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Uncertainty Assessment of Stochastic EM Problems via an Adaptive Anisotropic Polynomial-Chaos Technique

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Abstract—A novel polynomial-chaos (PC) technique is implemented based on anisotropic index sets. The proposed scheme takes advantage of the effect of each random variable on the output parameter of interest and adaptively constructs the PC expansion. Particularly, the algorithm starts by generating bases via low and high reliability heuristics and builds a PC representation, until an error criterion is satisfied or until the maximum desired polynomial order is reached. Our method is tested on a variety of uncertainty problems, where the statistical moments of the outputs of interest are estimated. Numerical results prove the efficiency of the proposed approach, since accurate outcomes are obtained in lower computational times than other techniques.

1. INTRODUCTION

The inherent uncertainties present in the fabrication process of an electromagnetic (EM) component, e.g., geometrical or electrical variations, may have a significant impact on the operation of the constructed device. For example, various manufacturing tolerances induce randomness, which eventually lead to uncertainty in the output response of the assembled component [1]. In such cases, an uncertainty assessment is of vital importance since the extracted simulated outcomes may differ significantly from the measured ones. Unfortunately, deterministic schemes are unable to perform a reliable analysis in stochastic problems; therefore, various methods have been developed, which are suitable for uncertainty quantification.

Monte Carlo (MC) techniques [2] are the standard approaches for dealing with stochastic cases, due to their accuracy and simple implementation. Specifically, those algorithms perform a high number of realizations, solving a given problem repeatedly using samples of random inputs. Then, the statistical moments of the outputs of interest are extracted based on the obtained data. However, the MC technique converges very slowly, and thus, a large number of simulations is required to produce reliable outcomes. As a result, this approach introduces elongated computational times for complex problems and eventually becomes impractical for those scenarios.

An alternative scheme utilizes Polynomial Chaos (PC) expansions [3]. This surrogate manages to extract reliable outcomes in problems with low or moderate numbers of random variables. The PC expansion has been applied in various EM cases, where uncertainty exists in the material [4–6] or the geometric characteristics [1, 7, 8] of a given dielectric. However, the computational cost of the PC scheme increases exponentially with the dimensionality of the examined problem, making it in some cases less efficient than the MC technique. For this reason, various suggestions exist in the literature, which propose improvements of the PC technique.

The proposed methodology is based on the works in [9] and [10]. Particularly, the novelty of this paper compared to [10] is the introduction of two heuristics: a highly reliable and a lowly reliable one.

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Next, the work in [9] is extended by adding a leave-one-out (LOO) error for the adaptive construction of a PC expansion from those heuristics, thus managing to extract accurate outcomes in a small computational cost. As a result, this algorithm estimates the statistical moments of the output of interest by taking into account the influence of each random input. It should also be noted that a proper order of the PC expansion is automatically determined via the LOO error, without requiring any excess computations. Additionally, the adaptive algorithm in [10] utilizes two extra parameters (which are not present in our technique), i.e., ϵ_1 and ϵ_2, which aim at retaining or discarding certain bases. However, the quantities of ϵ_1 and $\epsilon 2$ must be known beforehand, and thus, their exact values that produce the best outcomes are unknown. Our scheme is firstly tested on the Ishigami function, by assessing its statistical characteristics. Furthermore, this technique is also applied to a 1D transmission-line problem and to two microwave structure cases. Comparisons between the proposed approach and other methods prove that valid outcomes can be extracted in a more efficient manner.

2. RELATED WORK

Many popular approaches utilize sparse grids, based on the Smolyak algorithm [11]. Those schemes can significantly reduce the number of required simulations for the extraction of reliable results. However, Smolyak grids still suffer from the "curse of dimensionality" for problems with a high number of random variables. In [12], a weighted ℓ_1-minimization technique is presented which is able to construct sparse PC expansions. The proposed algorithm penalizes coefficients with small values, thus, increasing the overall efficiency. The work in [13] combines the Method of Moments (MoM) with the PC scheme. By considering the influence of each stochastic input on the output of interest, this technique can increase the efficiency and accuracy of both the PC approach and the MoM. In [14], an adaptive algorithm is presented that is based on non-isotropic Gauss-Paterson formulas. The proposed method utilizes nested sparse grids and takes into account the global sensitivity of the output of interest with respect to the stochastic inputs. Furthermore, the authors in [15] introduce a method that simultaneously performs basis adaptivity and sequential sampling. As a result, the presented numerical outcomes indicate that their PC algorithm can generate valid outcomes in a small computational burden. Moreover, a practical problem that remains unanswered is how to locate the best regression nodes that yield the most accurate results. This is partially tackled in [16], where a sparse linear regression technique is introduced, aiming to locate design of experiments that satisfy the D-optimality criterion. Particularly, the D-optimality criterion requires that the chosen nodes maximize the determinant of the information matrix in the least-squares problem. In [17], the authors present an adaptive algorithm, as well as similar heuristics to ours; however, their adaptive algorithm is not based on a LOO error. Additionally, the paper [18] presents an adaptive multi-fidelity PC scheme for solving Bayesian inverse problems. By combining low and high fidelity PC models, the authors prove that their approach can efficiently extract reliable outcomes. The work in [19] introduces a dimension-adaptive PC algorithm which is able to construct a nonuniform quadrature grid based on the significance of each random input. Also, this method is able to determine a proper order of the PC expansion, thus avoiding any overfitting or underfitting issues. The technique in [19] features several advantages compared to other schemes when a low number of influential stochastic variables exists. In [20], the authors propose an adaptive PC technique that enrich an existing basis set, of higher order bases, in chunks. An adaptive A-optimality design is utilized for the enrichment of the design of experiments. The paper [21] introduces an intrusive variant of the stochastic collocation technique. According to this work, the suggested algorithm utilizes fewer nodes than known stochastic collocation methods.

3. METHODOLOGY

According to [3], the PC expansion posits that a second-order random function y, which depends on the inputs $\boldsymbol{\xi} = [\xi_1,\ \xi_2,\ \ldots,\ \xi_N]$, can be represented via a series of orthogonal polynomials. Mathematically, this is expressed as:

$$y\left(\xi\right) = \sum_{r=0}^{\infty} c_r \Psi_r\left(\boldsymbol{\xi}\right) \tag{1}$$

where constants c_r denotes the PC coefficients, and Ψ_r are proper orthogonal basis functions. The approximation of (1) can be performed by truncating the summation into $P+1$ terms. In the traditional PC scheme, the number of terms is equal to $P+1 = (N+k)!/(N!k!)$, where k is the order of the expansion. For the multivariate case, the polynomial bases are constructed from the product of univariate ones. Additionally, the choice of those orthogonal polynomials depends on the distribution of each stochastic input. For example, Legendre functions are suitable for uniform distributions, and Hermite bases are proper for normal random variables. After the computation of the PC coefficients, the mean value and variance of y can be approximated as:

$$M\{y\} \approx c_0 \tag{2}$$

$$\sigma^2\{y\} \approx \sum_{r=1}^{P} c_r^2 \|\Psi_r\|^2 \tag{3}$$

where

$$\|\Psi_i\|^2 = \int_{\Omega_N} \Psi_i^2(\boldsymbol{\xi}) \,\mathrm{pdf}(\boldsymbol{\xi})\, \mathrm{d}\xi \tag{4}$$

