

ROTATIONAL STABILITY OF A CHARGED DIELECTRIC RIGID BODY IN A UNIFORM MAGNETIC FIELD

G.-Q. Zhou

Department of Physics
Wuhan University
Wuhan 430072, China

Abstract—Based on a new concept, i.e., charge moment tensor and the rotational equation of a charged dielectric rigid body about a fixed-point under a uniform external magnetic field, one symmetrical case has been rigorously solved. The rotational stability has been analyzed in detail for two cases, general and symmetrical, respectively, by means of some techniques of matrix analysis.

1. INTRODUCTION

A rotational charged body must generate magnetic moment, thus sustain a moment of force in an external magnetic field [1–5]. In view of electromagnetism, it is of both theoretical importance and broad application background to investigate the rotation dynamics of a rotational charged body under an external electromagnetic field [1–10]. For a general charged particle or a continuous charged medium, the electrodynamics has been well formulated [11–13]. Starting from a strict and delicate analogue relation, references [1–4] introduce a concept of magnetic-moment quadric, deduces and numerates some rules and examples about computing the magnetic moment of a rotational charged body. Meanwhile, the conditions of zero magnetic moment for an arbitrary rotational charged body have been formulated explicitly in [2]. Furthermore, without resorting to the analogue relation used in [1–3], a natural method of introducing the new concept of charge moment tensor has been proposed, and a more explicit dynamic theory for a rotational charged body has been constructed in [4]. As well known, movement of a rigid body can be viewed

Corresponding author: G.-Q. Zhou (zgq@whu.edu.cn).

as superposition of movement of its center of mass and the rotation around the center, and the latter is the topic of the present paper, especially that the stability of a dynamical system is of great interest for those researchers who work in this field. Reference [4] has dealt with a simple symmetric case for a charged dielectric rigid body purely rotating in an external uniform magnetic field. We take a charged dielectric rigid body as an instance to study its dynamical equation and its stability problem under an external uniform magnetic field plus an arbitrary time-dependent-only torque, and put our research object under limitation of slow rotation and no gravitation so that the mechanic damping effect of electromagnetic radiation and the relativistic effects caused by rotation can all be ignorable. In the process of rotation, the charge distribution is invariant with respect to the dielectric rigid body itself.

The paper is organized as follows. In Section 2, some fundamental concepts are reviewed, and the Euler's dynamic equations of a rotational charged dielectric rigid body under an external uniform magnetic field plus a time-dependent-only torque have been expressed in terms of tensor \tilde{T} . In Section 3, the stability of an autonomous dynamical system has been analyzed by means of matrix analysis. In Section 4, a symmetric case is rigorously solved, and its stability is analyzed. In the end, some concluding remarks have been given in Section 5.

2. THE EULER'S ROTATIONAL EQUATION FOR A ROTATIONAL CHARGED DIELECTRIC RIGID BODY UNDER ACTION OF A UNIFORM MAGNETIC FIELD AND A TIME-DEPENDENT-ONLY TORQUE

References [1–5] have defined and discussed a new concept — **charge moment tensor** $\tilde{T}(O)$ with respect to a fixed point O , so the **magnetic moment** with respect to this point O is

$$\vec{P}_m(O) = \frac{1}{2} \tilde{T}(O) \cdot \vec{\omega} \quad (1)$$

Based on the concepts of principal axes and principal-axis scalar charge moments [1, 2], charge moment tensor $\tilde{T}(O)$ has a definite meaning that is independent of the movement of this charged body and can be expressed in a diagonal form

$$\tilde{T}(O) = \text{diag}(T_1, T_2, T_3) \quad (2)$$

For the case of fixed point rotation about O , the magnetic moment

is

$$\vec{P}_m(O) = \frac{1}{2} \left(T_1 \omega_x \vec{i} + T_2 \omega_y \vec{j} + T_3 \omega_z \vec{k} \right) \quad (3)$$

Here $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors of axes X, Y, Z , respectively.

