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OSCILLATIONS IN SLOTTED RESONATORS WITH SEVERAL SLOTS: APPLICATION OF APPROXIMATE SEMI-INVERSION

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Abstract—We consider oscillations in cylindrical slotted resonators formed by combinations of rectangular domains with several slots cut in the walls using the methods of approximate semi-inversion of integral operator-valued functions with a logarithmic singularity of the kernel. The initial boundary value problems for the Helmholtz equation are reduced to Fredholm integral equations and systems of integral equations of the first kind with a logarithmic singularity. In the case of narrow slots, the dispersion equations are obtained and evaluated using perturbations and the small-parameter method. Eigenfrequencies and eigenfields are calculated explicitly. The values of geometrical and material parameters are determined that lead to the interaction of oscillations. The results obtained are used for improving the design of filters and switches on the basis of simple model prototype structures.

1. INTRODUCTION

Cylindrical slotted resonators formed by combinations of rectangular domains with several slots cut in the walls may serve as an excellent prototype family that (i) exhibit a number of important features inherent to much more sophisticated structures and (ii) can be treated using well-developed rigorous mathematical methods. On the other hand, structures with inhomogeneities of very small relative dimensions cannot be efficiently analyzed using purely numerical approach because the characteristic size of the mesh becomes comparable with the diameter of inclusion(s); in addition, a numerical method undergoes a computational collapse in the vicinity of various critical points (CPs), in particular, eigenfrequencies of the cavities formed by partial domains and certain degeneration points (DPs), where the most interesting phenomena are often observed [1, 2] that lead to restructuring of fields, etc. Among such phenomena, we note a resonance increase in the amplitude of the diffracted fields reported in [3] and [4] for a family of open slotted resonators with noncomact boundaries. The field distributions may become unstable with respect to parameters of the structure (geometric, permittivity, etc.) and the intertype interaction of oscillations [1] takes place when a geometric parameter is varied in the vicinity of a DP, where one or several resonant frequencies (eigenfrequencies) of the 'partial' resonators coupled by the aperture(s) coincide [5, 9]. Such DPs can be determined independently for each partial resonator (partial cross-sectional domain) and there are usually a few of them in the chosen frequency range. The knowledge of such points (in the given frequency range) and the character of the corresponding field perturbations may be considered as a type of data which can be used for the design of various devices, like filters and switches. The first step here may be to accumulate and study a sufficient amount of such data for a sufficiently rich family of prototype structures.

In this paper, we consider cylindrical slotted cavities with narrow slots and calculate eigenfrequencies, aperture fields, and field distributions in the cross-sectional domains (also in the form of segments of asymptotic series in powers of a characteristic small parameter, the relative slotwidth). We obtain explicit functional multi-parameter dependences and evaluate them analytically to determine various critical and extrema points that indicate an unstable behavior with respect to a chosen parameter (e.g., the dimension of a partial domain).

We develop the theory of integral operator-valued functions (OVFs) [7] defined on several intervals of integration and the methods of analytical semi-inversion [8] and apply the techniques to a specific family of closed cylindrical slotted resonators in order to calculate the field distributions in various critical modes, to explain and describe mathematically various instability phenomena, and to create the models of some electromagnetic devices. Particularly, we extend the method of approximate semi-inversion to the case of meromorphic integral OVFs with a logarithmic singularity of the kernel defined on several intervals of integration. In fact, general formulas of approximate inversion can be used when spectral parameter λ is sufficiently distant from the singular points because, according to [7], there exists a characteristic number (CN) (i.e., the point, at which the invertibility of the OVF is violated) in a neighbourhood of a pole of the integral OVF with a logarithmic singularity of the kernel. Below, we will take into account the closeness to a simple first-order pole and modify the inversion formulas for this case.

All geometric parameters are taken as relative (dimensionless) values normalized to the free-space wavelength.

2. SEMI-INVERSION OF INTEGRAL OPERATORS

2.1. Semi-Inversion of Meromorphic Integral OVFs: One Interval of Integration

Consider the integral OVF with a logarithmic singularity of the kernel

$$\begin{aligned} K(\lambda)\varphi &= \alpha L\varphi + N(\lambda)\varphi \\ &\equiv \int_\Gamma \left[\alpha\frac{1}{\pi}\ln\frac{1}{|t_0-t|} + N(t_0,t,\lambda)\right]\varphi(t)dt, \quad t_0\in\Gamma, \end{aligned} \tag{1}$$

where $\Gamma = (a,b) = (d-w, d+w)$ and $N(t_0,t,\lambda)$ is once continuously differentiable in $\Gamma\times\Gamma$ and a meromorphic function of λ, so that, in the vicinity of a pole λ_ν of $N(t_0,t,\lambda)$ OVF $K(\lambda)$ can be represented in the form

$$K(\lambda)\varphi = K_\nu(\lambda)\varphi + \frac{m_\nu}{\lambda_\nu-\lambda}(\varphi,\varphi_\nu)\varphi_\nu, \tag{2}$$

where the inner product

$$(\varphi,\varphi_\nu) = \int_\Gamma \varphi(t)\varphi_\nu(t)dt, \tag{3}$$

m_ν is a constant, and φ_ν is a given differentiable function. Note that often φ_ν is a member of a family of orthogonal basis functions used in bilinear eigenfunction expansions of traces of the Green functions (e.g., $\varphi_\nu(t) = \cos(w\nu t/a)$).