Ω_N is the N-dimensional random space, and pdf(ξ) denotes the joint probability-density function. In order to estimate the expansion coefficients, two types of methods can be applied: intrusive and non-intrusive ones. The first approaches compute the PC expansion terms by modifying the deterministic solver accordingly. On the other hand, non-intrusive schemes perform a number of deterministic simulations in the random domain, without requiring any changes to the existing solver. Then, the expansion coefficients can be estimated via linear regression methods. After performing a number of realizations at F collocation points in the random domain, the following overdetermined system of equations is constructed:

$$\underbrace{\begin{bmatrix} \Psi_0(\boldsymbol{\xi}_1) & \Psi_1(\boldsymbol{\xi}_1) & \dots & \Psi_P(\boldsymbol{\xi}_1) \\ \Psi_0(\boldsymbol{\xi}_2) & \Psi_1(\boldsymbol{\xi}_2) & \dots & \Psi_P(\boldsymbol{\xi}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_0(\boldsymbol{\xi}_F) & \Psi_1(\boldsymbol{\xi}_F) & \dots & \Psi_P(\boldsymbol{\xi}_F) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_P \end{bmatrix}}_{\mathbf{C}} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_F \end{bmatrix}}_{\mathbf{Y}} \tag{5}$$

where $\mathbf{A}$ is an $F \times (P+1)$ matrix, which stores the PC bases evaluated at each point. Also, the vector $\mathbf{C}$ contains the unknown coefficients, and $\mathbf{Y}$ consists of the deterministic outputs of y. According to [16], accurate results can be extracted with a number of simulations equal to $2 \times (P+1)$.

In this paper, the efficiency of the PC approach is enhanced by utilizing anisotropic index sets. These indices can be generated as follows [10]:

$$S_{\mathbf{w},q}^{N,k} := \left\{ \mathbf{d} \in \mathbb{N}^N : \|\mathbf{d}\|_{\mathbf{w},q} \le k \right\} \tag{6}$$

where $\mathbf{d} = (d_1, d_2, \dots, d_N)$ is a vector corresponding to the index of a given basis and

$$\|\mathbf{d}\|_{\mathbf{w},q} = \left(\sum_{i=1}^{N} |w_i d_i|^q \right)^{\frac{1}{q}}. \tag{7}$$

Note that $0 \le d_i \le k$ and $0 < q \le 1$. Furthermore, $\mathbf{w}$ is a vector consisting of N elements and is calculated as [10]:

$$w_i = 1 + K \frac{S_{\max} - S_i}{\sum_{k=1}^{N} S_i} \tag{8}$$

where S_i is the total Sobol index of the i-th input variable, $S_{\max} = \max\{S_i\}$, and K is a non-negative constant. The total Sobol index S_i is defined as:

$$S_i = \frac{\sum_{\mathbf{a} \in \#i} V_{\mathbf{a}}}{\sigma^2\{y\}} \tag{9}$$

where $\#\mathbf{i}$ is the set which contains all the subsets of $[1,\ldots,N]$ that include the index i, and $V_{\mathbf{a}}$ is the variance contribution of the inputs with corresponding indices in $\mathbf{a}$. The higher the weight w_i is, the less influential the i-th variable is. It should be noted that the dissertation in [10] sets K equal to 1 and does not provide any further investigation for its value. In this work, this issue is tackled by employing two heuristics: a highly and a lowly reliable one, which will be presented in this section. The variance contribution of each input alone (V_i) [22] is utilized, instead of the total Sobol indices, since the extraction of the later ones requires the knowledge of the variance of the output of interest, which in our case is not known beforehand. In order to estimate V_i, the conditional variance that corresponds only to the variable $\boldsymbol{\xi}_i$ is computed as:

$$V_i = \sum_{\mathbf{d}\in D_i} c_{\mathbf{d}}^2 \left\|\Psi_{\mathbf{d}}\right\|^2 \tag{10}$$

where $D_i = \{\mathbf{d} \in \mathbb{N}^N : d_i \neq 0, d_{i\neq j} = 0\}$. Additionally, $\Psi_{\mathbf{d}} \equiv \Psi_r : \Psi_r(\boldsymbol{\xi}) = \prod_i^N P_{a_i}(\xi_i)$, where P_{d_i} denotes the d_i-th univariate polynomial. The representation in (10) contains a small number of bases with a cardinality of $k \times N + 1$. However, the values of V_i change at every point in the computational domain (for example, distance points or frequencies); therefore, the averaged variance contribution at those points is utilized. Now, the constant K is replaced by the parameter k_i, which is the element of a new vector $\mathbf{K}$. The extraction of this vector will be done as follows. For every pair of stochastic inputs ξ_{j_1} and ξ_{j_2}, a matrix, called "significance matrix", with dimensions $N \times N$ is constructed with elements:

$$S_{j_1,j_2} = \frac{M\left\{V_{j_2}\right\}}{M\left\{V_{j_1}\right\}}. \tag{11}$$

$M\{V_{j_1}\}$ and $M\{V_{j_2}\}$ denote the averaged variance contribution of ξ_{j_1} and ξ_{j_2}, respectively. Then, $\mathbf{K}$ is obtained by computing the mean value in each row of sig_{j_1,j_2}, excluding the elements in the main diagonal. Furthermore, as the quantity of k_i increases, lower orders of the polynomials corresponding to the input ξ_i are preserved. In some cases, e.g., where the random parameters have a highly nonlinear effect on the outputs of interest, this truncation may lead to inaccurate outcomes compared to the MC or the traditional PC methods, and thus, additional bases must be included. However, as the number of bases increases, the efficiency of the PC approach is significantly reduced. Consequently, the above issues are tackled by employing two heuristics: a lowly and a highly reliable one. The lowly reliable heuristic is described as follows:

- We initially set $\mathbf{K}_l = \mathbf{K}$. Then, we select the elements of $\mathbf{K}_l$, with values greater than 1. Generally, an element of $\mathbf{K}$ with a value greater than 1 means that the corresponding random input may have less variance contribution than most of the other parameters and in this case is considered non-significant. Additionally, this could also mean that some random variables in the examined problem are much more influential than the remaining ones.
- We increase the selected quantities of $\mathbf{K}_l$ by adding to them the mean value of the elements which are greater than 1.

For the case of the highly reliable heuristic, the next steps are employed:

- We initially set $\mathbf{K}_h = \mathbf{K}$.
- We decrease the quantities of $\mathbf{K}_h$ which are greater than 1 by subtracting their mean value from them.