We select Cartesian coordinate system $O-XYZ$ in the body reference with the center-of-mass point O as its origin and the inertia principal axes as its three axes. Therefore, according to formula (3), the moment of force generated by the uniform magnetic field in a fixed reference is

$$\vec{M} = M_x \vec{i} + M_y \vec{j} + M_z \vec{k} = \vec{P}_m \times \vec{B} = \frac{1}{2} \left[\vec{T}(O) \cdot \vec{\omega} \right] \times \vec{B} \quad (4)$$

The rotation of the dielectric rigid body under action of an additional time-dependent-only torque $\vec{\Gamma}(t)$ satisfies Euler's equations in a fixed reference [14]

$$\begin{cases} J_x \dot{\omega}_x - (J_y - J_z) \omega_y \omega_z = M_x + \Gamma_x(t) \\ J_y \dot{\omega}_y - (J_z - J_x) \omega_z \omega_x = M_y + \Gamma_y(t) \\ J_z \dot{\omega}_z - (J_x - J_y) \omega_x \omega_y = M_z + \Gamma_z(t) \end{cases} \quad (5)$$

The concrete expression of Eq. (5) is

$$\begin{cases} J_x \dot{\omega}_x - (J_y - J_z) \omega_y \omega_z = \frac{1}{2} [(T_{21} B_z - T_{31} B_y) \omega_x + (T_{22} B_z - T_{32} B_y) \omega_y \\ \quad + (T_{23} B_z - T_{33} B_y) \omega_z] + \Gamma_x(t) \\ J_y \dot{\omega}_y - (J_z - J_x) \omega_z \omega_x = \frac{1}{2} [(T_{31} B_x - T_{11} B_z) \omega_x + (T_{32} B_x - T_{12} B_z) \omega_y \\ \quad + (T_{33} B_x - T_{13} B_z) \omega_z] + \Gamma_y(t) \\ J_z \dot{\omega}_z - (J_x - J_y) \omega_x \omega_y = \frac{1}{2} [(T_{11} B_y - T_{21} B_x) \omega_x + (T_{12} B_y - T_{22} B_x) \omega_y \\ \quad + (T_{13} B_y - T_{23} B_x) \omega_z] + \Gamma_z(t) \end{cases} \quad (6)$$

here $T_{ij} = T_{ji}$, $(i, j = 1, 2, 3)$ and $\tilde{J}(O) = \text{diag}(J_x, J_y, J_z)$ is the principal-axes inertia tensor of the rigid body with respect to point O . Note that although \tilde{J} is diagonal in the principal-axes coordinate system, generally the charge moment tensor \tilde{T} is not definitely of a diagonal form at the same time. It is due to a fact that the principal axes of the inertia moment tensor of the rigid body are not generally coincident with that of the charge moment tensor, just as that generally the center of mass is not coincident with the center of charge.

Equation (5) or (6) is universal and effective for an arbitrary rotational charged dielectric rigid body under the action of a uniform magnetic field plus an additional external time-dependent-only torque.

3. STABILITY OF A NONLINEAR AUTONOMOUS DYNAMICAL SYSTEM

Stability problem of a dynamical system involves discussing the implication of a tiny perturbed initial value upon the asymptotic behavior of a dynamical system (equation). It is of great theoretical significance and extensive application background.

According to theory of differential equation, when $\vec{\Gamma}(t) = 0$, Eq. (6) depicts a nonlinear autonomous dynamical system which can be rewritten in a standard form as follows

$$\frac{d\vec{\omega}}{dt} = \frac{1}{2}\Pi\vec{\omega} + \vec{R}(\vec{\omega}) \quad (7)$$

