + \varepsilon \tag{10}$$

where $\eta = \cos\theta$, d'_i and R'_i the location and complex excitation of the element, respectively. Note that here Q is smaller than or equal to M (ε is the approximation error). Sampling the above equation with $\eta_n = n\Delta = n/N$, we obtain

$$F[n] = \sum\nolimits_{i=1}^{Q} R'_i (z'_i)^n + \varepsilon_n \tag{11}$$

where $z'_i = e^{j\beta d'_i \Delta}$. The sampling condition can be found in [14]. Then the FBMPM-based synthesis method organizes the pattern data into a Hankel-Toeplitz matrix [15]

$$\mathbf{Y}^{fb} = \begin{bmatrix} \mathbf{y}_0 & \mathbf{y}_1 & \cdots & \mathbf{y}_L \\ \mathbf{y}^*_L & \mathbf{y}^*_{L-1} & \cdots & \mathbf{y}^*_0 \end{bmatrix} \tag{12}$$

where $\mathbf{y}_l = [y_l, y_{l+1}, \ldots, y_{2N-L+l}]^T$ and $y_l = F[l-N]$. Here L is called the pencil parameter.

To reduce the number of elements, the FBMPM-based synthesis method performs the optimal lower-rank approximation of $\mathbf{Y}^{fb}$ by using the singular value decomposition (SVD) and then discarding some small singular values. Assume that Q largest singular values are retained. Denote the lower-rank matrix by $\mathbf{Y}^{fb}_Q$ which corresponds to an approximate pattern that is produced by fewer elements. Then the FBMPM-based synthesis method finds the positions of new elements by solving the following generalized eigenvalue problem

$$\left(\mathbf{Y}^{fb}_{Q,f} - z'\mathbf{Y}^{fb}_{Q,l}\right)\mathbf{v} = 0 \tag{13}$$

where $\mathbf{Y}^{fb}_{Q,f}$ (resp. $\mathbf{Y}^{fb}_{Q,l}$) is obtained from $\mathbf{Y}^{fb}_Q$ deleting the first column (resp. deleting the last column). Denote the generalized eigenvalue by $\hat{z'}_i$. The new positions are given by

$$\hat{d'}_i = \frac{1}{j\beta\Delta}\ln\left(\hat{z'}_i\right) \tag{14}$$

It is proven in [15] that, the solution of (13) has the following property: if $\{\hat{z'}_i, \mathbf{v}_1\}$ is a pair of generalized eigenvalue and eigenvector, then $\{(1/\hat{z'}_i)^*, \mathbf{v}_2\}$ must be another pair of generalized eigenvalue and eigenvector. That is, all the eigenvalues must exist as a pair of $\{\hat{z'}_i, (1/\hat{z'}_i)^*\}$. This constraint has proven very useful for overcoming the problem of the eigenvalues lying off the unite circle which arises in the original MPM-based synthesis method for reconstructing the shaped-beam pattern [14, 15]. Once the element positions are obtained, the excitations can be calculated by solving the least-squares problems [14, 15]. Since the pattern to be synthesized in this paper may be very complicated, a weighted least-squares solution will be better for more accurate control of the sidelobe level. That is,

$$\hat{\mathbf{R}}_i' = \left(\hat{\mathbf{Z}}^H \mathbf{W} \hat{\mathbf{Z}}\right)^{-1} \hat{\mathbf{Z}}^H \mathbf{W} \mathbf{f} \tag{15}$$

where $\mathbf{f}$ and $\hat{\mathbf{Z}}$ are given by (17) and (18) of [14], respectively. In this paper, the weight is set to be the upper bound of the desired pattern, i.e., $U(\theta)$.

It is worth noting that for the same power pattern, choosing different field pattern will give different element excitations and locations. In addition, the choice of field patterns also affects the correlation of rows or columns of $\mathbf{Y}^{fb}$, and may finally change the minimum number of elements required for a satisfactory pattern synthesis. In other words, we actually have many degrees of freedom to choose the 'best' one from all the field synthesis results, for example, the one with minimum number of elements or with the lowest maximum-to-minimum excitation ratio.

