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AN APPROACH TO THE MULTIVECTORIAL APPARENT POWER IN TERMS OF A GENERALIZED POYNTING MULTIVECTOR

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Abstract—The purpose of this paper is to explain an exact derivation of apparent power in n-sinusoidal operation founded on electromagnetic theory, until now unexplained by simple mathematical models. The aim is to explore a new tool for a rigorous mathematical and physical analysis of the power equation from the Poynting Vector (PV) concept. A powerful mathematical structure is necessary and Geometric Algebra offers such a characteristic. In this sense, PV has been reformulated from a new *Multivectorial Euclidean Vector Space* structure ($\mathcal{CG}_n$-$\mathbb{R}^3$) to obtain a *Generalized Poynting Multivector* ($\tilde{\mathcal{S}}$). Consequently, from $\tilde{\mathcal{S}}$, a suitable multivectorial form ($\tilde{\mathcal{P}}$ and $\tilde{\mathcal{D}}$) of the Poynting Vector corresponds to each component of apparent power. In particular, this framework is essential for the clarification of the connection between a *Complementary Poynting Multivector* ($\tilde{\mathcal{D}}$) and the power contribution due to cross-frequency products. A simple application example is presented as an illustration of the proposed power multivector analysis.

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1. LIST OF SYMBOLS (NOMENCLATURE)

n-sinusoidal = non-sinusoidal or multi-sinusoidal.
$\mathbb{R}$ = real numbers
E^3 = Euclidean vector space
$\mathbf{C}$ = complex vector space
$\mathcal{V}^n$ = linear space over real numbers
$\mathcal{G}_n$ = Clifford algebra in n-dimensional real space
$\mathcal{CG}_n$ = complex Clifford Algebra
Φ = operator
Γ = time-domain frequency-domain transform
$\mathcal{CG}_n^t$-$\mathbb{R}^3$ = time generalized Euclidean space
$\mathcal{CG}_n$-$\mathbb{R}^3$ = frequency generalized Euclidean space
$\vec{\mathbf{1}}_{\mathbf{X}}, \vec{\mathbf{1}}_{\mathbf{Y}}, \vec{\mathbf{1}}_{\mathbf{Z}}$ = Euclidean canonical basis
$\vec{\mathbf{1}}_{X,Y,Z}$ = generic unitary vector of E^3
$\sigma_{1,\ldots,k}$ = Clifford algebra canonical basis
Id_C = identity operation
$\tilde{\mathbf{z}}(t)$ = instantaneous geometric vector ($\tilde{\mathbf{z}} \in \mathcal{CG}_n^t$-$\mathbb{R}^3$)
$\tilde{\mathbf{e}}(t)$ = instantaneous electric field geometric vector
$\tilde{\mathbf{h}}(t)$ = instantaneous magnetic field geometric vector
$\tilde{\mathbf{d}}(t)$ = instantaneous displacement field geometric vector
$\tilde{\mathbf{b}}(t)$ = instantaneous magnetic induction field geometric vector
$\tilde{\mathrm{z}}_{X,Y,Z}$ = components of $\tilde{\mathbf{z}}(t)$
$\tilde{\mathbf{z}}_p$ = p-th harmonic component of $\tilde{\mathbf{z}}(t)$
$\tilde{\mathbf{Z}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}$ = spatial components of $\tilde{\mathbf{Z}}$
$\tilde{\mathbf{Z}}$ = spatial geometric phasor ($\tilde{\mathbf{Z}} \in \mathcal{CG}_n$-$\mathbb{R}^3$)
$\tilde{\mathbf{Z}}_{\mathbf{p}}$ = spatial p-th harmonic component of $\tilde{\mathbf{Z}}$
$\tilde{Z}$ = geometric phasor ($\tilde{Z} \in \mathcal{CG}_n$)
$\tilde{Z}_p$ = p-th harmonic component of $\tilde{Z}$
$\tilde{Z}_{pq}$ = bivector component of $\tilde{Z}$
$\tilde{\mathbf{E}}$ = electric field geometric phasor
$\tilde{\mathbf{H}}$ = magnetic field geometric phasor
$\tilde{\mathbf{D}}$ = displacement field geometric phasor
$\tilde{\mathbf{B}}$ = magnetic induction field geometric phasor
$\tilde{\mathcal{S}}$ = generalized Poynting multivector (GPM)
$\tilde{\mathcal{P}}$ = Poynting multivector (PM)
$\tilde{\mathcal{D}}$ = complementary Poynting multivector (CPM)
U_p = p-th harmonic voltage rms value
I_p = p-th harmonic current rms value
$\otimes$ = classic geometric product
$\odot$ = generalized geometric product in $\mathcal{CG}_n$

$\circ$ = generalized geometric product in $\mathcal{CG}_n^t$-$\mathbb{R}^3$
$\cdot$ = inner product
$\wedge$ = outer product
$\oplus$ = direct sum
$+$ = classic sum for scalars and also direct sum for multivectors
j = imaginary unit
$*$ = conjugated operation
$\dagger$ = reverse operation
$\langle\,\rangle_0$ = scalar part
$\langle\,\rangle_2$ = bivector part
$\tilde{S}$ = apparent power multivector
$\|\tilde{S}\|$ = norm, value or magnitude of multivector $\tilde{S}$
$\tilde{\Omega}^{\cdot}$ = complex scalar
$\tilde{\Omega}^{\wedge}$ = complex bivector
ω_p, ω_q = harmonic frequencies
α_p = phase angle of p-th voltage geometric phasor
α_q = phase angle of q-th current geometric phasor
φ_q = phase angle between q-th voltage and q-th current geometric phasors
$\tilde{\delta}$ = relative quality index multivector (RQI)
PF = power factor

2. INTRODUCTION

2.1. Motivation

One of the fundamental issues in power system analysis is related with the electromagnetic theory in order to explain the energy transfer in an electric circuit. Hence, this paper establishes an electromagnetic foundation to the power equation representation. For this goal, a new Generalized Poynting Multivector is proposed.

2.2. Literature Review

The electrical circuits in n-sinusoidal operation can be analyzed by means of mathematical tools that are much simpler to handle than electromagnetic theory based on Maxwell equations [1]. However, it is also true that these equations fully explain interactions between electric and magnetic fields and therefore explain every electromagnetic phenomenon, including the energy transfer in an electric system. In n-sinusoidal operation, the distorted electromagnetic fields can be represented as sums or series of harmonics. Each harmonic component of the field is governed by Maxwell equations and satisfies the Poynting Theorem.

It is relevant to classify the contribution of these equations to the electric power theory into following lines of thought:

a) *Circuit theory analysis:* First, is the most commonly used approach. It analyzes currents, voltages and circuit element properties. In this sense, circuit theory, ruled by simple equations based on Ohm's law, can be regarded as a very particular case of electromagnetic theory, and power theory was developed mainly from circuit analysis. Electrical components of power systems are considered as elements of circuits and their electromagnetic behaviour is described by means of voltages and currents of element terminals. Circuit theory can explain only the power flows between components, and it is unable to reveal their spatial distribution. Nevertheless, no phenomena such as hysteresis losses or skin effects can be explained by circuit theory. These are phenomena of the electromagnetic field characteristics.