Here $\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$. The constant matrix Π and nonlinear term $\vec{R}(\vec{\omega})$ are written as follows

$$\Pi = \begin{pmatrix} (T_{21}B_z - T_{31}B_y)/J_x & (T_{22}B_z - T_{32}B_y)/J_x & (T_{23}B_z - T_{33}B_y)/J_x \\ (T_{31}B_x - T_{11}B_z)/J_y & (T_{32}B_x - T_{12}B_z)/J_y & (T_{33}B_x - T_{13}B_z)/J_y \\ (T_{11}B_y - T_{21}B_x)/J_z & (T_{12}B_y - T_{22}B_x)/J_z & (T_{13}B_y - T_{23}B_x)/J_z \end{pmatrix} \quad (8)$$

$$\vec{R}(\vec{\omega}) \equiv \begin{pmatrix} (J_y - J_z)\omega_y\omega_z/J_x \\ (J_z - J_x)\omega_z\omega_x/J_y \\ (J_x - J_y)\omega_x\omega_y/J_z \end{pmatrix} \quad (9)$$

with $\vec{R}(\vec{\omega})/\omega \rightarrow 0$, as $\vec{\omega} \rightarrow 0$. Generally Eq. (6) or (7) can not be rigorously solved, but its stability can be qualitatively analyzed by means of matrix theory.

Then for the case of $\vec{\Gamma}(t) = 0$, or an autonomous dynamical system, some discussion is in order. It can be proved that

$$\det(\Pi) = 0 \quad (10)$$

Thereby the **linear approximate equation** of (7) with $\vec{R}(\vec{\omega}) \rightarrow 0$ is

$$\frac{d\vec{\omega}}{dt} = \frac{1}{2}\Pi\vec{\omega}, \quad (11)$$

which permits nonzero constant solutions called the singular or equilibrium points of Eq. (7)

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \exp \left[\frac{1}{2J}\Pi(t - t_0) \right] \begin{pmatrix} \omega_{x0} \\ \omega_{y0} \\ \omega_{z0} \end{pmatrix} \quad (12)$$

where

$$\vec{\omega}_0|_{t=t_0} = (\omega_{x0}, \omega_{y0}, \omega_{z0}) \quad (13)$$

According to theory of matrix and linear differential equation, when each of the three roots $\{\lambda_1, \lambda_2, \lambda_3\}$ of following characteristic equation (E is a 3×3 identity matrix)

$$\det(\lambda E - \Pi) = 0 \quad (14)$$

has a negative real part, solution (12) must tend to zero as $t \rightarrow \infty$ and is called an asymptotic stable solution. If any one of the three roots $\{\lambda_1, \lambda_2, \lambda_3\}$ has a positive real part, the solution (12) must tend to infinite as $t \rightarrow \infty$. This must result in an unstable rotation. For such a case, the relativistic effects should be taken into consideration and invalidate our “non-relativistic approximation” supposed in Section 1. When some roots of the three roots $\{\lambda_1, \lambda_2, \lambda_3\}$ have zero real part, and the remaining roots have negative real part, Eq. (7) with $\vec{\Gamma}(t) = 0$ must have an oscillating solution which is called the critical case.

The theory of algebra equation gives relation of the roots and the coefficients of Eq. (14) as

$$\lambda_1 \lambda_2 \lambda_3 = \det(\Pi) = 0 \quad (15)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(\Pi) \equiv \alpha \quad (16)$$

$$\alpha = (T_{21}B_z - T_{31}B_y)/J_x + (T_{32}B_x - T_{12}B_z)/J_y + (T_{13}B_y - T_{23}B_x)/J_z \quad (17)$$

$$\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = \beta, \quad (18)$$

$$\begin{aligned} \beta \equiv & (T_{22}T_{33} - T_{23}^2)B_x^2/J_yJ_z + (T_{11}T_{33} - T_{13}^2)B_y^2/J_zJ_x \\ & + (T_{11}T_{22} - T_{12}^2)B_z^2/J_xJ_y - (T_{12}T_{33} - T_{13}T_{23})(J_x + J_y)B_xB_y/J_xJ_yJ_z \\ & - (T_{11}T_{23} - T_{12}T_{13})(J_y + J_z)B_yB_z/J_xJ_yJ_z \\ & - (T_{13}T_{22} - T_{12}T_{23})(J_z + J_x)B_zB_x/J_xJ_yJ_z \end{aligned} \quad (19)$$