OVFs $K(\lambda)$ and $K_\nu(\lambda)$ are Fredholm integral OVFs with a logarithmic singularity of the kernel. Therefore, according to the general theory of OVFs, they may have not more than a finite number of CNs in every ball $B_r = \{\lambda : |\lambda| < r\}$ in the complex λ-plane and are invertible at all (regular) points λ that differ from CNs. For every $r > 0$ there exists [7] a sufficient $w = w(r)$ such that the ball B_r contains only regular points λ of the OVF $K(\lambda)$ (and $K_\nu(\lambda)$).

Set

$$\beta = \left(\frac{1}{\pi}\ln\frac{1}{w}\right)^{-1} \tag{4}$$

and let β be a small parameter. Obtain approximate representations for the inverse operators $K(\lambda)^{-1}$ and $K_\nu(\lambda)^{-1}$ as segments of asymptotic series in powers of β using the method of approximate semi-inversion developed in [7].

We will proceed from the approximate representation

$$K(\lambda) = L_1(\lambda, \beta) + w^2 \ln w N(\lambda), \tag{5}$$

obtained in [7], where

$$\begin{aligned} L_1\varphi &\equiv \alpha L\varphi + g(\varphi, 1)1, \\ g = g(\lambda) &= \frac{\alpha}{\beta} + M_0(\lambda) = \text{const} \quad (\alpha \neq 0), \\ M_0(\lambda) &= N(d, d, \lambda), \end{aligned} \tag{6}$$

and $N(\lambda)$ is an integral OVF with the kernel $N(t_0, t, \lambda)$. Let us obtain the inverse OVF $L_1^{-1}(\lambda, \beta)$ explicitly. To this end, consider the integral equation of the first kind

$$L_1\varphi \equiv \alpha L\varphi + g(\varphi, 1)1 = f, \tag{7}$$

where the integral operator

$$L\varphi \equiv \frac{1}{\pi} \int_{-1}^{1} \ln \frac{1}{|t_0 - t|} \varphi(t) dt \tag{8}$$

and the inverse L^{-1} is defined according to [7]

$$L^{-1} f \equiv -\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - t_0^2} f'(t_0) dt_0}{(t_0 - t)\sqrt{1 - t^2}} + \frac{C}{\sqrt{1 - t^2}}, \tag{9}$$

with the constant

$$C = \frac{1}{\ln 2} \left[f(t_0) + \frac{1}{\pi^2} \int_{-1}^{1} \ln \frac{1}{|t_0 - t|} dt \int_{-1}^{1} \frac{\sqrt{1 - \tau^2} f'(\tau) d\tau}{\sqrt{1 - t^2}(\tau - t)} \right]. \tag{10}$$

Note that the homogeneous integral equation

$$L_1(\lambda)\varphi \equiv \alpha L\varphi + g(\lambda)(\varphi, 1)1 = 0 \tag{11}$$

where $g(\lambda)$ is a (given) analytical or meromorphic function of complex variable λ, constitutes the problem on CNs for the integral OVF $L_1(\lambda)$ with a logarithmic singularity of the kernel. Applying L^{-1} to both sides of the equation $L_1(\lambda)\varphi = f$ and using the formulas [7]

$$L^{-1}1 = p_0(t) = \frac{1}{\ln 2} \frac{1}{\sqrt{1 - t^2}}, \quad (L^{-1}1, 1) = \frac{\pi}{\ln 2}, \tag{12}$$

we obtain an equivalent equation of the second kind

$$\varphi = \frac{1}{\alpha}\left[L^{-1}f - g(\varphi,1)L^{-1}1\right]. \tag{13}$$

Multiplying both sides of (13) by the constant 1 and integrating over $(-1,1)$, we obtain a (linear) equation with respect to $(\varphi,1)$

$$[\alpha+\tilde{g}(\lambda)](\varphi,1) = (L^{-1}f,1), \quad \tilde{g}(\lambda) = \frac{\pi}{\ln 2}g(\lambda). \tag{14}$$

If $\alpha+\tilde{g}(\lambda)\neq 0$, then λ is a regular point of $L_1(\lambda)$ and equation (14) and hence (7) are uniquely solvable for arbitrary $f\in\widetilde{W}_2^1$, where $\widetilde{W}_2^1$ denotes the weighted Sobolev space defined in [5]. Dividing both sides of (14) by $\alpha+\tilde{g}(\lambda)$ and substituting $(\varphi,1)$ into (13) we obtain the expression for $L_1^{-1}f$:

$$\varphi = L_1^{-1}f = \frac{1}{\alpha}\left[L^{-1}f + B_0(L^{-1}f,1)\varphi_0\right], \quad B_0 = -\frac{\ln 2}{\pi}\frac{\tilde{g}}{\alpha+\tilde{g}}. \tag{15}$$

If $\alpha+\tilde{g}(\lambda)=0$ at $\lambda=\lambda^*$, then this λ^* is a CN of $L_1(\lambda)$; equations (14) and (7) are not solvable; the homogeneous integral equation (11) has a nontrivial solution (a characteristic element (eigenfunction) of $L_1(\lambda)$) $\varphi^*(t) = C_0p_0(t)$, where C_0 is an arbitrary constant. When performing the semi-inversion we will assume that λ is a regular point of $L_1(\lambda)$.