The less reliable bases S_l are generated by firstly replacing K with k_{l_i}, where k_{l_i} is the i-th element of $\mathbf{K}_l$, then applying the extracted weights to (6). The highly reliable ones, stored in S_h, are computed in a similar way. It should also be pointed out that S_l is a subset of S_h, with S_h containing higher order base indices. Initially, the PC representation contains only the polynomials of S_l with $k = 1$, and the corresponding bases are initially stored in S_{final}, e.g., $S_{\text{final}} = S_l$. Next, the LOO error is computed as [23]:

$$E_{\text{LOO}} = \frac{1}{F}\sum_{i=1}^{F}\left(\frac{y\left(\boldsymbol{\xi}_i\right) - \hat{y}\left(\boldsymbol{\xi}_i\right)}{h_i}\right)^2 \tag{12}$$

Algorithm 1: The pseudocode of the a-K-PC approach

Input: Mean variance contributions M $\{\mathbf{V}\}$, $D_{\max}$, q, ϵ_{t}, G
Output: PC coefficients $\mathbf{c}_{\mathrm{f}}$, S_{final}
$S_{\mathrm{final}} \leftarrow \phi$;
$\mathbf{A} \leftarrow \phi$;
Initialize the vector $\mathbf{y}_{\mathrm{col}} \leftarrow \phi$;
for $d \leftarrow 1$ **to** $D_{\max}$ **do**
  Compute the significance matrix from (11) for $k = d$;
  Calculate $\mathbf{K}_l$ and $\mathbf{K}_h$;
  if $d = 1$ **then**
    Extract the sets S_l and S_h from (6);
  else
    Extract S_l and S_h from (6) without including the bases from the $d-1$ step;
  end
  $S_{\mathrm{r}} \leftarrow S_h \setminus S_l$;
  Initialize the collocation point matrix ξ with $2 \times \mathrm{card}\{S_l\}$ rows via a Latin hypercube grid.;
  $\epsilon_{LOO} \leftarrow 1$;
  flag $\leftarrow 0$;
  Add to S_{final} the bases of S_l;
  while *true* **do**
    if $\epsilon_{LOO} = 1$ **then**
      Compute y for each row of ξ and add the extracted values to $\mathbf{y}_{\mathrm{col}}$;
    else
      Generate a set of G collocation points via a Latin hypercube grid;
      Find the two points ξ_{d_1} and $\mathbf{xi}_{d_2}$ of this set, which maximize the determinant $\left|\mathbf{A}^T\mathbf{A}\right|$;
      Compute y for the values of ξ_{d_1} and ξ_{d_2};
      Store the values of y computed in the previous step to $\mathbf{y}_{\mathrm{col}}$;
      Update matrix $\mathbf{A}$ with the basis of $S_{\min}$ and the points ξ_{d_1} and ξ_{d_2};
    end
    Extract $\mathbf{c}_{\mathrm{f}}$ by solving the system in (5);
    Compute ϵ_{LOO} from (13);
    if $\epsilon_{LOO} <= \epsilon_t$ **then**
      flag $\leftarrow 1$;
      break;
    end
    if $S_r = \phi$ **then**
      break;
    end
    Find the basis $S_{\min}$ of S_r with the minimum weighted q-norm and add it to S_l;
    Add $S_{\min}$ to S_{final}
  end
  if *flag* $= 1$ **then**
    break;
  end
end

where $y(\boldsymbol{\xi}_i)$ is the value of y at $\boldsymbol{\xi}_i$ point, $\hat{y}$ the metamodel built from S_{final}, and h_i the i-th diagonal term of the matrix $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$. The relative LOO can then be calculated as follows:

$$\epsilon_{\text{LOO}} = \frac{E_{\text{LOO}}}{\sigma^2\{y_{\text{col}}\}} \tag{13}$$

where y_{col} denotes the values of y at collocation points constructed via a Latin hypercube grid. Next, if ϵ_{LOO} is less than a target error ϵ_{t}, the algorithm stops. Otherwise, the basis of $S_h \setminus S_l$ with the minimum weighted q-norm is added in the PC expansion and therefore in S_{final}. In case of the latter one, this means that the expansion requires higher order bases. As a result, every time a new base index is inserted into S_{final}, the number of collocation points is increased by 2, and thus, the value of F is always equal to $2 \times \text{card}\{S_{\text{final}}\}$. In order to satisfy the D-optimality criterion, those two points are selected from a new Latin hypercube set of G cardinality, aiming to maximize the determinant $|\mathbf{A}^T\mathbf{A}|$. Then, the metamodel $\hat{y}(\xi)$ is built again from the updated S_{final} set, and the quantity ϵ_{LOO} is also computed from (13). The above steps are repeated until ϵ_{LOO} is less than ϵ_{t}, or if $S_h \setminus S_l$ becomes empty. In case of the latter, the expansion order is increased by 1, and the whole process restarts, excluding any indices already contained in S_l and S_h. The algorithm terminates if ϵ_{t} is satisfied, or if a maximum polynomial order $D_{\max}$ is achieved. From now on the proposed method will be called adaptive-K PC (a-K-PC) in the latter sections. In Algorithm 1, the pseudocode of the a-K-PC technique is presented.

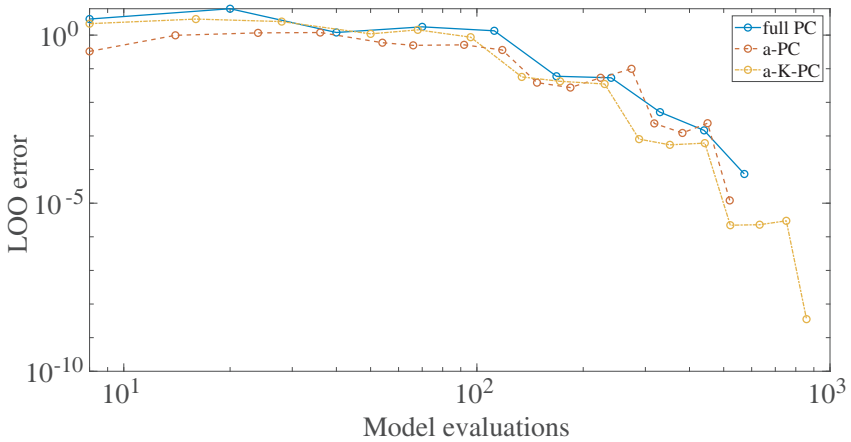
4. NUMERICAL RESULTS

4.1. Ishigami Function

The a-K-PC scheme is firstly tested on the Ishigami function [24], which is a widely known benchmark for uncertainty quantification. This function is defined as:

$$Y = \sin(X_1) + 7\sin^2(X_2) + 0.1X_3^4\sin(X_1) \tag{14}$$

where X_1, X_2, and X_3 are uniform random inputs distributed over $[-\pi, \pi]$. The proposed technique extracts the mean and standard deviation (std.) of Y by setting q to be equal to 0.85, while G has a value of 40. Fig. 1 presents the absolute LOO and std. errors as a function of the model evaluations (the mean error is relatively small, and thus it is neglected), computed with the full PC expansion, the anisotropic PC $K = 1$ (a-PC) along with the a-K-PC and the MC techniques. The depicted curves indicate that the proposed scheme can extract quite accurate outcomes in a small number of computations, rendering it more efficient than the other techniques. It is also worth mentioning that the estimation of V_i is performed with $k = 7$ and requires 44 (22 bases) computations. In Fig. 2, the probability density of the Ishigami function is depicted. The a-K-PC method estimates this statistical characteristic with a PC representation of only 68 bases, thus, a number of realizations equal to 2×68 are performed. It should also be noted that in this case q is equal to 0.85. Additionally, the algorithm terminates if the maximum



(a)

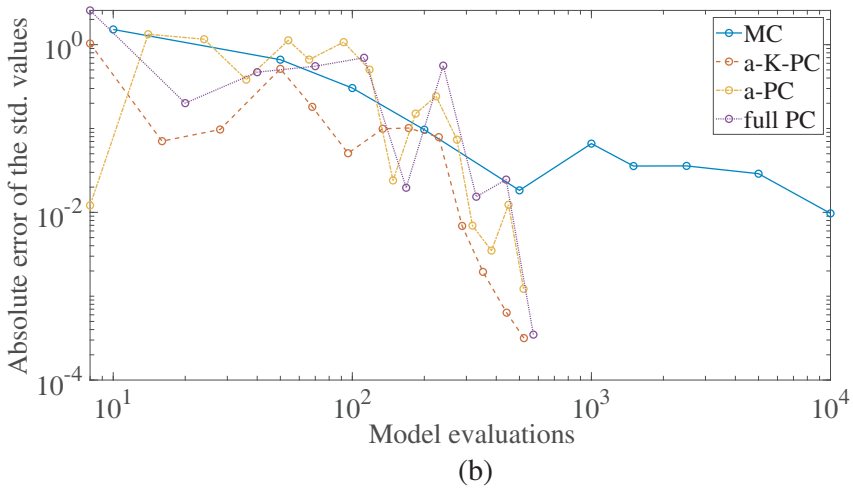


Figure 1. Convergence rate of the LOO and the absolute std. error for the Ishigami function problem. (a) Absolute LOO error of the Ishigami function. (b) Absolute error of the std. of the Ishigami function.