Then from (15)–(18), without loss of generality, we let $\lambda_3 = 0$, and

$$\lambda_1 = \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2}, \quad \lambda_2 = \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2}, \quad (\text{as } \alpha^2 - 4\beta \geq 0) \quad (20)$$

or

$$\lambda_1 = \frac{\alpha + i\sqrt{4\beta - \alpha^2}}{2}, \quad \lambda_2 = \frac{\alpha - i\sqrt{4\beta - \alpha^2}}{2}, \quad (\text{as } \alpha^2 - 4\beta < 0) \quad (21)$$

Due to $\lambda_3 = 0$, according to the stability theory, any one of the conditions listed below must cause (12) to be an unstable solution.

- i. $\alpha^2 - 4\beta \geq 0$ and $\lambda_1\lambda_2 = \beta < 0$. (One of λ_1, λ_2 is positive.)
- ii. $\alpha^2 - 4\beta \geq 0, \lambda_1\lambda_2 = \beta > 0$ and $\lambda_1 + \lambda_2 = \alpha > 0$. (Both λ_1 and λ_2 are positive.)
- iii. $\alpha^2 - 4\beta < 0$ and $\lambda_1 + \lambda_2 = \alpha > 0$. (The real parts of λ_1, λ_2 are positive.)

Also due to $\lambda_3 = 0$, any one of the listed below is a **critical stability condition** for solution (12)

- i. $\alpha^2 - 4\beta \geq 0$ and $\lambda_1\lambda_2 = \beta > 0$, but $\lambda_1 + \lambda_2 = \alpha < 0$. (Both λ_1 and λ_2 are negative.)
- ii. $\alpha^2 - 4\beta < 0, \lambda_1 + \lambda_2 = \alpha < 0$. (The real parts of λ_1, λ_2 are negative.)

On the other hand, signs of $\alpha^2 - 4\beta$, α and β are apparently determined by the distribution of charge and mass, the magnitude and direction of magnetic induction.

We will deal with another simple and symmetrical case in the subsequent section.

4. AN EXAMPLES OF STRICT SOLUTION AND ITS STABILITY PROBLEM

Let us consider a simple case of strict solution. A charged dielectric rigid body rotates around its center of mass O with a nonzero initial angular velocity $\vec{\omega}_0|_{t=t_0} = (\omega_{x0}, \omega_{y0}, \omega_{z0})$ expressed in the body Cartesian coordinate frame $O\text{-}XYZ$, which is defined by the three principal axes of the inertia tensor. Generally the charge moment tensor $\tilde{T}(O)$ is not definitely of a diagonal form as the inertia moment tensor $\tilde{J}(O)$ which is supposed here to have such a symmetry as $J_x = J_y = J_z \equiv J$. Then, under the action of a uniform magnetic field with magnetic induction $\vec{B} = (B_x, B_y, B_z)$ plus a time-dependent-only external torque $\vec{\Gamma}(t) = (\Gamma_x(t), \Gamma_y(t), \Gamma_z(t))$, some concrete conclusions will be drawn from Eq. (6) about the movement of the charged dielectric rigid body. This is a rigorously solvable example of our theory.

Equation (6) can be rewritten in a matrix form as $J_x = J_y = J_z \equiv J$

$$\frac{d}{dt} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \frac{1}{2J} \Lambda \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} + \frac{1}{J} \begin{pmatrix} \Gamma_x(t) \\ \Gamma_y(t) \\ \Gamma_z(t) \end{pmatrix} \quad (22)$$

with

$$\Lambda = \begin{pmatrix} T_{21}B_z - T_{31}B_y & T_{22}B_z - T_{32}B_y & T_{23}B_z - T_{33}B_y \\ T_{31}B_x - T_{11}B_z & T_{32}B_x - T_{12}B_z & T_{33}B_x - T_{13}B_z \\ T_{11}B_y - T_{21}B_x & T_{12}B_y - T_{22}B_x & T_{13}B_y - T_{23}B_x \end{pmatrix} \quad (23)$$

where Λ is a time-independent matrix. Thus, by use of the matrix theory, we have

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \exp \left[\frac{1}{2J} \Lambda (t - t_0) \right] \begin{pmatrix} \omega_{x0} \\ \omega_{y0} \\ \omega_{z0} \end{pmatrix} + \frac{1}{J} \int_{t_0}^t \exp \left[\frac{1}{2J} \Lambda (t - \tau) \right] \begin{pmatrix} \Gamma_x(\tau) \\ \Gamma_y(\tau) \\ \Gamma_z(\tau) \end{pmatrix} d\tau \quad (24)$$