For this first approach, Complex Algebra [2] provides an initial procedure to solve the problem, despite its limitation to the purely sinusoidal case. The n-sinusoidal operation imposes the substitution of the Complex Algebra approach with a new representation model and the reformulation of the energy balance. Considerable research efforts have been directed towards the representation of apparent power in various ways [3–9]. Specifically, in [9], the authors use Geometric Algebra to define a multivector power based on the decomposition of the instantaneous current into the active and reactive components. It should be noted that their approach does not distinguish between reactive and distortion power from a mathematical viewpoint. Furthermore, none of the aforementioned papers leads to a representation that could be considered universally satisfactory.

b) *Electromagnetic theory analysis:* This second valid method analyzes the energy flow using the Poynting Theorem (PT), and therefore the Poynting Vector (PV) should be considered, since it represents the bridge between electromagnetic theory and circuit theory [10]. These tools are fundamental concepts of electromagnetic theory with respect to energy flow. The goal is to investigate the nature of the non-active power and some progress has undeniably been made. Numerous valuable contributions have appeared in the literature [11–17], each shedding more light on some aspects of the problem. From among them, [12, 13] masterfully explain the physical mechanism of energy propagation in electric power systems, [15] reconsiders the bases of electromagnetism in order to find a physical interpretation for the power equation, and [16] uses the PV to illustrate the nature of power flow in electric circuits using electromagnetic fields. However, critics of PV calculations [17] argue that electromagnetic theory is useless for practical applications of electric power theory. Against this reference, it is our view that the power equation can be based and interpreted through a new formulation of the Poynting Vector and that other aspects concerning the electromagnetic field in n-sinusoidal operation and their direct relation with power theory have yet to be thoroughly investigated. Thus, the purpose of this paper is to advance energy flow analysis by using a new mathematical structure for the representation of the power equation in single-phase circuits under n-sinusoidal operation. In this way, a complete solution to the power equation analysis problem for linear/non-linear circuits based on a *Generalized Poynting Multivector* ($\tilde{\mathcal{S}}$), is presented.

To this end, primarily our work introduces a $\mathcal{CG}_n^t$-$\mathbb{R}^3$ mathematical structure based on Clifford Algebra for the definition of the distorted electric and magnetic field intensities ($\tilde{\mathbf{e}}$, $\tilde{\mathbf{h}}$), which are time geometric fields associated to an Euclidean direction. From these definitions it is possible to obtain the quantities called *spatial geometric phasors* ($\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$) in the $\mathcal{CG}_n$-$\mathbb{R}^3$ structure in frequency domain. Consequently, our work is aimed at showing how an electric and magnetic field can be associated with the elements of Clifford Algebras [21, 22] for a new formulation and interpretation of power theory in this framework.

This second approach is more general and fundamental that the first approach based on circuit theory, and it has the additional advantage of providing a physical insight into the spatial distribution of the power flow.

Finally, this paper addresses the need to understand the multidimensional character of electric power theory and its relation to the electromagnetic theory.

2.3. Contributions

The paper is concerned with a representation of the power equation under non-sinusoidal conditions from electromagnetic theory. The apparent power concept is better understood if a Clifford vector space is used for the representation of the distorted electric and magnetic field intensities. This generates a larger linear space called *Generalized Euclidean Space* $\mathcal{CG}_n^t$-$\mathbb{R}^3$, which will be utilized in this paper for a new representation of the power equation. This objective cannot be reached on the Complex Algebra framework.

3. MATHEMATICAL FOUNDATIONS: GEOMETRIC EUCLIDEAN SPACES

3.1. Time Domain: Generalized Euclidean Space $\mathcal{CG}_n^t$-$\mathbb{R}^3$

In order to introduce the instantaneous quantities of electric and magnetic fields, in this section we define a new structure for the time domain that we have named *Generalized Euclidean Space*, $\mathcal{CG}_n^t$-$\mathbb{R}^3$, whose coefficients belong to the Complex Geometric Algebra $\mathcal{CG}_n$ constructed in [18]. Let $\left\{\vec{\mathbf{1}}_X, \vec{\mathbf{1}}_Y, \vec{\mathbf{1}}_Z\right\}$ be the "canonic" basis of the Euclidean space E^3. A generic element of $\mathcal{CG}_n^t$-$\mathbb{R}^3$ is given by

$$\tilde{\mathbf{z}}(t) = \tilde{z}_X\vec{\mathbf{1}}_X + \tilde{z}_Y\vec{\mathbf{1}}_Y + \tilde{z}_Z\vec{\mathbf{1}}_Z \tag{1}$$

where each component in (1) is in the form $\tilde{z}(t) = ke^{j[\alpha(t)+\theta]}\sigma_a \in \mathcal{CG}_n^t$, $k \geq 0$ and σ_a is a basis element of $\mathcal{CG}_n$ structure [18].

Thus, the $\mathcal{CG}_n^t$-$\mathbb{R}^3$ structure is a $\mathcal{CG}_n^t$ vector space whose inner product is defined by

$$\tilde{\mathbf{z}}(t) \cdot \tilde{\mathbf{w}}(t) = \langle \tilde{z}_X, \tilde{w}_X^* \rangle_0 + \langle \tilde{z}_Y, \tilde{w}_Y^* \rangle_0 + \langle \tilde{z}_Z, \tilde{w}_Z^* \rangle_0 \tag{2}$$

where, $\tilde{\mathbf{z}}(t) = \tilde{z}_X\vec{\mathbf{1}}_X + \tilde{z}_Y\vec{\mathbf{1}}_Y + \tilde{z}_Z\vec{\mathbf{1}}_Z$, $\tilde{\mathbf{w}}(t) = \tilde{w}_X\vec{\mathbf{1}}_X + \tilde{w}_Y\vec{\mathbf{1}}_Y + \tilde{w}_Z\vec{\mathbf{1}}_Z$. Moreover, from (D1) the norm of $\tilde{\mathbf{z}}(t)$ is given by

$$\|\tilde{\mathbf{z}}(t)\| = \sum_{i=X,Y,Z} \langle \tilde{z}_i, \tilde{z}_i^* \rangle_0 \tag{3}$$

Now we define the outer product in this structure as

$$\begin{aligned}
\tilde{\mathbf{z}}(t) \wedge (-\tilde{\mathbf{w}}(t)) &= \begin{bmatrix} \vec{\mathbf{1}}_X & \vec{\mathbf{1}}_Y & \vec{\mathbf{1}}_Z \\ \tilde{z}_X & \tilde{z}_Y & \tilde{z}_Z \\ -\tilde{w}_X & -\tilde{w}_Y & -\tilde{w}_Z \end{bmatrix} \\
&= \left(\langle -\tilde{z}_Y, \tilde{w}_Z \rangle_2 + \langle \tilde{z}_Z, \tilde{w}_Y \rangle_2\right) \vec{\mathbf{1}}_X \\
&\quad + \left(\langle \tilde{z}_X, \tilde{w}_Z \rangle_2 + \langle -\tilde{z}_Z, \tilde{w}_X \rangle_2\right) \vec{\mathbf{1}}_Y \\
&\quad + \left(\langle -\tilde{z}_X, \tilde{w}_Y \rangle_2 + \langle \tilde{z}_Y, \tilde{w}_X \rangle_2\right) \vec{\mathbf{1}}_Z
\end{aligned} \tag{4}$$

Based on (3) and (4), the Geometric Algebra $\mathcal{CG}_n^t$-$\mathbb{R}^3$ is defined by the following geometric product

$$\tilde{\mathbf{z}}(t) \circ \tilde{\mathbf{w}}(t) = \tilde{\mathbf{z}}(t) \cdot \tilde{\mathbf{w}}(t) + \tilde{\mathbf{z}}(t) \wedge \tilde{\mathbf{w}}(t) \tag{5}$$