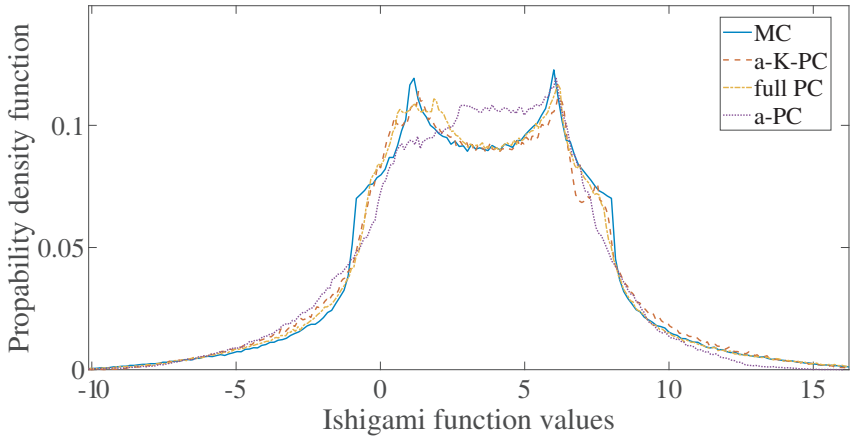


Figure 2. Probability density of the Ishigami function computed with the MC and the PC approaches.

polynomial order $D_{\max}$ becomes 12, or if $\epsilon_t \leq 0.06$. The full PC ($k = 7$) and the MC approaches need 240 (120 bases) and 10^6 model evaluations, respectively, making them less efficient than the a-K-PC technique. The a-PC scheme constructs an expansion with 128 computations (64 bases); however, it estimates the probability density function less accurately than the other algorithms.

4.2. 1D Transmission-Line

The second test case consists of the 1D transmission-line of Fig. 3 with length equal to 4.5 m. Particularly, the line comprises eight uniform random dielectrics, whose statistical properties are depicted in Table 1. It should be noted that the magnetic permeability of all materials is considered deterministic and has a value of μ_0. Also, an incident Gaussian pulse, with a maximum frequency of 2 GHz, is emitted at 0.015 m for around 50 ns. The examined scenario is solved via the finite-difference time-domain (FDTD) technique [25] in a computational space of 1200 cells. Moreover, the discretization density is equal to 40 cells per wavelength in the vacuum at 2 GHz, and the time-step size has a value of 12.5 ps. Finally, the first-order Mur's absorbing boundary condition [26] is applied at the two ends of the domain, aiming to minimize unwanted reflections.

In Figs. 4 and 5, the mean and std. of the electric field intensity are displayed for this problem

Table 1. Mean values and standard deviations of the dielectric materials for the transmission-line problem.

Dielectric permittivities	Mean	Std.
ϵ_1	$8\epsilon_0$	$0.23\epsilon_0$
ϵ_2	$2.4\epsilon_0$	$0.07\epsilon_0$
ϵ_3	$5.6\epsilon_0$	$0.16\epsilon_0$
ϵ_4	$4.2\epsilon_0$	$0.12\epsilon_0$
ϵ_5	$6\epsilon_0$	$0.17\epsilon_0$
ϵ_6	$7.2\epsilon_0$	$0.2\epsilon_0$
ϵ_7	$9.4\epsilon_0$	$0.27\epsilon_0$
ϵ_8	$8.7\epsilon_0$	$0.25\epsilon_0$

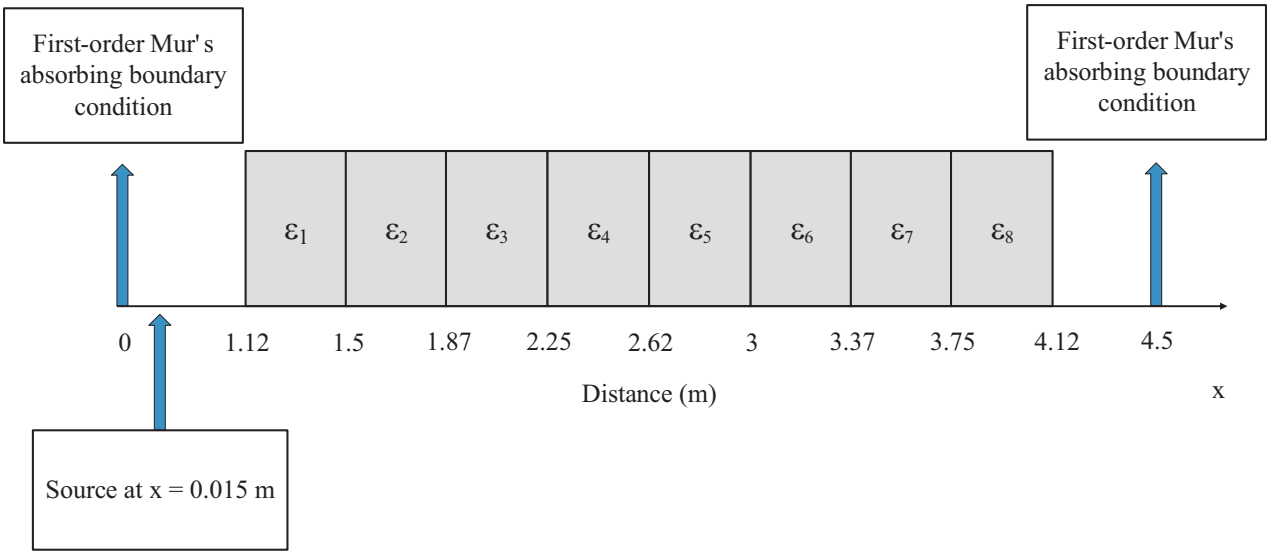


Figure 3. The schematic of the transmission-line problem.