Some comments should be given for the choice of parameter L for the FBMPM. For the problem of estimating multiple complex exponentials in noise, the MPM/FBMPM has the optimal value of L depending on the number of data length in terms of the minimum estimation variance, and this optimal value can be theoretically given in the Gaussian noise model [18]. However, for the synthesis problem we concerned here, the FBMPM is actually used to approximate the sum of multiple exponentials with fewer exponentials. For such a numerical approximation problem, the optimal value of L is hard to be predicted. In general, we choose L within $[2N/3, 4N/3]$, and the best value of L can be determined by checking the synthesis results.

2.4. The Proposed Array Synthesis Procedure

The proposed procedure for the synthesis of nonuniformly spaced arrays is shown in Figure 1. As can be seen, at the beginning of the procedure we need to set the functions $T(\theta)$, $L(\theta)$ and $U(\theta)$ in (7), which are combined to represent the desired radiation requirement. The procedure at first finds a satisfactory power pattern by solving the inequality-constrained least-squares problem described in (7). Then the polynomial rooting is utilized to find all the roots of the polynomial associated with the power pattern function. Then we can pick up the

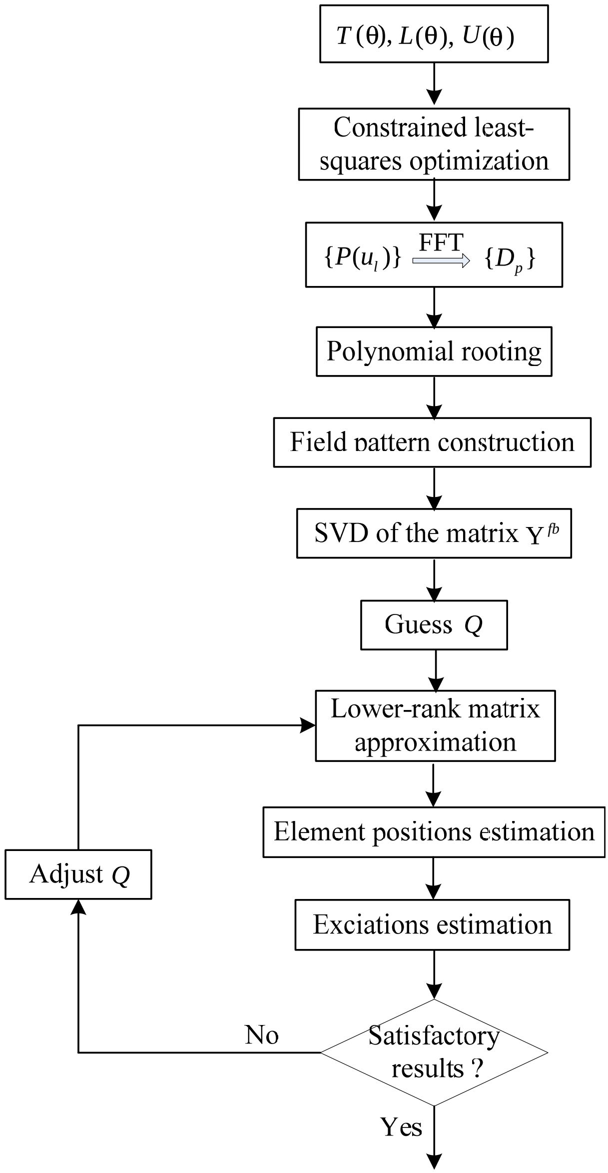

Figure 1. The proposed array synthesis procedure.

roots from each pair of $\{\alpha_i, 1/\alpha_i^*\}$, and construct the field pattern according to (9). As pointed out previously, many field patterns with different phase distributions are available, depending on the choice of roots. The field pattern obtained at this step corresponds to a uniformly spaced array. Hence, the forward-backward matrix pencil method (FBMPM) is then used to reduce the number of elements by optimizing both the element locations and excitations. At this stage, the initial can be automatically determined by using (7) of [15]. The minimum number of elements required for a satisfactory pattern reconstruction should be within a small range of this initial guess [14, 15]. Note that since many field patterns for the same power pattern are available which correspond to different synthesis results in the number of elements, element excitations and locations, we can choose the best one in the sense of the number of elements or the maximum-to-minimum excitation ratio.