3.2. Frequency Domain: Generalized Euclidean Space $\mathcal{CG}_n$-$\mathbb{R}^3$

Let $\Phi : \mathcal{CG}_n^t \to \mathcal{CG}_n$, $\Phi\left(k\, e^{j[\alpha(t)+\theta]}\sigma_a\right) = k\, e^{j\theta}\sigma_a$, be the operator that enables the transformation between time-domain and frequency-domain. We define $\mathcal{CG}_n$-$\mathbb{R}^3$ as

$$\Phi(\mathcal{CG}_n^t)\vec{\mathbf{1}}_X + \Phi(\mathcal{CG}_n^t)\vec{\mathbf{1}}_Y + \Phi(\mathcal{CG}_n^t)\vec{\mathbf{1}}_Z \tag{6}$$

where a generic element of this space is $\Phi(\tilde{z}_a) = \tilde{Z}_a$. Note that $\mathcal{CG}_n$-$\mathbb{R}^3$ can also be seen as

$$\mathcal{CG}_n\text{-}\mathbb{R}^3 = \left\{ \tilde{Z}_X \vec{\mathbf{1}}_X + \tilde{Z}_Y \vec{\mathbf{1}}_Y + \tilde{Z}_Z \vec{\mathbf{1}}_Z : \tilde{Z} \in \mathcal{CG}_n \right\}$$

where

$$\tilde{Z} = \sum_p \bar{Z}_p \sigma_p, \quad \bar{Z}_p \in \mathcal{C} \quad \text{and} \quad \sigma_p \in \mathcal{G}_n$$

Obviously, $\mathcal{CG}_n$-$\mathbb{R}^3$ is a $\mathcal{CG}_n$ (*complex-geometric*) vector space and the multiplication rule for two vectors $\tilde{\mathbf{Z}}$, $\tilde{\mathbf{W}} \in \mathcal{CG}_n$-$\mathbb{R}^3$ is given by

$$\tilde{\mathbf{Z}} \circ \tilde{\mathbf{W}} = \tilde{\mathbf{Z}} \cdot \tilde{\mathbf{W}} + \tilde{\mathbf{Z}} \wedge \left(-\tilde{\mathbf{W}} \right) \tag{7}$$

where (7) is the restriction from $\mathcal{CG}_n^t$-$\mathbb{R}^3 \to \mathcal{CG}_n$-$\mathbb{R}^3$.

The nesting of the geometric Euclidean spaces denoted by $\mathcal{CG}_n^t$-$\mathbb{R}^3$, $\mathcal{CG}_n$-$\mathbb{R}^3$, and $\mathcal{CG}_n$ are graphically illustrated in Fig. 1.

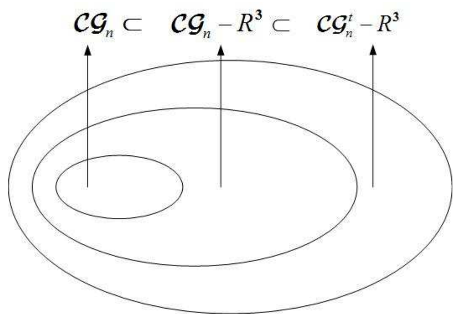

Figure 1. Nested geometric Euclidean vector spaces.

The fundamental concepts of Generalized Complex Geometric Algebra $\mathcal{CG}_n$ are given in [18] and further research about Geometric Algebra can be found in [21, 22].

4. DISTORTED PERIODIC ELECTRIC AND MAGNETIC FIELDS: BASIC CONCEPTS

A periodic electromagnetic field is distorted if, simultaneously with the fundamental harmonic of the field, the highest harmonic components are present. In this way, if distorted vector field functions satisfy Dirichlet's conditions, then they can be developed into Fourier series, namely:

$$\mathbf{e}(t) = \sum_p \mathbf{e}_p(t), \ \mathbf{d}(t) = \sum_p \mathbf{d}_p(t), \ \mathbf{h}(t) = \sum_q \mathbf{h}_q(t), \ \mathbf{b}(t) = \sum_q \mathbf{b}_q(t) \tag{8}$$

where $\mathbf{e}_p, \mathbf{d}_p, \mathbf{h}_q, \mathbf{b}_q$, are harmonics of the field vectors. Each harmonic component in the frequency domain of a periodic electromagnetic field satisfies Maxwell's equations

$$\begin{aligned} \nabla \times \mathbf{H}_p &= \mathbf{J}_p + j\omega_p \mathbf{D}_p \\ \nabla \times \mathbf{E}_p &= -j\omega_p \mathbf{B}_p \\ \nabla \cdot \mathbf{D}_p &= \rho_p \\ \nabla \cdot \mathbf{B}_p &= 0 \end{aligned} \tag{9}$$

where $\mathbf{E}_p, \mathbf{H}_p, \mathbf{D}_p, \mathbf{B}_p$ are complex phasors of the p-th harmonic of the fields.

One of the most important consequences of the first two Maxwell equations is Poynting's theorem, which describes the flow of electromagnetic energy in space and for a volume v enclosed by a surface s. This can be stated as

$$\iint -(\mathbf{e} \times \mathbf{h})\mathbf{n}\, ds = \iiint \mathbf{e} \cdot \mathbf{j} dv + \iiint \left(\mathbf{h} \cdot \frac{\partial \mathbf{b}}{\partial t} + \mathbf{e} \cdot \frac{\partial \mathbf{d}}{\partial t} \right) dv \tag{10}$$

where $\mathbf{n}$ is the unit vector orthogonal to the infinitesimal surface ds, $\mathbf{e}$ and $\mathbf{h}$ are the instantaneous intensity of the electric and magnetic fields, $\mathbf{d}$ and $\mathbf{b}$ are the instantaneous flux densities of these fields respectively, and $\mathbf{j}$ is the instantaneous current density. The theorem simply means that the increase in stored energy in the fields plus the ohmic losses within a volume, equal the inflow of a vector $\mathbf{e} \times \mathbf{h}$ across the surface bounding that volume. The vector $\mathbf{e} \times \mathbf{h}$ is known as the Poynting Vector (PV), and gives the power density at a point on the surface in terms of the electric and magnetic fields at that point. Its physical meaning is also known [23].

The equivalent complex Poynting theorem for a system in linear media is given by

$$-\iint (\mathbf{E}_p \times \mathbf{H}_p)\mathbf{n}\, ds = \iiint \mathbf{E}_p \mathbf{J}_p^* dv + j\omega_p \iiint [\mathbf{B}_p \mathbf{H}_p^* - \mathbf{E}_p \mathbf{D}_p^*] dv \tag{11}$$

The energetic interpretation of (11) is as follows

$$\bar{S}_p = P_p + jQ_p \tag{12}$$

where

- $\bar{S}_p$ is a complex apparent power of the p-th harmonic received by the system enclosed in the surface "s".
- P_p is the active power of the p-th harmonic received by the system.
- Q_p is the reactive power of the p-th harmonic received by the system.