If β is sufficiently small, then B_0 can be expanded in powers of the small parameter β

$$B_0 = -1 + \frac{\ln 2}{\pi}\beta - \frac{\ln 2}{\pi}\beta^2\left(\frac{\ln 2}{\pi} + \frac{M_0(\lambda)}{\alpha}\right) + O(\beta^3), \tag{16}$$

which yields a similar expansion for L_1^{-1}

$$L_1^{-1}(\lambda,\beta)f = \frac{1}{\alpha}L^{-1}f - \frac{\ln 2}{\alpha\pi}\left(1-\beta\frac{\ln 2}{\pi} + \beta^2\frac{\ln 2}{\pi}\left(\frac{M_0(\lambda)}{\alpha} + \frac{\ln 2}{\pi}\right)\right) \cdot (L^{-1}f,1)L^{-1}1 + O(\beta^3). \tag{17}$$

Write the approximate representation [7] for $K_\nu(\lambda)$

$$K_\nu(\lambda) = L_1(\lambda,\beta) + w^2\ln w N_\nu(\lambda), \tag{18}$$

similar to (5). Here $N_\nu(\lambda)$ coincides with OVF $N(\lambda)$ in (1) minus the singularity at λ_ν. In the vicinity of λ_ν, the approximate representation [7] for the inverse $K_\nu^{-1}(\lambda)$ is valid

$$K_\nu^{-1}(\lambda)f = L_1^{-1}f - w^2\ln w L_1^{-1}N_\nu L_1^{-1}f + O(w^4\ln^2 w), \quad \text{as} \quad w\to 0. \tag{19}$$

2.2. Two Intervals of Integration: Approximate Semi-Inversion of the Principal Part

When the set of integration in (1) consists of two intervals,

$$\Gamma = \bigcup_{j=1}^{2} \Gamma_i, \quad \Gamma_i = (d_j - w_j, d_j + w_j), \quad j = 1, 2, \tag{20}$$

we introduce the vector-functions $\mathbf{f} = (\varphi_1, \varphi_2)^T$ and $\mathbf{F} = (f_1, f_2)^T$, where $\varphi_i(x) \equiv \varphi(x)$, $f_i(t) \equiv f(t)$, $x \in \Gamma_i$ and write the integral operator (1) in the matrix form

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \tag{21}$$

where

$$\begin{aligned} K_{ij}(\lambda)\varphi_j &= \alpha L_{ij}\varphi_j + N_{ij}(\lambda)\varphi_j \\ &\equiv \int_{d_j - w_j}^{d_j + w_j} \left[\frac{\alpha}{\pi} \ln \frac{1}{|x_0 - x|} + N(x_0, x, \lambda) \right] \varphi_j(x) dx. \end{aligned} \tag{22}$$

Using the change of variables

$$x = w_j t + d_j, \quad x_0 = w_i t_0 + d_i, \quad i, j = 1, 2, \tag{23}$$

we transform (22) to the equivalent form with the integration in each operator K_{ij} over $(-1, 1)$:

$$\begin{aligned} \widetilde{K}_{jj}(\lambda)\varphi_j = \int_{-1}^{1} \Bigg[& \frac{\alpha}{\pi} \ln \frac{1}{|t_0 - t|} + \frac{\alpha}{\pi} \ln \frac{1}{w_j} \\ & + N(w_i t_0 + d_i, w_j t + d_j, \lambda) \Bigg] \tilde{\varphi}_j(t) dt, \\ & t_0 \in (-1, 1), \quad j = 1, 2; \end{aligned} \tag{24}$$

$$\begin{aligned} \widetilde{K}_{ij}(\lambda)\varphi_j = \int_{-1}^{1} \Bigg[& \frac{\alpha}{\pi} \ln \frac{1}{|w_i t_0 - w_j t + (d_i - d_j)|} \\ & + N(w_i t_0 + d_i, w_j t + d_j, \lambda) \Bigg] \tilde{\varphi}_j(t) dt, \\ & t_0 \in \Gamma_i, \quad i \neq j, \quad i, j = 1, 2. \end{aligned} \tag{25}$$

Here $\tilde{\varphi}_j(t) = w_j \varphi_j(w_j t + d_j)$, $j = 1, 2$.

As follows from the estimate (1.128) in [7], when w_j are taken as small parameters, the diagonal operators $\widetilde{K}_{jj} = \widetilde{K}_{jj}(\lambda, w_j)$ can be represented in the form

$$\widetilde{K}_{jj}\tilde{\varphi} = \alpha L\tilde{\varphi} + \alpha\frac{1}{\pi}\ln\frac{1}{w_j}(\tilde{\varphi},1)1 + M_{jj}(\lambda)(\tilde{\varphi},1)1 + w_j^2\ln w_j l R_{jj}(\lambda, w_j)\tilde{\varphi}, \tag{26}$$

where $M_{jj} = M_{jj}(\lambda) = N(d_j, d_j, \lambda)$; for the nondiagonal operators we have

$$K_{ij}(\lambda)\tilde{\varphi} = M_{ij}(\tilde{\varphi},1)1 + w_j^2\ln w_j R_{ij}, \quad i \neq j, \quad i,j = 1,2, \tag{27}$$

where

$$M_{ij} = M_{ij}(\lambda) = \frac{\alpha}{\pi}\ln\frac{1}{|d_i - d_j|} + N(d_i, d_j, \lambda), \quad i \neq j, \quad i,j = 1,2, \tag{28}$$

and R_{ij} are integral OVFs holomorphic with respect to λ and uniformly bounded as $w_j \to 0$, $j = 1,2$.