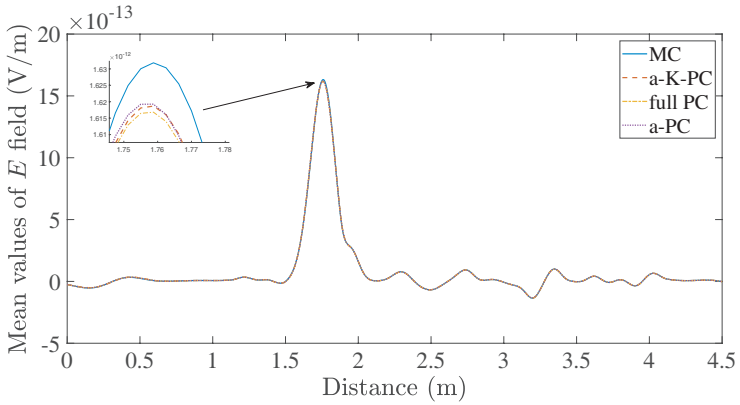


Figure 4. Mean values of the electric field intensity for the transmission-line problem.

at 50 ns. The examined output is of particular interest, since the uncertainty inside the dielectric permittivities has a significant impact on the field components, thus in the reflection or the transmission coefficients. Also, Fig. 6 depicts the absolute error of those moments between each PC approach and the

MC algorithm. The a-K-PC technique is compared with 1000 MC realizations and also with the a-PC and full PC approaches. Parameter q has a value of 0.85, for both the a-K-PC and a-PC schemes, while the quantity G is equal to 10. Additionally, the proposed method terminates if $\epsilon_t = 0.2$, and the order

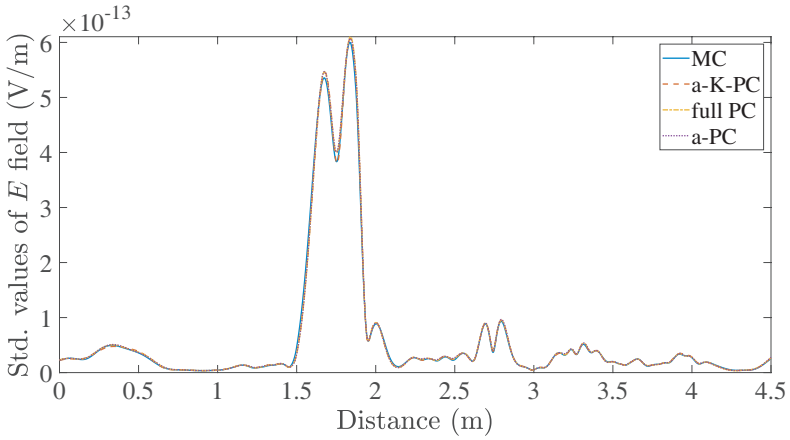


Figure 5. Std. of the electric field intensity for the transmission-line problem.

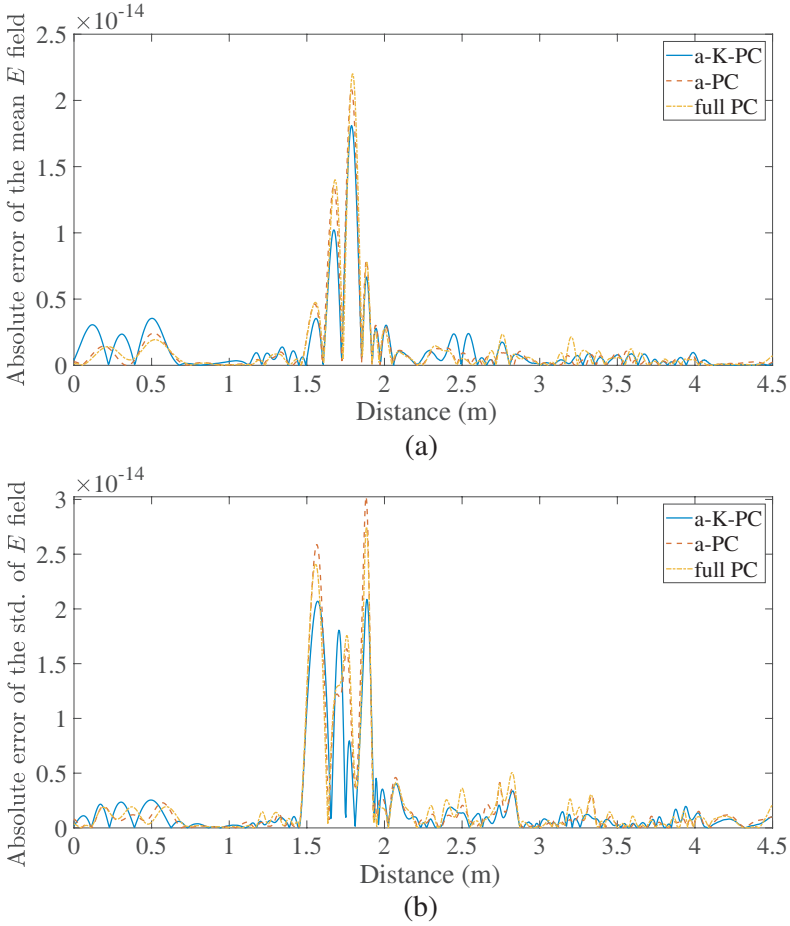


Figure 6. Absolute error of the mean and the std. of the electric field intensity for the transmission-line problem. (a) Absolute error of the mean electric field intensity. (b) Absolute error of the std. of the electric field intensity.

of all PC expansions is $k = 3$. The numerical results obtained via the PC techniques present a good agreement compared to the ones of the MC algorithm. The approximation of V_i is performed with $k = 4$ and takes about 5.56 sec (66 simulations). The a-PC and full PC approaches require around 56 sec (320 simulations) and 57 sec (330 simulations), respectively. It should be noted that the duration of those simulations includes the time needed for the construction of matrix $\mathbf{A}$. Furthermore, the MC scheme takes about 75 sec for the simulations to be completed. However, the suggested technique manages to extract the simulated outcomes in 30 sec (170 simulations), thus rendering it more efficient than the aforementioned methods.

The transmission-line problem is reexamined with $k = 3$, $q = 0.15$, and $\epsilon_t = 0.75$, while all the other parameters remain the same. In Fig. 7, the std. of the electric field intensity is displayed, and Fig. 8 illustrates the absolute error of the same quantity between each PC technique and the MC algorithm. The results obtained with the a-PC method are less accurate than the ones in the a-K-PC and full PC approaches. Concerning the simulations times, the a-PC scheme (50 simulations) and proposed algorithm (46 simulations) both require around 4 sec. However, the a-K-PC approach can extract more accurate outcomes than the a-PC scheme with around the same number of realizations. Finally, the full PC technique takes about 7.2 sec (90 simulations) for the simulations to be completed.

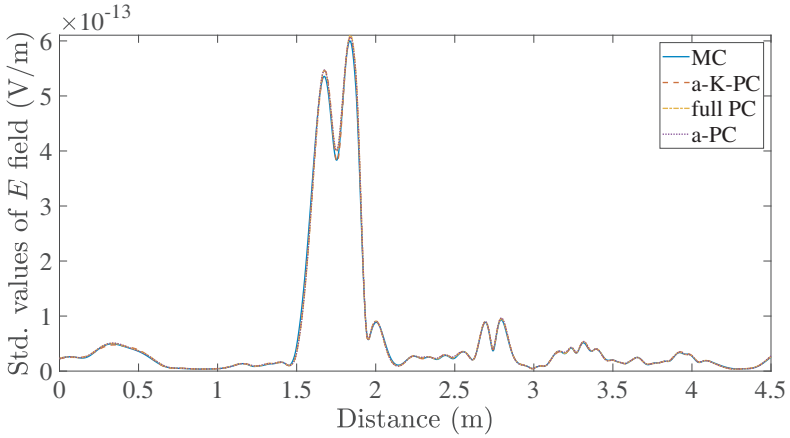


Figure 7. Std. of the electric field intensity for the transmission-line problem with $q = 0.15$ and $k = 3$.