One can readily observe that

$$P_p = \iiint \mathbf{E}_p \mathbf{J}_p^* dv \tag{13}$$

$$Q_p = 2\omega_p \iiint \left[\frac{\mathbf{B}_p \mathbf{H}_p^*}{2} - \frac{\mathbf{E}_p \mathbf{D}_p^*}{2} \right] dv \tag{14}$$

where P_p represents harmonic losses in Joules and Q_p is associated to the average values of the p-th harmonic magnetic and electric energies accumulated in the volume v [15]. Another representation of (13) and (14) is given

$$P_p = \mathrm{Re}\left[- \oint (\mathbf{E}_p \times \mathbf{H}_p^*) \right] \mathbf{n}\, ds \tag{15}$$

$$Q_p = \mathrm{Im}\left[- \oint (\mathbf{E}_p \times \mathbf{H}_p^*) \right] \mathbf{n}\, ds \tag{16}$$

5. POWER FLOWS IN DISTORTED ELECTROMAGNETIC FIELDS: GENERALIZED POYNTING MULTIVECTOR ($\tilde{\mathcal{S}}$)

The following notation is adopted to define the electric and magnetic fields in the $\mathcal{CG}_n^t$-$\mathbb{R}^3$ framework:

$$\tilde{\mathbf{e}}_p = |\tilde{\mathbf{e}}_p| e^{j(\omega_p t + \theta_p)} \sigma_p \vec{\mathbf{1}}_{X,Y,Z}, \quad \tilde{\mathbf{h}}_q = |\tilde{\mathbf{h}}_q| e^{j(\omega_q t + \gamma_q)} \sigma_q \vec{\mathbf{1}}_{X,Y,Z} \tag{17}$$

where $\tilde{\mathbf{e}}_p(t)$ and $\tilde{\mathbf{h}}_q(t)$ are called *instantaneous electric and magnetic complex-geometric fields* respectively.

Observe that the classic instantaneous fields can be derived from the real (or imaginary) part of the projections given by the scalar product (2) as follows:

$$\mathbf{e}_p(t) = \mathrm{Im}(\tilde{\mathbf{e}}_p \cdot \sigma_p) = \mathrm{Im}\{|\tilde{\mathbf{e}}_p| e^{j(\omega_p t + \theta_p)} \vec{\mathbf{1}}_{X,Y,Z}\} \tag{18}$$

$$\mathbf{h}_q(t) = \mathrm{Im}(\tilde{\mathbf{h}}_q \cdot \sigma_q) = \mathrm{Im}\{|\tilde{\mathbf{h}}_q| e^{j(\omega_q t + \gamma_q)} \vec{\mathbf{1}}_{X,Y,Z}\} \tag{19}$$

In order to obtain the *geometric phasors*, it is necessary to apply the Φ operator on these quantities (see Section 3.2).

$$\Phi(\tilde{\mathbf{e}}_p) = |\tilde{\mathbf{e}}_p| e^{j\theta_p} \sigma_p \vec{\mathbf{1}}_{X,Y,Z} = \tilde{\mathbf{E}}_p \tag{20}$$

$$\Phi(\tilde{\mathbf{h}}_q) = |\tilde{\mathbf{h}}_q| e^{j\gamma_q} \sigma_q \vec{\mathbf{1}}_{X,Y,Z} = \tilde{\mathbf{H}}_q \tag{21}$$

where $\tilde{\mathbf{E}}_p$ and $\tilde{\mathbf{H}}_q$ are called harmonic "*spatial geometric phasors*" of the electric and magnetic harmonic fields respectively and verify

Maxwell's equations [20]. In this framework, a *Generalized Poynting Multivector* ($\tilde{\mathcal{S}}$) is defined as

$$\tilde{\mathcal{S}} = \sum_{p,q} \left(\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_q^* \right) \tag{22}$$

where $\tilde{\mathbf{H}}_q^*$ is the conjugate of the *q-th harmonic spatial geometric phasor* $\tilde{\mathbf{H}}_q$ and the symbol "$\odot$" is the generalized complex geometric product (B7). Consequently, the *Generalized Poynting Multivector* ($\tilde{\mathcal{S}}$) for a volume v enclosed by a surface s is given by

$$\begin{aligned} \iint_s \mathbf{n} \cdot \tilde{\mathcal{S}}\, ds &= \iint_s \sum_p \mathbf{n} \cdot \tilde{\mathcal{P}}\, ds + \iint_s \sum_{p \neq q} \mathbf{n} \cdot \tilde{\mathcal{D}}\, ds \\ &= \iint_s \sum_p \mathbf{n} \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_p^*)\, ds + \iint_s \sum_{p \neq q} \mathbf{n} \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_q^* + \tilde{\mathbf{E}}_q \odot \tilde{\mathbf{H}}_p^*)\, ds \end{aligned} \tag{23}$$

where the unitary vector $\mathbf{n}$ is the unitary vector orthogonal to the infinitesimal surface ds, $\tilde{\mathcal{P}}$ is the *Poynting Multivector* and $\tilde{\mathcal{D}}$ is the *Complementary Poynting Multivector*. Equation (23) expands the flux of the $\tilde{\mathcal{S}}$ into two terms

- The first term $\iint_s \sum_p \mathbf{n} \cdot \tilde{\mathcal{P}}\, ds = \iint_s \sum_p \mathbf{n} \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_p^*)\, ds$ represents the power contribution due to like-frequency products.
- The second term $\iint_s \sum_{p \neq q} \mathbf{n} \cdot \tilde{\mathcal{D}}\, ds = \iint_s \sum_{p \neq q} \mathbf{n} \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_q^* + \tilde{\mathbf{E}}_q \odot \tilde{\mathbf{H}}_p^*)\, ds$ represents the power contribution due to cross-frequency products.

6. POWER MULTIVECTOR APPROACH

6.1. Multivector Representation of Power in Terms of Generalized Poynting Multivector

The considerations stated in the above section have been derived from an electric field $\mathbf{e}(t)$ and a magnetic field $\mathbf{h}(t)$. To understand energy balance, one must touch base with electromagnetic field theory and reformulate the classic Poynting Vector concept. In this way, we consider an elementary single-phase transmission line (two conductors), where an n-sinusoidal voltage source

$$u(t) = \sqrt{2}\,\mathrm{Im} \sum_p U_p e^{j(\omega_p t + \alpha_p)} \tag{24}$$

is connected at the sending end in order to supply either a linear or non-linear load. The same conductors carry a current responsible for the generation of the magnetic field, given by

$$i(t) = \sqrt{2}\,\mathrm{Im}\sum_q I_q e^{j(\omega_q t+\beta_q)} \tag{25}$$

where $\beta_q = \alpha_q - \varphi_q$ for linear operation, φ_q is the harmonic impedance phase angle and U_p, I_q represent rms values of $u_p(t)$ and $i_q(t)$ respectively.