Combining (26) and (27) we can separate the principal part L_P of the operator (1) in the matrix form

$$L_P(\lambda) = \begin{pmatrix} L_{1,1} & M_{12}(\cdot,1) \\ M_{21}(\cdot,1) & L_{2,1} \end{pmatrix}, \tag{29}$$

and write

$$L_P(\lambda)\mathbf{f} = \begin{pmatrix} L\varphi_1 + g_1(\varphi_1,1)1 + M_{12}(\varphi_2,1)1 \\ L\varphi_2 + g_2(\varphi_2,1)1 + M_{21}(\varphi_1,1)1 \end{pmatrix}. \tag{30}$$

where

$$g_j = g_j(\lambda, \beta_j) = \frac{\alpha}{\beta_j} + M_{jj,\nu}(\lambda), \quad j = 1,2. \tag{31}$$

In order to obtain an explicit representation for the inverse $L_P^{-1}(\lambda)$ we repeat componentwise the proof (7)–(15). Apply L^{-1} given by (9) and (10) to both sides of every line in integral equation $L_P(\lambda)\mathbf{f} = \mathbf{F}$ written in the form (30). As a result, we obtain an equivalent equation of the second kind. Using formulas (12), we write this equation componentwise

$$\begin{aligned} \varphi_1(t_0) &= \frac{1}{\alpha}\left\{L^{-1}f_1 - [g_1(\varphi_1,1) + M_{12}(\varphi_2,1)]p_0(t)\right\}, \\ \varphi_2(t_0) &= \frac{1}{\alpha}\left\{L^{-1}f_2 - [M_{21}(\varphi_1,1) + g_2(\varphi_2,1)]p_0(t)\right\}. \end{aligned} \tag{32}$$

Denoting

$$\tilde{g}_j = \frac{\pi}{\ln 2} g_j, \quad \widetilde{M}_{ij} = \frac{\pi}{\ln 2} M_{ij}, \quad i, j = 1, 2, \tag{33}$$

and multiplying both sides of (32) by the constant 1 and integrating over $(-1, 1)$, we obtain a linear equation system with respect to $(\varphi_1, 1)$ and $(\varphi_2, 1)$. The determinant of the system matrix

$$\frac{1}{\alpha} \begin{pmatrix} \alpha + \tilde{g}_1 & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \alpha + \tilde{g}_2 \end{pmatrix} \tag{34}$$

is

$$D_P(\lambda) = (\alpha + \tilde{g}_1)(\alpha + \tilde{g}_2) - \widetilde{M}_{12}\widetilde{M}_{21}. \tag{35}$$

Solving the system under the assumption $D_P(\lambda) \neq 0$ and substituting the solution to (32) we obtain the componentwise expression for $L_P^{-1}(\lambda)\mathbf{F}$ with $\mathbf{F} = (f_1, f_2)$:

$$\begin{aligned} \varphi_1(t_0) &\equiv \widetilde{L}_1^{-1}\mathbf{F} = \frac{1}{\alpha}\left\{L^{-1}f_1 + \left[B_{11}(L^{-1}f_1, 1) + B_{12}(L^{-1}f_2, 1)\right] p_0(t)\right\}, \\ \varphi_2(t_0) &\equiv \widetilde{L}_2^{-1}\mathbf{F} = \frac{1}{\alpha}\left\{L^{-1}f_2 + \left[B_{21}(L^{-1}f_1, 1) + B_{22}(L^{-1}f_2, 1)\right] p_0(t)\right\}, \end{aligned} \tag{36}$$

where the quantities $B_{ij} = B_{ij}(\lambda)$ are defined using the matrix

$$\begin{aligned} B &= \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ &= \frac{1}{D_P(\lambda)} \begin{pmatrix} M_{12}\widetilde{M}_{21} - g_1(\alpha + \tilde{g}_2) & -\alpha M_{12} \\ -\alpha M_{21} & M_{21}\widetilde{M}_{12} - g_2(\alpha + \tilde{g}_1) \end{pmatrix}. \end{aligned} \tag{37}$$

Introducing the diagonal matrix integral operator

$$\mathrm{L} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \tag{38}$$

we write (36) in the vector notation and obtain the definition of the inverse $L_P^{-1}(\lambda)$

$$\mathbf{f} = L_P^{-1}(\lambda)\mathbf{F} = \frac{1}{\alpha}\mathrm{L}^{-1}\mathbf{F} + \frac{1}{\alpha} p_0(t) B \langle \mathrm{L}^{-1}\mathbf{F}, 1 \rangle, \tag{39}$$

where the vectors

$$\mathrm{L}^{-1}\mathbf{F} = \begin{pmatrix} L^{-1}f_1 \\ L^{-1}f_2 \end{pmatrix}, \quad \langle \mathrm{L}^{-1}\mathbf{F}, 1\rangle = \begin{pmatrix} (L^{-1}f_1, 1) \\ (L^{-1}f_2, 1) \end{pmatrix}. \tag{40}$$