About the stability problem of (22), we only need to discuss the stability of solution at the equilibrium point for the homogeneous equation of Eq. (22) as $\vec{\Gamma}(t) = 0$, because the inhomogeneous case, Eq. (22), can be changed into a homogeneous case by a simple transformation

$$\begin{aligned} \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix} &= \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} - \frac{1}{J} \int_{t_0}^t \exp \left[\frac{1}{2J} \Lambda (t - \tau) \right] \begin{pmatrix} \Gamma_x(\tau) \\ \Gamma_y(\tau) \\ \Gamma_z(\tau) \end{pmatrix} d\tau \\ &= \exp \left[\frac{1}{2J} \Lambda (t - t_0) \right] \begin{pmatrix} \omega_{x0} \\ \omega_{y0} \\ \omega_{z0} \end{pmatrix}, \end{aligned} \quad (25)$$

with $\vec{\Omega}$ satisfies following linear equation

$$\frac{d}{dt} \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix} = \frac{1}{2J} \Lambda \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix}, \quad \begin{pmatrix} \Omega_x(t_0) \\ \Omega_y(t_0) \\ \Omega_z(t_0) \end{pmatrix} = \begin{pmatrix} \omega_{x0} \\ \omega_{y0} \\ \omega_{z0} \end{pmatrix} \quad (26)$$

Some discussion similar to that in Section 3 is then in order about Eq. (22) for the homogeneous case (26), i.e., the case of $\vec{\Gamma}(t) = 0$. It can be found that $\det(\Lambda) = 0$. Thereby, Eq. (26) permits nonzero constant vector solutions in addition to the trite zero solution. They are called the singular points or equilibrium points of Eq. (26). According to the stability theory of a dynamical system, when each of the three roots $\{\eta_1, \eta_2, \eta_3\}$ of following characteristic equation

$$\det(\eta E - \Lambda) = 0, \quad (27)$$

has a negative real part, solution (25) is called an asymptotic stable solution. If at least one of the three roots $\{\eta_1, \eta_2, \eta_3\}$ has a positive real part, the solution (25) is unstable. When some of the three roots $\{\eta_1, \eta_2, \eta_3\}$ have zero real parts, and the remaining roots have negative real parts, the linear approximate part of Eq. (22) (i.e., $\vec{\Gamma}(t) = 0$) must have an oscillating solution which is called the critical case between the asymptotic stable and unstable solutions.

The theory of algebra equation gives relation of the roots and coefficients of Eq. (27) as

$$\eta_1 + \eta_2 + \eta_3 = \text{tr}(\Lambda) = 0, \quad (28)$$

$$\eta_1 \eta_2 \eta_3 = \det(\Lambda) = 0, \quad (29)$$

$$\eta_1 \eta_2 + \eta_2 \eta_3 + \eta_3 \eta_1 = K, \quad (30)$$

where

$$\begin{aligned} K \equiv & (T_{22}T_{33} - T_{23}^2)B_x^2 + (T_{11}T_{33} - T_{13}^2)B_y^2 + (T_{11}T_{22} - T_{12}^2)B_z^2 \\ & - 2(T_{12}T_{33} - T_{13}T_{23})B_xB_y - 2(T_{11}T_{23} - T_{12}T_{13})B_yB_z \\ & - 2(T_{13}T_{22} - T_{12}T_{23})B_zB_x \end{aligned} \quad (31)$$

Then from (28)–(31), without loss of generality, we let $\eta_3 = 0$, and

$$\eta_1 = -\eta_2 = \pm i\sqrt{K}, \quad (\text{as } K \geq 0) \quad (32)$$

or

$$\eta_1 = -\eta_2 = \pm \sqrt{-K}, \quad (\text{as } K < 0) \quad (33)$$

Thus, when $K \geq 0$, (then $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = 0$), solution (25) is a constant vector (for $K = 0$), or an oscillating one (for $K > 0$), which is the critical case between the asymptotic stable and unstable ones. Its stability depends on the concrete form of Λ .