The energy balance can thus be expressed as a multivector $\tilde{S}$ in $\{\mathcal{CG}_n\}$, generated by "$\odot$" of the voltage and conjugate current geometric phasors (B2) given by the following set

$$\tilde{S} = \tilde{U} \odot \tilde{I}^* = \left\{ \underbrace{\tilde{U} \cdot \tilde{I}^*}_{\tilde{\Omega}^{\cdot}} \oplus \underbrace{\tilde{U} \wedge \tilde{I}^*}_{\tilde{\Omega}^{\wedge}} \right\} \tag{26}$$

This quantity consists of complex scalar ($\tilde{\Omega}^{\cdot}$) and complex bivector ($\tilde{\Omega}^{\wedge}$) parts. Note from (26) that $\tilde{\Omega}^{\cdot} = \sum\limits_{p\in N} U_p I_p e^{j\alpha_p}\sigma_0$. Clearly, $\|\tilde{P}\| = \|\mathrm{Re}\{\tilde{\Omega}^{\cdot}\}\| = \|\sum\limits_{p\in N} U_p I_p \cos\varphi_p \sigma_0\|$ is the active power or average value of the instantaneous power in the time domain. $\|\tilde{Q}\| = \|\mathrm{Im}\{\tilde{\Omega}^{\cdot}\}\| = \|\sum\limits_{p\in N} U_p I_p \sin\varphi_p \sigma_0\|$ is called reactive power and is merely the geometric complement of the active component. The complex bivector, deduced from (B7), is given by

$$\tilde{\Omega}^{\wedge} = \sum_{\substack{p\neq q \\ Linear}} \left\{ (U_p I_q e^{j\varphi_q} - U_q I_p e^{j\varphi_p}) e^{j(\alpha_p-\alpha_q)} \right\} \sigma_{pq} + \sum_{\substack{p\neq q \\ Non\ Linear}} U_p I_q e^{j(\alpha_p-\beta_q)} \sigma_{pq} \tag{27}$$

and it is associated to distortion power.

The components $\mathrm{Im}\{\tilde{\Omega}^{\cdot}\}$ and $\tilde{\Omega}^{\wedge}$ have a non-independent physical nature: they constitute non-active power. Note that, consistent with (D1), the squared value $\|\tilde{S}\|^2$ in (26), may be represented as $\|\tilde{S}\|^2 = \|\tilde{U} \odot \tilde{I}^*\|^2 = |\tilde{U}|^2 |\tilde{I}|^2$ and $\|\tilde{S}\|^2 = \|\{\tilde{\Omega}^{\cdot}\}\|^2 + \|\{\tilde{\Omega}^{\wedge}\}\|^2$. This expression is identical to any classic squared value of the apparent power.

From electromagnetic theory, (26) can be verified to have not only a formal meaning, but also a physical meaning. For clarity of presentation and without loss of generality, it is possible to consider that the electric field $\mathbf{e}_p(t)$ vector is parallel with the X axis and associate it with $u_p(t)$. Similarly, the magnetic field $\mathbf{h}_q(t)$ vector is parallel with the Y axis and associated to the harmonic current $i_q(t)$.

Since the harmonic electric and magnetic fields are supposed to vary in time and space, these quantities (18), (19) are given by

$$\begin{aligned}\mathbf{e}_p(t) &= \mathrm{Im}(\tilde{\mathbf{e}}_p \cdot \sigma_p) = \mathrm{Im}\left\{|\tilde{\mathbf{e}}_p| e^{j(\omega_p t+\alpha_p)} \vec{\mathbf{1}}_X\right\} \\ &= |\tilde{\mathbf{e}}_p| \sin(\omega_p t + \alpha_p)\vec{\mathbf{1}}_X \Rightarrow \mathbf{e}(t) = \left[\sum_p \mathbf{e}_p(t)\right] \vec{\mathbf{1}}_X \qquad (28)\\ \mathbf{h}_q(t) &= \mathrm{Im}(\tilde{\mathbf{h}}_q \cdot \sigma_q) = \mathrm{Im}\left\{|\tilde{\mathbf{h}}_q| e^{j(\omega_q t+\beta_q)} \vec{\mathbf{1}}_Y\right\} \\ &= |\tilde{\mathbf{h}}_q| \sin(\omega_q t + \beta_q)\vec{\mathbf{1}}_Y \Rightarrow \mathbf{h}(t) = \left[\sum_p \mathbf{h}_p(t)\right] \vec{\mathbf{1}}_Y \qquad (29)\end{aligned}$$

By considering the conservation law of the electrical charge and magnetic flux, $i_q(t) = -\frac{dq_q}{dt} = h_q(t) \cdot l_H$ and $u_p(t) = -\frac{d\phi_p}{dt} = e_p(t) \cdot l_E$, then the harmonic *vector phasors* $\tilde{\mathbf{e}}_p(t)$ and $\tilde{\mathbf{h}}_q^*(t)$ of (28) and (29) can be expressed as

$$\tilde{\mathbf{e}}_p(t) = \frac{\sqrt{2}}{l_E}\left[U_p e^{j(\omega_p t+\alpha_p)}\sigma_p\right]\vec{\mathbf{1}}_X \qquad (30)$$

$$\tilde{\mathbf{h}}_q^*(t) = \frac{\sqrt{2}}{l_H}\left[I_q e^{-j(\omega_q t+\beta_q)}\sigma_q\right]\vec{\mathbf{1}}_Y \qquad (31)$$

The corresponding harmonic *spatial geometric phasors* are therefore given by

$$\tilde{\mathbf{E}}_p = \left(\frac{l}{l_E} U_p e^{j\alpha_p}\sigma_p\right)\vec{\mathbf{1}}_X \qquad (32)$$

$$\tilde{\mathbf{H}}_q^* = \left(\frac{l}{l_H} I_q e^{-j\beta_q}\sigma_q\right)\vec{\mathbf{1}}_Y \qquad (33)$$

where l_E and l_H are average lengths of flux lines of the vector fields.

Thus, returning to the *Generalized Poynting Multivector* ($\tilde{\mathcal{S}}$) concept (23), when $\mathbf{n} = \vec{\mathbf{1}}_Z$, this equation becomes

$$\begin{aligned}\iint_s \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{S}}\, ds &= \iint_s \sum_p \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{P}}\, ds + \iint_s \sum_{p\neq q} \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{D}}\, ds \\ &= \iint_s \sum_p \vec{\mathbf{1}}_Z \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_p^*)\, ds + \iint_s \sum_{p\neq q} \vec{\mathbf{1}}_Z \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_q^* + \tilde{\mathbf{E}}_q \odot \tilde{\mathbf{H}}_p^*)\, ds \qquad (34)\end{aligned}$$

and by combining (32), (33), and (34), two cases can be identified:

- First case: If $p = q$, then (34) can be written as

$$\iint_s \sum_p \vec{\mathbf{1}}_Z \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_p^*)\, ds = \iint_S \vec{\mathbf{1}}_Z \cdot \left(\frac{1}{l_E l_H} \sum_p U_p I_p e^{j\phi_p} \sigma_0 \right) ds$$
$$= \sum_P U_p I_p e^{j\phi_p} \sigma_0 = \tilde{\Omega}^{\cdot} \tag{35}$$

Hence, by virtue of (35), one obtains

$$\iint_s \sum_p \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{P}}\, ds = \iint_s \sum_p \vec{\mathbf{1}}_Z \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_p^*)\, ds = \sum_P U_p I_p e^{j\phi_p} \sigma_0 \tag{36}$$

The multivector $\tilde{\mathcal{P}} = \sum_p \tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_p^*$, (*Poynting Multivector*), is associated to the power density at a point on the surface, in terms of the harmonic spatial geometric phasor of electric and magnetic fields at that point. The real part of (36) permits a direct interpretation in terms of average power flow, (i.e., *active power*), P. Consequently, a net energy flow occurs in any linear or non-linear system when voltage and current components of the same frequency exist.