3. SYNTHESIS EXAMPLES

As a first example, consider the synthesis of a flat-top pattern that was produced by [8]. For all synthesis examples, we set the sampling parameter $N = 2M$, and the pencil parameter L is equal to $2N/3$, $N/3$ or $4N/3$ whichever gives the best synthesis results. According to the radiation requirement, we set the upper and lower bounds as depicted by thick lines in Figure 2(a). At the region of shaped beams, we set the desired pattern function as $T(\theta) = [L(\theta) + U(\theta)]/2$. By solving the constrained least-squares problem of (8), we obtain a satisfactory power pattern $P(u)$. The result is shown by the thin real line in Figure 2(a).

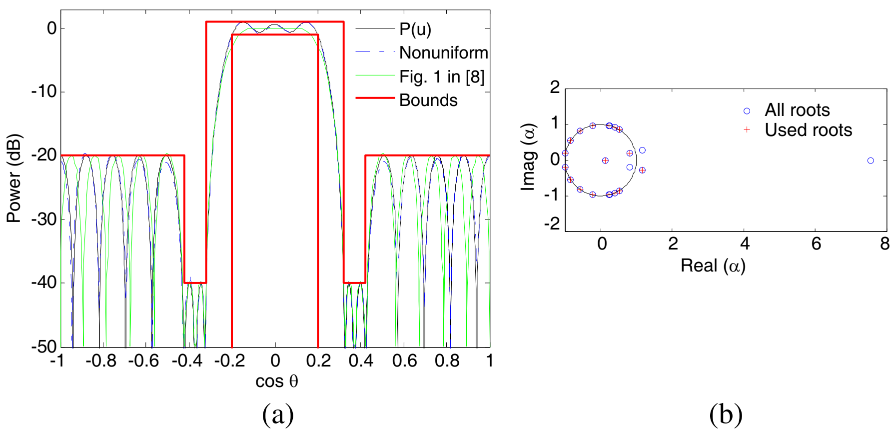

Figure 2. Synthesis of a flap-top pattern: (a) the pattern $P(u)$, the pattern reconstructed by 12 nonuniformly spaced elements, and the pattern synthesized by [8]; (b) distribution of the roots of the polynomial associated with the power pattern.

We can observe that the synthesized power pattern has nearly equal ripple within the upper and lower bounds at the shaped region. In addition, it has narrower transition band than the pattern synthesized by [8]. Figure 2(b) shows all the roots of the polynomial associated with the power pattern and the ones that are chosen for constructing the field pattern. The uniformly spaced array obtained at this step has 18 elements. Then the forward-backward matrix pencil method (FBMPM) is used to reconstruct the field pattern by fewer elements with optimized locations and excitations. The new array requires only 12 nonuniformly spaced elements. Table 1 shows element locations and excitations for this nonuniformly spaced array. The reconstructed pattern is shown also in Figure 2(a) for comparison. As can be seen, the reconstructed pattern is almost the same as the pattern $P(u)$. Note that the array synthesized by [8] requires 20 uniformly spaced elements with the maximum-to-minimum excitation ratio of 6.83, while the proposed synthesis requires only 12 elements with excitation ratio of 3.60. The saving in the number of elements is 40%.