When $K < 0$, then one of the roots is positive, and the solution (25) is unstable and therefore invalidate our “non-relativistic approximation” and the dynamic Eq. (6). On the other hand, sign of K is apparently determined by the sign and distribution of charge, the magnitude and direction of magnetic induction, but not affected direct by the distribution of mass.

5. CONCLUSION

Research on the dynamic behaviors of a charged rigid body in an electromagnetic field is an important and valuable pursuit involved in

many disciplines. By use of a new concept, i.e., charge moment tensor, which is related to magnetic moment of a rotational charged body, the Euler's equation of a rotational charged dielectric rigid body, under a uniform magnetic field plus an additional time-dependent torque, is analyzed, especially at its equilibrium point. Some concrete and simple conclusions about a case with symmetric mass distribution but arbitrary charge distribution are drawn. The corresponding stability about the system has been discussed in detail, and the stability test rules are also explicitly manifested.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China under Grant No. 10775105.

REFERENCES

1. Zhou, G.-Q., "Charge moment tensor and the magnetic moment of rotational charged bodies," *Progress In Electromagnetics Research*, PIER 68, 156–160, 2007.
2. Zhou, G.-Q. and W.-J. Zhou, "The magnetic-moment quadric and conditions of vanishing magnetic moment for a rotational charged body," *Progress In Electromagnetics Research*, PIER 70, 211–223, 2007.
3. Zhou, G.-Q., "Several rules about the magnetic moment of rotational charged bodies," *PIERS Online*, Vol. 3, No. 6, 812–816, The Electromagnetics Academy, Cambridge, 2007.
4. Zhou, G.-Q. and C. Guan, "Charge moment tensor and its application to a rotational charged rigid body in a uniform magnetic field," *Journal of Electromagnetic Waves and Applications*, Vol. 22, 2179–2190, 2008.
5. Zhou, G.-Q. and X. Xiao, "Dynamical problem of a rotational charged dielectric rigid body in a uniform magnetic field," *Progress In Electromagnetics Research C*, Vol. 1, 229–240, 2008.
6. Krasheninnikov, S. I., V. I. Shevchenko, and P. K. Shukla, "Spinning of charged dust particle in a magnetized plasma," *Physics Letter A*, Vol. 361, 133–135, 2007.
7. Kroh, H. J. and B. U. Felderhof, "Lorentz torque on a charged sphere rotating in a dielectric fluid in the presence of a uniform magnetic field," *Phys. Rev. E*, Vol. 62, No. 4, 7309–7314, 2000.
8. Dolinsky, Y. and T. Elperin, "Rotation of the leaky dielectric

- particle in a rotating electric field,” *Phys. Rev. E*, Vol. 75, 026611, 2007.
9. Tchernyi, V. V. and A. Y. Pospelov, “Possible electromagnetic nature of the Saturn’s rings: Superconductivity and magnetic levitation,” *Progress In Electromagnetics Research*, PIER 52, 277–299, 2005.
 10. Dmtriev, V., “Tables of the second rank constitutive tensors for linear homogeneous media described by point magnetic groups of symmetry,” *Progress In Electromagnetics Research*, PIER 28, 43–95, 2000.
 11. Landau, L. D. and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, Oxford, 1984.
 12. Jackson, J. D., *Classical Electrodynamics*, John Wiley & Sons, Inc., New York, 1962.
 13. Liu, J.-P., *Electrodynamics*, Higher Education Press, Beijing, 2004 (in Chinese).
 14. Zhou, Y.-B., *Theoretical Mechanics*, People’s Education Press, Beijing, 1974.