- Second case: If $p \neq q$ then (34) yields

$$\iint_S \sum_{p\neq q} \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{D}}\, ds = \iint_S \sum_{p\neq q} \vec{\mathbf{1}}_Z \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_q^* + \tilde{\mathbf{E}}_q \odot \tilde{\mathbf{H}}_p^*)\, ds = \tilde{\Omega}^{\wedge} \tag{37}$$

where $\tilde{\mathcal{D}} = \sum_p (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_q^* + \tilde{\mathbf{E}}_q \odot \tilde{\mathbf{H}}_p^*)$ is the *Complementary Poynting Multivector*. Combining (32), (33) with (37) yields

$$\begin{aligned} \tilde{\Omega}^{\wedge} = & \sum_{\substack{p\neq q \\ Linear}} \left\{ \left(U_p I_q e^{j\varphi_q} - U_q I_p e^{j\varphi_p} \right) e^{j(\alpha_p - \alpha_q)} \right\} \sigma_{pq} \\ & + \sum_{p\neq q} U_p I_q e^{j(\alpha_p - \beta_q)} \sigma_{pq} \end{aligned} \tag{38}$$

Equation (38) represents the power contribution due to the cross-frequency products. In this way, the power multivector that originates from the surface of the source equals the power that enters in the load surface. Finally, through (36) and (37), it can be observed that

$$\iint_S \sum_p \vec{\mathbf{1}}_Z \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_p^*) ds + \iint_S \sum_{p\neq q} \vec{\mathbf{1}}_Z \cdot (\tilde{\mathbf{E}}_p \odot \tilde{\mathbf{H}}_q^* + \tilde{\mathbf{E}}_q \odot \tilde{\mathbf{H}}_p^*) ds$$
$$= \tilde{\Omega}^{\cdot} \oplus \tilde{\Omega}^{\wedge} \tag{39}$$

Equation (39) coincides with (26) and represents to *apparent power multivector* given by

$$\tilde{S} = \tilde{\Omega}^{\cdot} \oplus \tilde{\Omega}^{\wedge} \tag{40}$$

It is important to notice that in our framework, the $\mathrm{Re}\{\tilde{\Omega}^{\cdot}\}$ and $\mathrm{Im}\{\tilde{\Omega}^{\cdot}\}$ components are associated to a scalar plane perpendicular to the Z axis and each $\tilde{\Omega}^{\wedge}_{pq}$ is associated to a pq complex bivector plane pertaining to the set of planes that contains the Z axis. Under these conditions, is obvious that the average value of the second integral on the right-hand side of (39) is zero. However, this component must be considered in order to understand the complete physical meaning of the power equation. Consequently, this new analysis demonstrates that (39) not only justifies the net flow of energy, but also provides powerful information for the analysis of power theory. In short, the complex scalar in (35) and (36), is similar to the complex power equation in sinusoidal operation. However, a new quantity, given by (38), is identified with the *Complementary Poynting Multivector* ($\tilde{\mathcal{D}}$).

6.2. Relative Quality Index and Power Factor

Regarding the power factor improvement, the suggested representation in (34) can be particularly useful. Thus, the multivectorial *relative quality index* (*RQI*) [18] expressed in terms of $\tilde{\mathcal{S}}$, $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{D}}$ is reduced to

$$\begin{aligned}\tilde{\delta} &= \frac{\iint\limits_{s} \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{S}}\, ds}{\mathrm{Re}\left\{\iint\limits_{s} \sum\limits_{p} \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{P}}\, ds\right\}} \\ &= 1 + j\frac{\mathrm{Im}\left\{\iint\limits_{s} \sum\limits_{p} \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{P}}\, ds\right\}}{\mathrm{Re}\left\{\iint\limits_{s} \sum\limits_{p} \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{P}}\, ds\right\}} + \frac{\iint\limits_{s} \sum\limits_{p\neq q} \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{D}}\, ds}{\mathrm{Re}\left\{\iint\limits_{s} \sum\limits_{p} \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{P}}\, ds\right\}}\end{aligned} \tag{41}$$

and the *power factor* (PF) can be written as

$$\mathrm{PF} = \frac{1}{\|\tilde{\delta}\|} = \frac{\left\|\mathrm{Re}\{\iint\limits_{s} \sum\limits_{p} \vec{\mathbf{1}_Z} \cdot \tilde{\mathcal{P}}\, ds\}\right\|}{\left\|\iint\limits_{s} \vec{\mathbf{1}_Z} \cdot \tilde{\mathcal{S}}\, ds\right\|} \tag{42}$$

Equation (41) shows that on this index all its electromagnetic quantities, with their direction and sense, are accessible for possible control of the power factor improvement.

7. NUMERICAL EXAMPLE

In this section, a numerical example is developed. Units of physical quantities are the standard units of the MKSA system and thus are omitted.

In order to validate this approach, an elementary circuit, Fig. 2, constituted by a non-linear load supplied from a simple transmission line [12], is analyzed in the $\mathcal{CG}_n$-$\mathbb{R}^3$ framework.

Two parallel plane conductors in linear media are considered. Both conductors, of thickness λ and width γ, are separated by a dielectric material of thickness ρ. We suppose that $\gamma \gg \lambda, \rho$. Consider that a non-sinusoidal voltage $u(t) = \sqrt{2}(200\sin\omega_1 t + 100\sin\omega_2 t)$, applied at the sending end, supplies a non-linear load. The resulting current has hypothetic instantaneous value given by $i(t) = \sqrt{2}[10\sin(\omega_1 t - 30°) + 5\sin(\omega_2 t + 45°) + 10\sin(\omega_3 t + 60°)]$.

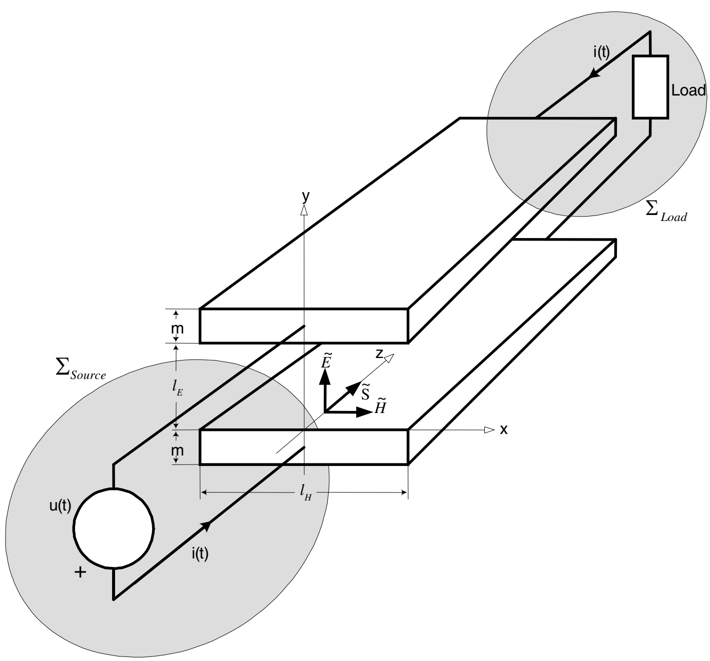

Figure 2. Elementary circuit.