Write the integral OVF $K(\lambda)$ in (1) with the the principal part L_P separated according to (22)–(29)

$$K(\lambda) = L_P(\lambda) + qN_p(\lambda), \tag{41}$$

where the small parameter $q = \max_{i=1,2} w_i^2 |\ln w_i|$. Following [7], it is easy to verify that at any fixed (regular) point $\lambda = \lambda_0$ which does not coincide with a pole of OVF $K(\lambda)$, the inverse $K^{-1}(\lambda)$ exists and admits the representation in the form of the Neumann series:

$$K^{-1}(\lambda) = \sum_{n=0}^{\infty} (-1)^n q^n (L_P^{-1}N)^n L_P^{-1}, \tag{42}$$

which converges in the operator norm uniformly with respect to λ and q in the vicinity of λ_0 and in the interval $(0, q_0)$ for a certain $q_0 > 0$, respectively. Indeed, at any regular points λ

$$K^{-1}(\lambda) = (L_P + qN_P)^{-1} = (I + qL_P^{-1}N)^{-1}L_P^{-1}, \tag{43}$$

where $\|qL_P^{-1}N_P\| < 1$ for sufficiently small q because $\|L_P^{-1}N_P\|$ is uniformly bounded in the vicinity of λ_0 and for $q \in (0, q_0)$. In applications, it is often sufficient to use only the first two terms of expansion (42)

$$K^{-1}(\lambda)\mathbf{f} = L_P^{-1}\mathbf{f} - qL_P^{-1}NL_P^{-1}\mathbf{f} + O(q^2) \quad \text{as} \quad q \to 0. \tag{44}$$

2.3. Estimates for the Inner Product

For two-dimensional vector-functions $\mathbf{f} = (f_1, f_2)^T$ and $\mathbf{g} = (g_1, g_2)^T$ we define the inner product associated with the set of two integration intervals in (22), (24), and (25),

$$(\mathbf{f}, \mathbf{g}) = \int_{-1}^{1} \left[\sum_{j=1}^{2} f_j(t) g_j(t) \right] dt. \tag{45}$$

Using formulas (12) we can express the inner products $(\widetilde{L}_i^{-1}\mathbf{p}, 1)$, $i = 1, 2$, in (36) explicitly:

$$(\widetilde{L}_i^{-1}\mathbf{p}, 1) = \frac{1}{\alpha} \frac{\pi}{\ln 2} \left(1 + \frac{\pi}{\ln 2} B_i \right), \quad B_i = B_{i1} + B_{i2}, \quad i = 1, 2, \tag{46}$$

where $\mathbf{p} = (1,1)^T$. Expanding B_i in powers of the small parameter β_i and retaining linear terms, we obtain, after some algebra, the expansions for $(\widetilde{L}_i^{-1}\mathbf{p}, 1)$, $i = 1,2$:

$$(\widetilde{L}_i^{-1}\mathbf{p}, 1) = \frac{\beta_i}{\alpha} + O(\hat{\beta}^2), \quad \hat{\beta} = \beta_1 + \beta_2, \quad i = 1,2. \tag{47}$$

Let $f_i(t)$ be differentiable functions defined in a vicinity of $t = d_i$, and w_i be small parameters, $i = 1,2$. Then

$$f(w_i t + d_i) = c_i + O(w_i), \quad c_i = f(d_i), \quad i = 1,2, \tag{48}$$

or, in the vector notation,

$$\mathbf{f}(w_i t + d_i) = \mathbf{c} + O(w)\mathbf{p}, \quad w = \max_{i=1,2} w_i, \quad \mathbf{c} = (c_1, c_2)^T. \tag{49}$$

Then, according to (46),

$$\begin{aligned}(\widetilde{L}_i^{-1}\mathbf{f}, f_i) &= |c_i|^2(\widetilde{L}_i^{-1}\mathbf{p}, 1) + O(w)\\ &= \frac{|c_i|^2}{\alpha}\frac{\pi}{\ln 2}\left(1 + \frac{\pi}{\ln 2}B_i\right) + O(w), \quad i = 1,2.\end{aligned} \tag{50}$$

The expansions in powers of the small parameters β_i, $i = 1,2$, are, respectively,

$$(\widetilde{L}_i^{-1}\mathbf{f}, f_i) = \frac{|c_i|^2}{\alpha}\beta_i + O(\hat{\beta}^2), \quad i = 1,2. \tag{51}$$

Using the definition (26) of the diagonal operators $\widetilde{K}_{ii} = \widetilde{K}_{ii}(\lambda, w_i)$ and formulas (42), (43), and (47), we obtain

$$(\widetilde{K}_{ii}^{-1}\mathbf{f}, f_i) = |c_i|^2(\widetilde{L}_i^{-1}\mathbf{p}, 1) + O(w), \quad i = 1,2. \tag{52}$$

so that their asymptotic expansions in powers of β_i coincide with (51),

$$(\widetilde{K}_{ii}^{-1}\mathbf{f}, f_i) = \frac{|c_i|^2}{\alpha}\beta_i + O(\hat{\beta}^2), \quad i = 1,2. \tag{53}$$