The second example is given for the synthesis of a nonsymmetrical shaped power pattern. For comparison, the pattern synthesized by [3] is considered. The desired pattern has the maximum at $\theta = 100°$, and is equal to $T(\theta) = \csc^2(\theta - 90°)\cos(\theta - 90°)$ for $\theta \in [100°, 140°]$. Here, we set the upper and lower bounds as shown by the thick lines in Figure 3(a). Figure 3(a) also gives the comparison of the

Table 1. The element locations and excitations synthesized by the proposed method for the flat-top pattern shown in Figure 2.

i	d'_i/λ	$\lvert R'_i \rvert$	$\angle R'_i(°)$
1	−4.2222	0.27748	1.0741
2	−3.5416	0.34377	7.0909
3	−2.8671	0.50031	16.377
4	−2.1796	0.41221	51.078
5	−1.4624	0.5942	114.71
6	−0.7486	0.95783	133.11
7	−0.02566	1	134.18
8	0.6949	0.67907	120.96
9	1.4203	0.40474	63.212
10	2.1299	0.52789	17.993
11	2.8408	0.42523	6.5106
12	3.6231	0.30366	0.44723

pattern function $P(u)$, the pattern reconstructed by the FBMPM with nonuniformly spaced elements, and the pattern synthesized by [3]. As can be seen, the reconstructed pattern by the FBMPM has slightly

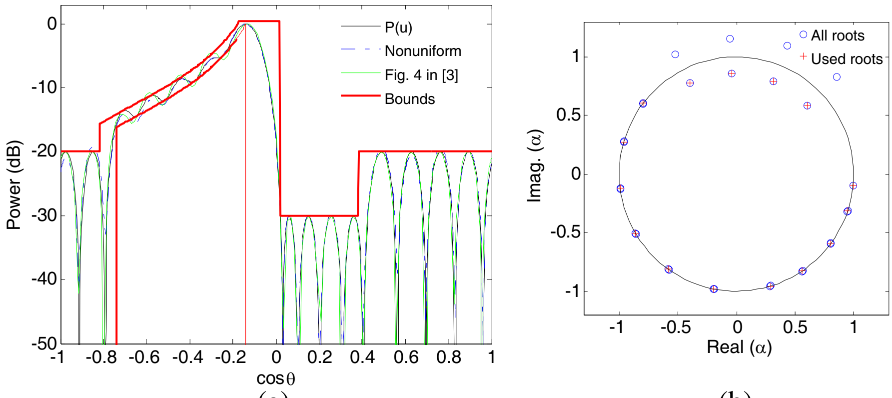

Figure 3. Synthesis of a nonsymmetrical pattern: (a) the pattern $P(u)$, the pattern reconstructed by 13 nonuniformly spaced elements, and the pattern synthesized by [3]; (b) distribution of the roots of the polynomial associated with the power pattern.

Table 2. The element locations and excitations synthesized by the proposed method for the nonsymmetrical pattern shown in Figure 3.

i	d'_i/λ	$\lvert R'_i\rvert$	$\angle R'_i(^\circ)$
1	−3.7487	0.5706	0.4968
2	−3.2394	0.79232	78.366
3	−2.7173	1	132.9
4	−2.141	0.90363	−177.17
5	−1.4688	0.81698	−145.42
6	−0.7949	0.75715	−114.92
7	−0.0912	0.60235	−83.919
8	0.6200	0.56128	−58.481
9	1.3009	0.3924	−31.911
10	2.0304	0.31161	−7.6628
11	2.6541	0.26747	11.033
12	3.2221	0.09028	2.747
12	3.7513	0.29844	70.422

lower ripple level than the pattern synthesized by [3]. Figure 3(b) shows the roots of the polynomial associated with $P(u)$ and the ones used for the field pattern construction. Table 2 shows the locations and excitations of the synthesized nonuniformly spaced array. Note that this array requires only 13 elements, while the array synthesized by [3] requires 16 elements. For this example, we save 18.75% elements.