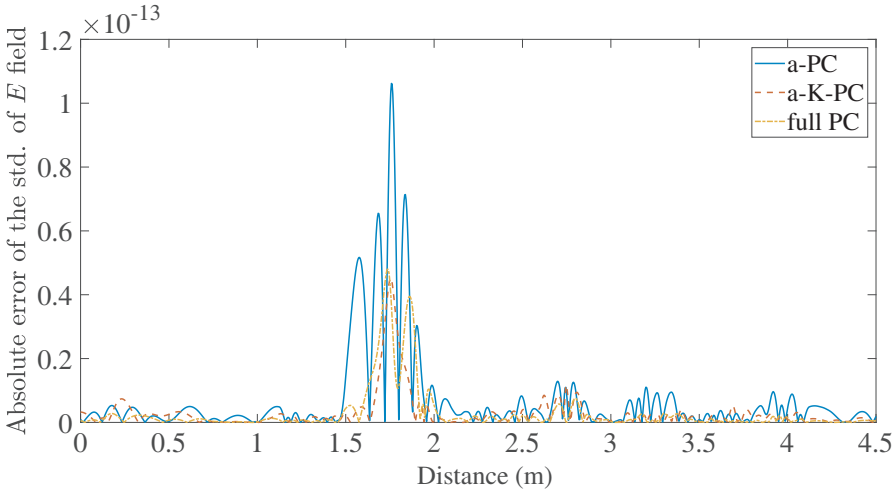


Figure 8. Absolute error of the std. of the electric field intensity for the transmission-line problem with $q = 0.15$ and $k = 3$.

4.3. Dual-Band 5G Antenna

The third scenario comprises the patch 5G antenna [27] of Fig. 9, which resonates in the bands of 27 GHz and 37 GHz. In this example, the effects of various random design parameters on the reflection coefficient are examined. Particularly, those stochastic variables present a uniform distribution with their statistical properties depicted in Table 2. The dual-band 5G antenna problem has been designed and solved in the CST Microwave Studio 2019 [28] in a computational lattice of $58 \times 59 \times 24$ cells. Furthermore, the order of all PC schemes is equal to 3, while the parameter q in the anisotropic techniques has a value of 0.85. It is also worth mentioning that the a-K-PC method estimates the statistical moments of the reflection coefficient by setting $G = 40$, in order to further maximize the determinant of the matrix $\mathbf{A}^T\mathbf{A}$. Lastly, the boundaries are terminated via a 4-cell Perfectly-Matched Layer (PML) [29]. Fig. 10 displays the mean of the absolute value of the reflection coefficient, and Fig. 11 illustrates the std. of the same quantity. In Fig. 12, the absolute error of those statistical moments is presented. Fig. 13

Table 2. Mean values and standard deviations of the design parameters for the 5G antenna problem.

Design parameters	Mean	Std.
D_f	0.9 mm	0.02 mm
L_1	3.1 mm	0.09 mm
L_2	2.5 mm	0.07 mm
W_f	0.2 mm	0.005 mm
substrate dielectric constant	$2.2\epsilon_0$	$0.06\epsilon_0$
substrate thickness	0.254 mm	0.007 mm
t_1	0.1 mm	0.003 mm
t_2	0.7 mm	0.02 mm
t_3	0.4 mm	0.01 mm
t_4	0.4 mm	0.01 mm

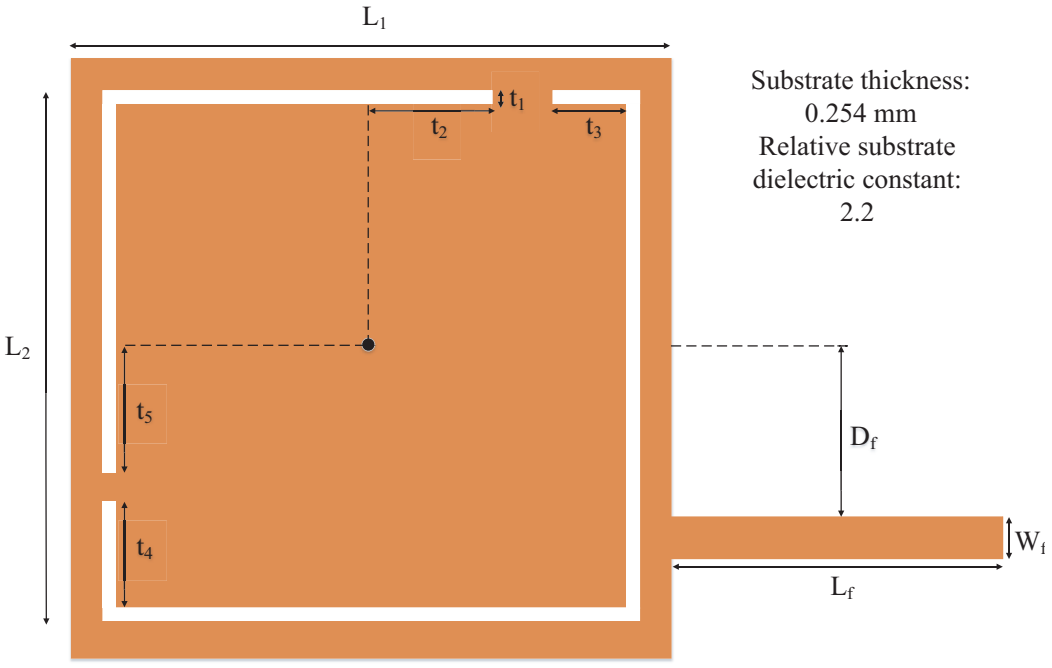


Figure 9. The schematic for the dual-band 5G antenna problem.

depicts the probability density function of the 6-dB roll-off frequency. The PC techniques generate reliable outcomes compared to the ones in the MC scheme in this case too. Concerning the simulation times of each method, the MC approach requires approximately 76.65 hours (5000 simulations), while the full PC and e a-PC techniques need around 30.7 hours (2002 realizations) and 7.7 hours (502 realizations). Nonetheless, the a-K-PC algorithm takes about 2.2 hours (144 realizations), making it again less computationally expensive than the other approaches. Additionally, the values of V_i are estimated in 1.34 hours (82 realizations) with $k = 4$.

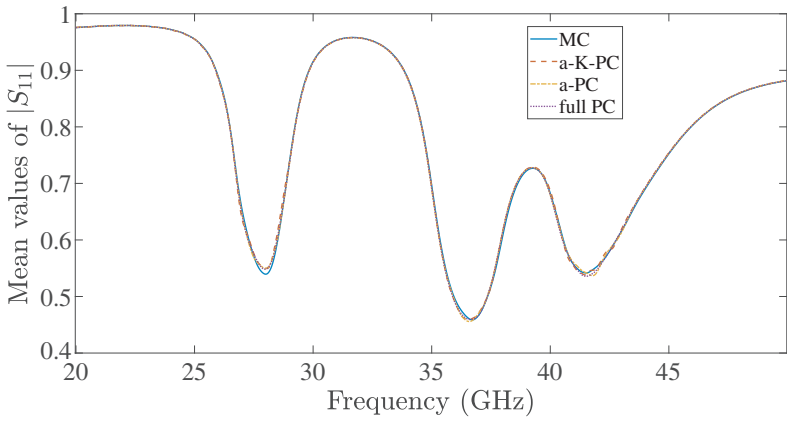


Figure 10. Mean values of the reflection coefficient for the dual-band 5G antenna problem.