Then from (44) and the definition (45) it follows that

$$(K^{-1}(\lambda)\mathbf{f}, \mathbf{f}) = (L_P^{-1}\mathbf{f}, \mathbf{f}) + O(q) \quad \text{as} \quad q \to 0, \tag{54}$$

which yields

$$(K^{-1}(\lambda)\mathbf{f}, \mathbf{f}) = \sum_{j=1}^{2}(\widetilde{L}_j^{-1}\mathbf{f}, f_j) + O(q) = \frac{c_0}{\alpha}\frac{\pi}{\ln 2}\left(1 + \frac{\pi}{\ln 2}\hat{B}\right) + O(q), \tag{55}$$

where

$$c_0 = \|\mathbf{c}\|_2^2 = \sum_{j=1}^{2} |c_j|^2, \quad \hat{B} = \frac{|c_1|^2 B_1 + |c_2|^2 B_2}{c_0}. \tag{56}$$

Expanding B_i and then $\hat{B}$ in powers of small parameters β_i, $i = 1, 2$, and retaining the first- and second-order terms involving β_i^n, $n = 0, 1, 2$, $i = 1, 2$, and $\beta_1\beta_2$ and discarding the third-order terms involving β_i^n with $n \geq 3$ and $\beta_1^k\beta_2^m$ with $k + m \geq 3$, we obtain

$$\begin{aligned}(K^{-1}(\lambda)\mathbf{f}, \mathbf{f}) &\equiv G_{\mathbf{f}}(\lambda)\\ &= \frac{1}{\alpha}(c_1\beta_1 + c_2\beta_2) + \frac{c_0}{\alpha}\left(\frac{\pi}{\ln 2}\right)^2 (d_3\beta_1^2 + d_4\beta_2^2 + d_5\beta_1\beta_2) + O(q),\end{aligned} \tag{57}$$

where $d_j = d_j(\lambda)$, $j = 3, 4, 5$, are given in Appendix A.

Introducing the weighted small parameter $\widetilde{\beta} = \dfrac{|c_1|^2 B_1 + |c_2|^2 B_2}{c_0}$ and discarding in (57) all terms of the order higher than one, we obtain

$$(K^{-1}(\lambda)\mathbf{f}, \mathbf{f}) = \frac{c_0}{\alpha}\widetilde{\beta} + O(\widetilde{\beta}^2). \tag{58}$$

2.4. Approximate Determination of Characteristic Numbers

In this section we will use the approximate semi-inversion to reduce the determination of CNs of logarithmic integral OVFs to functional dispersion equations (DEs).

The case of one interval. Consider the local representation of the integral equation $K(\lambda)\varphi = f$ in the vicinity of the chosen pole λ_ν according to (2)

$$K_\nu(\lambda)\varphi + \frac{m_\nu}{\lambda_\nu - \lambda}(\varphi, \varphi_\nu)\varphi_\nu = f. \tag{59}$$

Applying the operator $K_\nu^{-1}(\lambda)$ to both sides of (59), we obtain the equivalent equation

$$\varphi + \frac{m_\nu}{\lambda_\nu - \lambda}(\varphi, \varphi_\nu)K_\nu^{-1}(\lambda)\varphi_\nu = K_\nu^{-1}(\lambda)f. \tag{60}$$

The solution to (60) is uniquely defined if (φ, φ_ν) is uniquely defined. Calculating the inner product of both sides of equation (60) with φ_ν, we obtain

$$(\varphi, \varphi_\nu) = \frac{(\lambda_\nu - \lambda)(K_\nu^{-1} f, \varphi_\nu)}{\lambda_\nu - \lambda + m_\nu (K_\nu^{-1} \varphi_\nu, \varphi_\nu)}. \tag{61}$$

Substituting expression (61) into (60), we obtain the local representation of the inverse $K^{-1}(\lambda)$ in the vicinity of the pole λ_ν:

$$\varphi = K_\nu^{-1}(\lambda) f - \frac{m_\nu (K_\nu^{-1}(\lambda) f, \varphi_\nu)}{\lambda_\nu - \lambda + m_\nu (K_{NM}^{-1}(\lambda) \varphi_\nu, \varphi_\nu)} K_\nu^{-1}(\lambda) \varphi_\nu. \tag{62}$$

Quantity (φ, φ_ν) is uniquely defined for any f if and only if the denominator of the fraction in (62) is not equal to zero. The zeros of the denominator are the points at which the invertibility of $K(\lambda)$ is lost. Since the integral operator $K(\lambda)$ is a Fredholm the holomorphic OVF, these points are its CNs. Thus, CNs of $K(\lambda)$ are the roots of the equation

$$\lambda = \lambda_\nu + m_\nu (K_\nu^{-1}(\lambda) \varphi_\nu, \varphi_\nu). \tag{63}$$