The last example is to apply the proposed method to the synthesis of a transmitting antenna array used for an outdoor antenna test system. The geometry of the antenna test range is shown in Figure 4. The antenna under test is located within 5 $\sim$ 9 meters from the ground. The transmitting antenna array to be synthesized will be mounted at a position that is 45 meters from the test antenna and 7 meters from the ground. The radiation requirement of the transmitting array is translated into the upper and lower bounds of the desired pattern. Figure 5 shows the pattern bounds. The sidelobe level

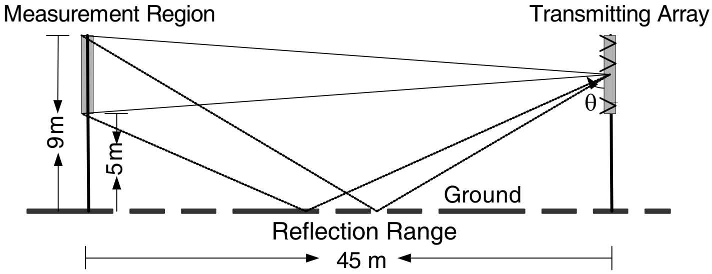

Figure 4. Geometry of an outdoor antenna test range.

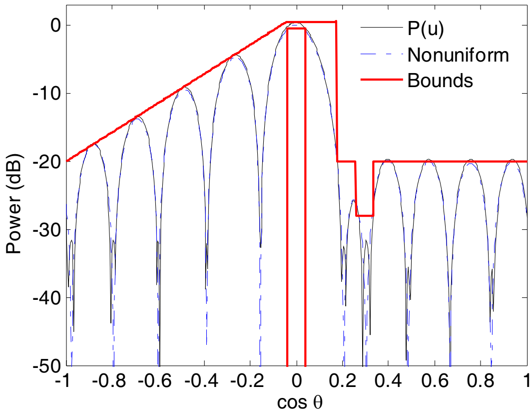

Figure 5. Pattern synthesis of the transmitting antenna array.

Table 3. The element locations and excitations synthesized by the proposed method for the transmitting antenna array.

i	d'_i/λ	$\lvert R'_i \rvert$	$\angle R'_i(°)$
1	-2.4935	0.84266	0.321
2	-1.9602	0.99781	43.053
3	-1.3582	0.97502	62.55
4	-0.6912	0.85897	55.853
5	-0.0003	0.80626	36.573
6	0.6905	0.85888	17.155
7	1.3580	0.97451	10.447
8	1.9602	1	29.914
9	2.4935	0.84465	72.454

(SLL) is required to be less than $-20\,\text{dB}$ for $\theta \leq 80°$ except at the reflection region of $[70.4°, 75.1°]$ where the SLL should be less than $-30\,\text{dB}$. The synthesized power pattern $P(u)$ and the pattern reconstructed by the FBMPM are shown in Figure 5. The pattern $P(u)$ requires 11 antenna elements, and the reconstructed pattern requires 9 nonuniformly spaced elements. Table 3 shows the element positions and excitations for the reconstructed pattern.

4. CONCLUSION

We have presented a new method for the synthesis of nonuniformly spaced antenna arrays with shaped power patterns. The proposed method factorizes the original power pattern synthesis process into three steps, which avoids the nonlinearities related with the power pattern synthesis and element position optimization. For any desired radiation requirement even with a complicated sidelobe level control, we only need to set suitable upper and lower pattern bounds, and the proposed method can efficiently find the satisfactory antenna array pattern. This makes the proposed synthesis very robust for different radiation requirements. The synthesized antenna array has optimized element positions, excitation amplitudes and phases, and therefore requires a smaller number of elements than the uniformly spaced array or the array in which part of parameters are not well optimized. In addition, multiple choices of field pattern for the same power pattern are available in the proposed synthesis process, and therefore we can choose the best synthesis results, for example, the array with the minimum number of elements or with the minimum excitation amplitude ratio. The proposed synthesis method should be very useful for many shaped power pattern applications.