By ignoring eddy currents, line impedance, fringing effects, then (32) and (33) can be expressed as

$$\tilde{\mathbf{E}} = \frac{1}{l_E}(200e^{j0}\sigma_1 + 100e^{j0}\sigma_2)\vec{\mathbf{1}}_X \tag{43}$$

$$\tilde{\mathbf{H}}^* = \frac{1}{l_H}(10e^{j30}\sigma_1 + 5e^{-j45}\sigma_2 + 10e^{-j60}\sigma_3)\vec{\mathbf{1}}_Y \tag{44}$$

However, from (35) and (36), it follows that

$$\begin{aligned}\iint_s \sum_p \vec{\mathbf{1}}_Z \cdot \tilde{\mathcal{P}}\, ds &= [(1732 + 353.5) + j(1000 - 353.5)]\sigma_0\vec{\mathbf{1}}_Z \\ &= (2085.5 + j646.5)\sigma_0\vec{\mathbf{1}}_Z \end{aligned} \tag{45}$$

Therefore, from (38),

$$\iint_S \sum_{p\neq q} \tilde{\mathbf{1}}_{\mathbf{Z}} \cdot \tilde{\mathcal{D}}\, ds = (-158.9 - j1207.1)\sigma_{12} + (1000 - j1732)\sigma_{13} + (500 - j866)\sigma_{23} \tag{46}$$

This example states that $\mathrm{Re}\{\tilde{\Omega}^{\cdot}_1\} = 1732\sigma_0$, $\mathrm{Re}\{\tilde{\Omega}^{\cdot}_2\} = 353.5\sigma_0$, $\mathrm{Im}\{\tilde{\Omega}^{\cdot}_1\} = j1000\sigma_0$, $\mathrm{Im}\{\tilde{\Omega}^{\cdot}_2\} = -j353.5\sigma_0$ and that the linear complex bivector component becomes $\tilde{\Omega}^{\wedge}_{12} = (-158.9 - j1207.1)\sigma_{12}$, as well as the nonlinear complex bivector components $\tilde{\Omega}^{\wedge}_{13} = (1000 - j1732)\sigma_{13}$, $\tilde{\Omega}^{\wedge}_{23} = (500 - j866)\sigma_{23}$ with their corresponding directions and senses. On the other hand, the rms values of voltage and current are given by $\|\tilde{U}\|^2 = 200^2 + 100^2 = 5 \cdot 10^4$ and $\|\tilde{I}\|^2 = 10^2 + 5^2 + 10^2 = 225$ respectively. The values of $P = \|\mathrm{Re}\{\tilde{\Omega}^{\cdot}\}\|^2$, $\|\mathrm{Im}\{\tilde{\Omega}^{\cdot}\}\|^2$, $\|\tilde{\Omega^{\wedge}}\|^2$ are found to add up to

$$\|\tilde{S}\|^2 = P^2 + \|\mathrm{Im}\{\tilde{\Omega}^{\cdot}\}\|^2 + \|\tilde{\Omega^{\wedge}}\|^2 = 11.25 \cdot 10^6 \tag{47}$$

Therefore, apparent volt-amperes $\|\tilde{S}\|$ at the terminals are found from the relation $\|\tilde{S}\|^2 = \|\tilde{U}\|^2\|\tilde{I}\|^2 = 11.25 \cdot 10^6$. Finally, from (41) and (42) we obtain the relative quality index and power factor respectively

$$\begin{aligned}\tilde{\delta} &= 1 + \frac{j646.5\sigma_0}{2085.5\sigma_0} \\ &\quad + \frac{(-158.9 - j1207.1)\,\sigma_{12} + (1000 - j1732)\,\sigma_{13} + (500 - j866)\,\sigma_{23}}{2085.5\sigma_0} \\ \left\|\tilde{\delta}\right\| &= 1.608 \end{aligned}$$

$$PF = \frac{1}{\left\|\tilde{\delta}\right\|} = 0.62 \tag{48}$$

The methodology in the above example differs greatly to that of circuit theory. Furthermore, unlike the circuit theory approach, it can be applied to solve and understand the operation of electric systems designed to work in the frequency domain.

8. CONCLUSION

The suggestion that the power equation should be founded on electromagnetic theory is analyzed in this paper. This goal remains unexplained by simple mathematical models used in classical theory. To this end, we propose a *Generalized Poynting Multivector* ($\tilde{\mathcal{S}}$) based on Clifford Algebras, which is decomposed into a *Poynting Multivector* ($\tilde{\mathcal{P}}$) and a *Complementary Poynting Multivector* ($\tilde{\mathcal{D}}$). From Equations (34)–(40), both quantities are considered as the keystone of the bridge between electromagnetic theory and circuit theory. Thus, the real part of the flow of *Poynting Multivector* ($\tilde{\mathcal{P}}$) coincide with active power, and imaginary part of the complex scalar coincides with the power contribution due to like-frequency products. The *Complementary Poynting Multivector* ($\tilde{\mathcal{D}}$) is associated to the complex bivector or to the power contribution due to cross-frequency products. This analysis demonstrates that the power equation can be founded on the multivectorial concept of the *Generalized Poynting Multivector* ($\tilde{\mathcal{S}}$). Consequently, the *apparent, active, and non-active powers* can be expressed and differentiated in terms of $\tilde{\mathcal{S}}$. The application of the proposed *Generalized Poynting Multivector* ($\tilde{\mathcal{S}}$) to power theory should indicate important advances for any real future research in this area.

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APPENDIX A. GENERALIZED COMPLEX GEOMETRIC PRODUCT IN $\mathcal{CG}_n$

We define as $\mathcal{C}$ the complex-vector space, and $\mathcal{G}_n$, the Clifford algebra on n-dimensional real space $\mathcal{V}^n$. We define the set

$$\mathcal{CG}_n^{=} \left\{ \sum_{k=1,2\ldots n} \bar{Z}_{1\ldots k}\sigma_{1\ldots k} \right\} \tag{A1}$$

where the coefficients $\bar{Z}_{1\ldots k} \in \mathcal{C}$ and the basis $\sigma_{1\ldots k} \in \mathcal{G}_n$. Obviously $\mathcal{CG}_n$ is a vector space over $\mathbb{R}$. According to (A1) definition, in the complex-vector case, we obtain the vector subspace $[\mathcal{CG}_n]_1 = \sum_{p=1}^{n} \bar{Z}_p\sigma_p$, where $\bar{Z}_p \in \mathcal{C}$ and $\sigma_p \in \mathcal{G}_n$. The generic element $\bar{Z}_p\sigma_p$, is a p-th complex-vector, and can be represented by the *geometric phasor* $\tilde{Z}_p = (a_p + jb_p)\,\sigma_p$. In the complex-bivector case, we obtain the vector subspace $[\mathcal{CG}_n]_2 = \sum_{p \neq q} \bar{Z}_{pq}\sigma_{pq}$. The generic element $\bar{Z}_{pq}\sigma_{pq}$, is a pq-th complex-bivector, and can be represented by $\tilde{Z}_{pq} = (a_{pq} + jb_{pq})\,\sigma_{pq}$. In the most general form, complex-multivectors, we obtain the vector subspace $[\mathcal{CG}_n]_k = \sum \bar{Z}_{12\ldots k}\sigma_{12\ldots k}$. The element $\bar{Z}_{12\ldots k}\sigma_{12\ldots k}$, is the $12\ldots k$-th complex-multivector, and may be represented by $\tilde{Z}_{12\ldots k} = (a_{12\ldots k} + j\,b_{12\ldots k})\,\sigma_{12\ldots k}$. Therefore, $\mathcal{CG}_n$ (A1), also can be represented as

$$\mathcal{CG}_n = \underbrace{\mathcal{C}}_{\substack{complex\\ scalar}} \oplus \underbrace{[\mathcal{CG}_n]_1}_{\substack{complex\\ vectors}} \oplus \underbrace{[\mathcal{CG}_n]_2}_{\substack{complex\\ bivectors}} \oplus \cdots \oplus \underbrace{[\mathcal{CG}_n]_n}_{\substack{complex\\ pseudoscalar}}$$

The structure $\{\mathcal{CG}_n, \odot\}$ is a complex geometric algebra since the following properties are fulfilled: associative, distributive with respect to the sum and contraction.