Using (7)–(18) and (51)–(53), one can obtain the formula

$$(K_\nu^{-1}(\lambda) \varphi_\nu, \varphi_\nu) = \frac{|c_\nu|^2}{\alpha} \frac{\pi}{\ln 2} \left(1 + \frac{\pi}{\ln 2} B_{0,\nu}\right) + O(w), \tag{64}$$

and then the estimate

$$(K_\nu^{-1}(\lambda) \varphi_\nu, \varphi_\nu) = \beta \frac{|c_\nu|^2}{\alpha} + O(\beta^2), \tag{65}$$

where $K_\nu(\lambda)$ is defined in (2); $B_{0,\nu}$ is given by (15) where M_0 defined by (7) should be replaced by the corresponding quantity $M_{0,\nu}$ minus the singularity at λ_ν,

$$M_{0,\nu} = M_{0,\nu}(\lambda) = N(d, d, \lambda) - \frac{m_\nu c_\nu}{\lambda_\nu - \lambda}; \tag{66}$$

and

$$\varphi_\nu(t) = c_\nu + O(w), \quad c_\nu = \varphi_\nu(d), \tag{67}$$

where w is a small parameter and $\varphi_\nu(t)$ is a differentiable function in the vicinity of $t = d$. When $c_\nu = 0$, then the estimate

$$(K_\nu^{-1}(\lambda) \varphi_\nu, \varphi_\nu) = O(w) \quad \text{as} \quad w \to 0, \tag{68}$$

holds.

Equation (63) can be written in the form

$$\lambda = F_\nu(\lambda), \quad F_\nu(\lambda) \equiv \lambda_\nu + m_\nu \frac{|c_\nu|^2}{\alpha} \frac{\pi}{\ln 2} \left(1 + \frac{\pi}{\ln 2} \hat{B}_{0,\nu}\right). \tag{69}$$

Expanding in powers of small parameters β we obtain

$$F_\nu(\lambda) = \lambda_\nu + m_\nu \frac{|c_\nu|^2}{\alpha} \beta + O(\beta^2). \tag{70}$$

Therefore $F_\nu(\lambda)$ is a contraction mapping which yields the existence of a root λ_ν^* to the equation (63) $\lambda = F_\nu(\lambda)$ in the vicinity of λ_ν. This root can be obtained as a segment of an asymptotic series in powers of β directly from (69):

$$\lambda_\nu^* = \lambda_\nu + \beta \frac{m_\nu |c_\nu|^2}{\alpha} + \beta^2 \frac{m_\nu |c_\nu|^2}{\alpha} \left(\frac{\ln 2}{\pi} + \frac{M_{0,\nu}(\lambda_\nu)}{\alpha}\right) + O(\beta^3), \quad \text{as} \quad \beta \to 0, \tag{71}$$

when $c_\nu \neq 0$ is given by (67). This result is based on the estimate (65). When $c_\nu = 0$, the estimate (68) holds and

$$\lambda_\nu^* = \lambda_\nu + O(w), \quad \text{as} \quad w \to 0. \tag{72}$$

Since the roots of equation (63) are CNs of OVF $\widetilde{K}(\lambda)$, we have proved that for sufficiently small w, there exists a CN of OVF $K(\lambda)$ which lies in the vicinity of the pole λ_ν and is expressed by formulas (71) or (72).

In addition to estimate (72), a more accurate formula for the CN is obtained in [7] when $c_N = 0$:

$$\lambda_\nu^* = \lambda_\nu - w^2 \frac{\pi m_\nu |c_\nu|^2}{2\alpha} + O(w^3) \quad \text{as} \quad w \to 0.$$

For $\varphi_\nu = \cos \dfrac{\pi N}{b} t$, we have

$$\begin{aligned} K_\nu^{-1}(\lambda)\varphi_\nu &= K_\nu^{-1}(\lambda)1 + O(w^2) = L_1^{-1}1 + O(w^2) \\ &= \frac{1}{\alpha \ln 2} \frac{1}{1 + \dfrac{\pi}{\ln 2}\left(\dfrac{1}{\beta} + \dfrac{M_{0,\nu}(\lambda)}{\alpha}\right)} \frac{1}{\sqrt{1-t^2}} + O(w^4 \ln w) \\ &\qquad \text{as} \quad w \to 0. \end{aligned}$$

For sufficiently small w,

$$(K_\nu^{-1}(\lambda)\varphi_\nu, \varphi_\nu) = \frac{1}{\alpha}\beta\left(1 - \beta\left(\frac{\ln 2}{\pi} + \frac{C_\nu(\lambda)}{\alpha}\right)\right) + O(\beta^3) \quad \text{as} \quad \beta \to 0,$$

and the corresponding solution to equation (61) is

$$\lambda_\nu^* = \lambda_\nu + \beta\frac{m_\nu}{\alpha} + \beta^2\frac{m_\nu}{\alpha}\left(\frac{\ln 2}{\pi} + \frac{M_{0,\nu}(\lambda_\nu)}{\alpha}\right) + O(\beta^3), \quad \text{as} \quad \beta \to 0.$$