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REFERENCES

1. Woodward, P. M., "A method of calculating the field over a plane aperture required to produce a given polar diagram," *J. Inst. Elect. Eng.*, Vol. 93, 1554–1558, 1947.
2. Stutzman, W. L., "Synthesis of shaped-beam radiation patterns using the iterative sampling method," *IEEE Trans. Antennas Propagat.* Vol. 19, No. 1, 36–41, 1971.
3. Elliott, R. S. and G. J. Stern, "A new technique for shaped beam synthesis of equispaced arrays," *IEEE Trans. Antennas Propagat.*, Vol. 32, No. 10, 1129–1133, 1984.
4. Milne, K., "Synthesis of power radiation patterns for linear array antennas," *IEE Proceedings H*, Vol. 134, No. 3, 285–296, 1987.
5. Zhang, W.-X. and Y.-M. Bo, "Pattern synthesis for linear equal-spaced antenna array using an iterative eigenmodes method," *IEE Proceedings H*, Vol. 135, No. 3, 167–170, 1988.
6. Werner, D. H. and A. J. Ferraro, "Cosine patterns synthesis for single and multiple main beam uniformly spaced linear arrays," *IEEE Trans. Antennas Propagat.*, Vol. 37, No. 11, 1480–1484, 1989.
7. Isernia, T., O. M. Bucci, and N. Fiorentino, "Shaped beam antenna synthesis problems: Feasibility criteria and new strategies," *Journal of Electromagnetic Waves and Applications*, Vol. 12, No. 1, 103–138, 1998.
8. Cid, J. M., J. A. Rodriguez, and F. Ares, "Shaped power patterns produced by equispaced linear arrays: Optimized synthesis using orthogonal $\sin(Nx)/\sin(x)$ beams," *Journal of Electromagnetic Waves and Applications*, Vol. 13, No. 7, 985–992, 1999.

9. Azevedo, J. A. R., "Shaped beam pattern synthesis with nonuniform sample phases," *Progress In Electromagnetics Research B*, Vol. 5, 77–90, 2008.
10. Akdagli, A. and K. Guney, "Shaped-beam pattern synthesis of equally and unequally spaced linear antenna arrays using a modified tabu search algorithm," *Microwave Opt. Technol. Lett.*, Vol. 36, No. 1, 16–20, 2003.
11. Kazemi, S., H. R. Hassani, G. R. Dadashzadeh, and F. Geran, "Performance improvement in amplitude synthesis of unequally spaced array using least mean square method," *Progress In Electromagnetics Research B*, Vol. 1, 135–145, 2008.
12. Marcano, D. and F. Duran, "Synthesis of antenna arrays using genetic algorithms," *IEEE Antennas Propag. Mag.*, Vol. 42, No. 3, 12–22, 2000.
13. Dritschel, M. A., "On factorization of trigonometric polynomials," *Integr. Equ. Oper. Theory*, Vol. 49, No. 1, 11–42, 2004.
14. Liu, Y.-H., Z.-P. Nie, and Q. H. Liu, "Reducing the number of elements in a linear antenna array by the matrix pencil method," *IEEE Trans. Antennas Propagat.*, Vol. 56, No. 9, 2955–2962, 2008.
15. Liu, Y.-H., Q. H. Liu, and Z.-P. Nie, "Reducing the number of elements in the synthesis of shaped-beam patterns by the forward-backward matrix pencil method," *IEEE Trans. Antennas Propagat.*, Vol. 58, No. 2, 604–608, 2010.
16. Miller, E. K. and D. M. Goodman, "A pole-zero modeling approach to linear array synthesis I: The unconstrained solution," *Radio Sci.*, Vol. 18, 57–69, 1983.
17. Oppenheim, A. V. and R. W. Schafer, *Digital Signal Processing*, Prentice-Hall, Englewood Ciffs, 1975.
18. Hua, Y. and T. K. Sarkar, "Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 38, No. 5, 814–824, 1990.