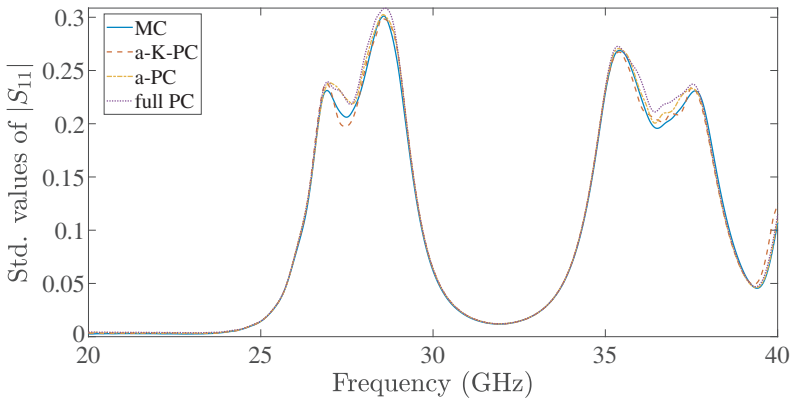
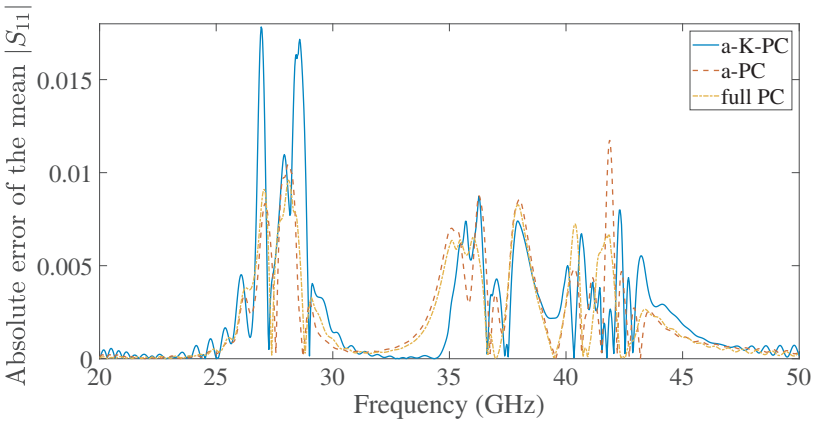
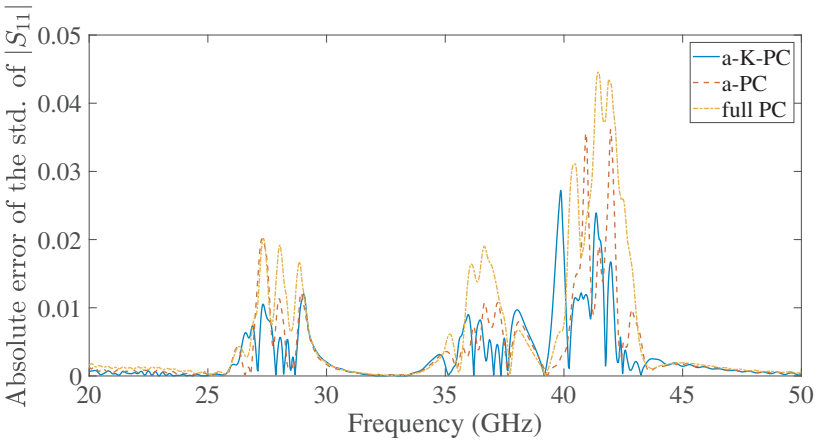


Figure 11. Std. of the reflection coefficient for the dual-band 5G antenna problem.



(a)



(b)

Figure 12. Absolute error of the mean and the std. of the reflection coefficient for the dual-band 5G antenna problem. (a) Absolute error of the mean reflection coefficient. (b) Absolute error of the std. of the reflection coefficient.

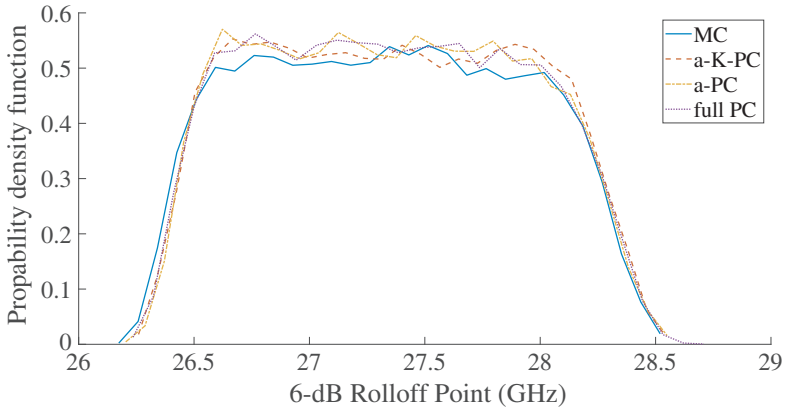


Figure 13. Probability density function of the 6 dB roll-off point for the dual-band 5G antenna problem.

4.4. Lowpass Filter

The proposed technique is applied again in the lowpass filter of Fig. 14. This problem incorporates eight uniform random variables with statistical characteristics presented in Table 3. In particular, the

Table 3. Mean values and standard deviations of the design parameters for the lowpass filter problem.

Design parameters	Mean	Std.
d_1	5.69 mm	0.32 mm
d_2	5.69 mm	0.32 mm
d_3	5.69 mm	0.32 mm
h	2.54 mm	0.14 mm
s_1	2.43 mm	0.14 mm
s_2	2.43 mm	0.14 mm
substrate thickness	0.79 mm	0.04 mm
substrate dielectric constant	$2.12\epsilon_0$	$0.12\epsilon_0$

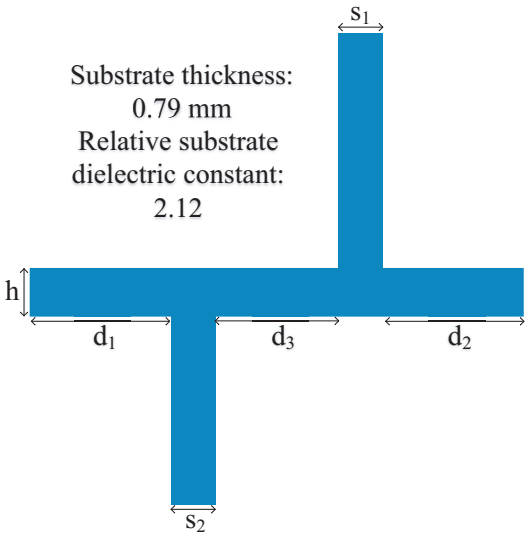


Figure 14. The schematic of the lowpass filter problem.

effects of those inputs to the variability of the transmission coefficient is examined. The CST Microwave Studio 2019 has been utilized as a deterministic solver in this case too. Additionally, the computational grid consists of $103 \times 71 \times 22$ cells, and also, the boundaries are terminated via a 10-cell PML. In this test case, the polynomial order of both the a-PC and full PC techniques is equal to $k = 4$, and q has a value of 0.9. The a-K-PC algorithm terminates with the criterion $\epsilon_t = 0.15$, resulting in a PC expansion of $k = 3$.