APPENDIX B. PARTICULAR CASE: GENERALIZED COMPLEX GEOMETRIC PRODUCT FOR COMPLEX VECTORS (GEOMETRIC PHASORS)

Let $\{\sigma_1, \ldots, \sigma_n\}$ be a vector basis of $\mathcal{CG}_n$. For two vectors $\tilde{Z}_p = \bar{Z}_p\sigma_p\,(p \in \Omega)$ and $\tilde{Z}'_q = \bar{Z}'_q\sigma_q\,(q \in \Psi)$ where $\Omega, \Psi \subseteq \{1, 2, \ldots, n\}$, and where complex numbers associated to each vector are

$$\begin{aligned} \bar{Z}_p &= Z_p e^{j\alpha_p} \\ \bar{Z}'_q &= Z'_q e^{j\beta_q} = Z'_q e^{j(\alpha_q - \varphi_q)} \end{aligned} \tag{B1}$$

we define a new geometric product termed *"generalized complex geometric product"*, $\odot$:

$$\odot\colon \left(\Re_{\alpha_p,\alpha_q}, \otimes\right) \tag{B2}$$

The symbol "$\otimes$" represents the classic geometric product [21] and $\Re_{\alpha_p,\alpha_q}$ is an application in the complex planes associated to any multivector product when $\alpha_p \neq \alpha_q$, and is given by

$$\Re_{\alpha_p,\alpha_q}\left(\bar{Z}'_p, \bar{Z}'_q\right) = \begin{cases} e^{-2j\left(\alpha_q - \alpha_p\right)} \text{ if } p > q, & p, q \in N \\ 1 \text{ otherwise}, & p \text{ and/or } q \notin N \end{cases} \tag{B3}$$

where $N = \Omega \cap \Psi$.

This new product for vectors $\tilde{Z}_p$ and $\tilde{Z}'_q$ is given by

$$\bar{Z}_p\sigma_p \odot \bar{Z}'_q\sigma_q = \bar{Z}_p\bar{Z}'_q\sigma_{pq} \tag{B4}$$

and the basis transposition states

$$\left(\bar{Z}'_q\bar{Z}_p\sigma_{qp}\right) = (-1)\Re_{\alpha_p,\alpha_q}\bar{Z}_p\bar{Z}'_q\sigma_{pq} \tag{B5}$$

Note that the transposition operation is involutive.

If $\alpha_p = \alpha_q \forall p$, $q \in N$, then

$$\Re_{\alpha_p,\alpha_p} = Id_C \tag{B6}$$

and "$\odot$", (B2), will then become the classic geometric product "$\otimes$". It should be noted that when $\mathcal{C}$ is restricted to real numbers, the classic Clifford Algebra is obtained.

In particular, for two complex vectors

$$\tilde{Z} = \sum_p Z_p e^{j\alpha_p}\sigma_p \quad \text{and} \quad \tilde{Z}' = \sum_q Z'_q e^{j(-\alpha_q+\varphi_q)}\sigma_q,$$

where the angles α_p and $(-\alpha_q + \varphi_q)$ identify the phase of the p-th and q-th harmonics respectively, the generalized complex geometric product in linear operation $(p, q \in N)$, can be written

$$\begin{aligned}\tilde{Z} \odot \tilde{Z}' =& \sum_p Z_p Z'_p e^{j\varphi_p} + \sum_{p<q} e^{j(\alpha_p-\alpha_q)} Z_p Z'_q e^{j\varphi_q}\sigma_{pq} \\ &+\sum_{q<p} e^{j(\alpha_q-\alpha_p)} Z_q Z'_p e^{j\varphi_p}\sigma_{qp} = \sum_p Z_p Z'_p e^{j\varphi_p} \\ &+\sum_{p<q}\left\{e^{j(\alpha_p-\alpha_q)} Z_p Z'_q e^{j\varphi_q} - \Re_{\alpha_p,\alpha_q} e^{j(\alpha_q-\alpha_p)} Z_q Z'_p e^{j\varphi_p}\right\}\sigma_{pq}\end{aligned} \tag{B7}$$

where

$$\Re_{\alpha_p,\alpha_q} e^{j(\alpha_q-\alpha_p)} Z_q Z'_p e^{j\phi_p}\sigma_{q\,p} = e^{j(\alpha_p-\alpha_q)} Z_q Z'_p e^{j\phi_p}\sigma_{qp}$$

APPENDIX C. REVERSE AND CONJUGATED OPERATIONS

We define the bivector *reverse* element as

$$\left(\bar{Z}_{q\,p}\sigma_{q\,p}\right)^\dagger = (-1)\bar{Z}_{pq}\sigma_{pq} \tag{C1}$$

where $(\dagger)$ is the "*reverse*" operation.

The "*conjugated*" operation $(*)$ is given by

$$\left(\bar{Z}_p\sigma_p\right)^* = \bar{Z}^*_p\sigma_p \tag{C2}$$

APPENDIX D. NORM DEFINITION

The *norm, value or magnitude*, of a multivector $\tilde{Z}$ is the unique scalar $\left\|\tilde{Z}\right\|$, Z calculated by

$$\left\|\tilde{Z}\right\|^2 = \langle \tilde{Z}(\tilde{Z}^{\dagger})^{*}\rangle_0 \tag{D1}$$

where we apply $(*)$ in C, and $(\dagger)$ in $\mathcal{G}_n$.

APPENDIX E. TIME-DOMAIN FREQUENCY-DOMAIN TRANSFORM: Γ-TRANSFORM

Let $f_k : R \to \mathcal{CG}_n^t,\ f_k(t) = X_k e^{j(\omega_k t+\theta_k)}\sigma_k$ be a continuous signal. The Γ-transform of f_k is given by

$$\Gamma\left\{f_k(t)\right\}(\omega) = \frac{1}{T}\int_T f_k(t)e^{-j\omega_k t}dt = X_k e^{j\theta_k}\sigma_k \tag{E1}$$

where $j^2 = -1$.

Let $\tilde{f} : \mathbb{R} \to \mathcal{CG}_n^t$ be a real-valued multivector function. Therefore $\tilde{f}(t) = \sum\limits_{A\in P(\{1,\dots,n\})\cup 0} f_A(t)$ with $f_A(t) = k_A e^{j(\omega_A t+\theta_A)}\sigma_A$, where $P(\{1,\dots,n\})$ is the set of all the subsets of $\{1,\dots,n\}$.

According to the linearity of the Γ-transform:

$$\Gamma\left\{\tilde{f}(t)\right\}(\omega) = \sum_{A\in P(\{1,\dots,n\})\cup 0} \Gamma\left\{f_A(t)\right\} \tag{E2}$$

and

$$\Gamma\left\{\tilde{f}(t)\right\}(\omega) = \sum_{A\in P(\{1,\dots,n\})\cup 0} k_A e^{j\theta_A}\sigma_A = \tilde{F}(\omega) \tag{E3}$$

where $\tilde{F}(\omega)$ is a *geometric phasor*.

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