Figures 15 and 16 depict the mean and the std. of the absolute value of the transmission coefficient, respectively. In Fig. 17, the absolute error between each PC technique and the MC method is illustrated for the aforementioned quantities. Furthermore, Fig. 18 displays the probability density function of the 6 dB roll-off point. The PC approaches present a satisfying accuracy compared to the MC method in this case too. The MC scheme takes about 86.1 hours (5000 simulations), while the a-PC and full PC approaches require approximately 7.3 hours (424 simulations) and 17.05 hours (990 simulations). The simulations in the a-K-PC technique last for about 1.61 hours (94 simulations), rendering it again more efficient than the other PC schemes. It should be noted that the number of realizations in the a-PC and full PC algorithms for $k = 3$ is equal to 122 and 330, respectively. As a result, those two methods are still more computationally expensive than the proposed technique. Finally, V_i are approximated in 1.13 hours (66 simulations) with $k = 4$.

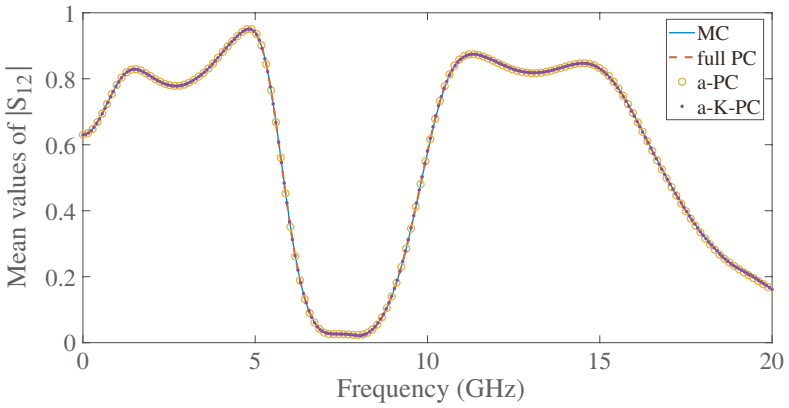


Figure 15. Mean values of the transmission coefficient for the lowpass filter problem.

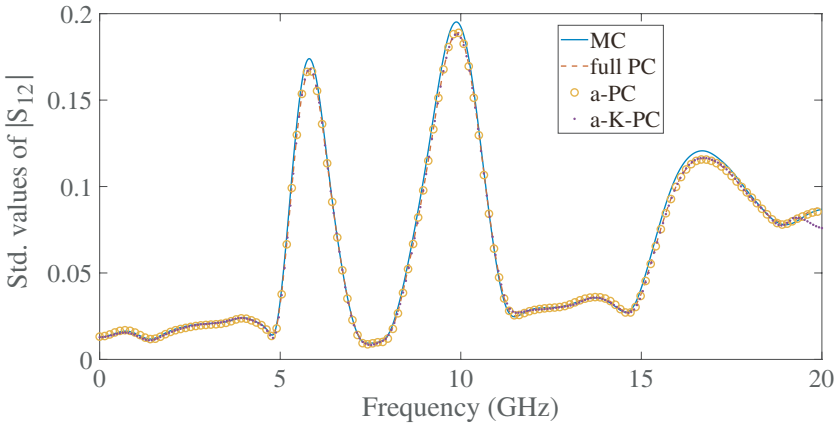


Figure 16. Std. values of the transmission coefficient for the lowpass filter problem.

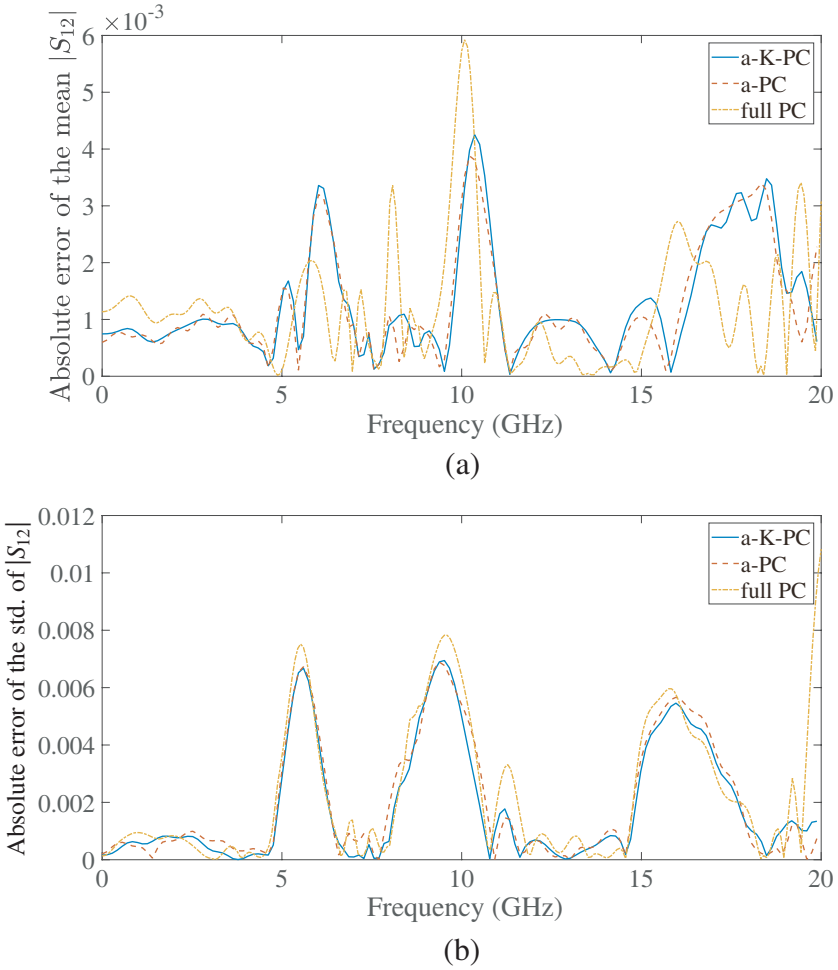


Figure 17. Absolute error of the mean and the std. of the transmission coefficient for the lowpass filter problem. (a) Absolute error of the mean transmission coefficient. (b) Absolute error of the std. of the transmission coefficient.

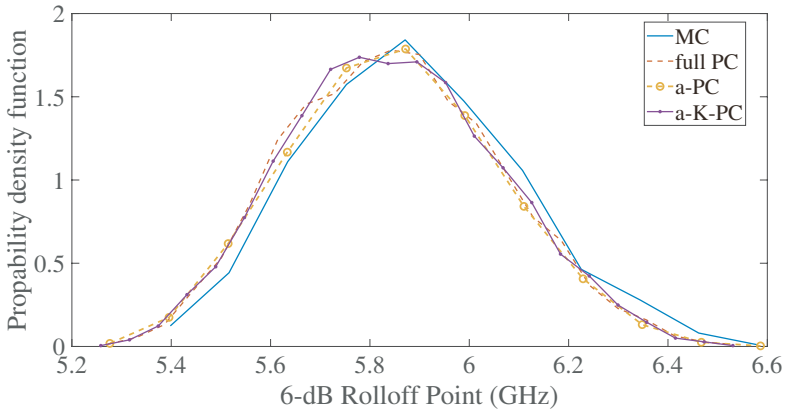


Figure 18. Probability density function of the 6 dB roll-off point for the lowpass filter problem.

5. CONCLUSION

A novel anisotropic PC approach has been implemented that manages to estimate the PC coefficients in small computational times. Specifically, the presented scheme utilizes low and high fidelity heuristics for constructing a PC expansion. Numerical outcomes and comparisons with other techniques prove the validity and efficiency of this technique. In future works, different sampling algorithms besides the Latin hypercube grid will be examined, in order to further improve the accuracy of the proposed method.

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