The case of n intervals. Consider the integral OVF

$$\begin{aligned} K(\lambda)\varphi &= \alpha L\varphi + N(\lambda)\varphi \\ &\equiv \int_\Gamma \left[\frac{\alpha}{\pi}\ln\frac{1}{|t_0 - t|} + N(t_0, t, \lambda)\right]\varphi(t)dt \quad (t_0 \in \Gamma), \end{aligned} \tag{73}$$

with

$$\begin{aligned} \Gamma &= \bigcup_{j=1}^{n} \Gamma_j, \\ \Gamma_j &= (a_j, b_j) = (d_j - w_j, d_j + w_j), \\ j &= 1, 2, \dots, n, \quad n \ge 2, \end{aligned} \tag{74}$$

where $N(t_0, t, \lambda)$ is once continuously differentiable in $\Gamma \times \Gamma$ and a meromorphic function of λ. In the vicinity of a chosen pole λ_ν of $N(\lambda; t_0, t)$, OVF $K(\lambda)$ can be written as a matrix operator and represented componentwise with the inner product

$$(\varphi_j, \varphi_\nu)_j = \int_{\Gamma_j} \varphi_j(t)\varphi_\nu(t)dt \quad (j = 1, 2, \dots, n).$$

Introduce the column vector-function

$$\mathbf{f} = (\varphi_1, \varphi_2, \dots, \varphi_n)^T \tag{75}$$

and the auxiliary n-dimensional vector-function

$$\mathbf{f}_\nu = \varphi_\nu(1, 1, \dots, 1)^T, \tag{76}$$

and define, as in (45), the inner product associated with the set of the integration intervals

$$(\mathbf{f}, \mathbf{f}_\nu) = \sum_{j=1}^{n}(\varphi_j, \varphi_\nu)_j = \sum_{j=1}^{n}\int_{\Gamma_j} \varphi_j(t)\varphi_\nu(t)dt. \tag{77}$$

Then OVF $K(\lambda)$ written in the matrix form can be represented as

$$K(\lambda)\mathbf{f} = K_\nu(\lambda)\mathbf{f} + \frac{m_\nu}{\lambda_\nu - \lambda}(\mathbf{f}, \mathbf{f}_\nu)\mathbf{f}_\nu, \tag{78}$$

where $K_\nu(\lambda)$ is a matrix OVF with the entries $K_{ij,\nu}(\lambda)$, $i, j = 1, 2, \ldots, n$.

Use (78) and consider the local representation of the integral equation $K_\nu(\lambda)\mathbf{f} = \mathbf{F}$ in the vicinity of the chosen pole λ_ν (similar to (59))

$$K(\lambda)\mathbf{f} = K_\nu(\lambda)\mathbf{f} + \frac{m_\nu}{\lambda_\nu - \lambda}(\mathbf{f}, \mathbf{f}_\nu)\mathbf{f}_\nu = \mathbf{F}, \tag{79}$$

where the n-dimensional right-hand side $\mathbf{F} = (F_1, F_2, \ldots, F_3)^T$. Applying the operator $K_\nu^{-1}(\lambda)$ to both sides of (79), we obtain the equivalent equation

$$\mathbf{f} + \frac{m_\nu}{\lambda_\nu - \lambda}(\mathbf{f}, \mathbf{f}_\nu)K_\nu^{-1}(\lambda)\mathbf{f}_\nu = K_\nu^{-1}(\lambda)\mathbf{F}. \tag{80}$$

The solution to (80) is uniquely defined if the inner product (scalar quantity) $(\mathbf{f}, \mathbf{f}_\nu)$ is uniquely defined. Calculating the inner product of both sides of equation (80) with $\mathbf{f}_\nu$, we obtain

$$(\mathbf{f}, \mathbf{f}_\nu) + \frac{m_\nu}{\lambda_\nu - \lambda}(\mathbf{f}, \mathbf{f}_\nu)(K_\nu^{-1}\mathbf{f}_\nu, \mathbf{f}_\nu) = (K_\nu^{-1}\mathbf{F}, \mathbf{f}_\nu). \tag{81}$$

Resolving (81) with respect to $(\mathbf{f}, \mathbf{f}_\nu)$ and substituting the result into (80), we obtain the local representation of the inverse $K^{-1}(\lambda)$ in the vicinity of the pole λ_ν:

$$\mathbf{f} = K_\nu^{-1}(\lambda)\mathbf{F} - \frac{m_\nu(K_\nu^{-1}(\lambda)\mathbf{F}, \mathbf{f}_\nu)}{\lambda_\nu - \lambda + m_\nu(K_\nu^{-1}(\lambda)\mathbf{f}_\nu, \mathbf{f}_\nu)}K_\nu^{-1}(\lambda)\mathbf{f}_\nu. \tag{82}$$

The quantity $(\mathbf{f}, \mathbf{f}_\nu)$ is uniquely defined for any $\mathbf{F}$ if and only if the denominator of the fraction in (82) is not equal to zero. The zeros of the denominator are the points at which $K(\lambda)$ is not invertible. Since the matrix integral operator $K(\lambda)$ is a Fredholm and holomorphic OVF, these points are its CNs. Thus, CNs of $K(\lambda)$ are the roots of the equation

$$\lambda = \lambda_\nu + m_\nu(K_\nu^{-1}(\lambda)\mathbf{f}_\nu, \mathbf{f}_\nu). \tag{83}$$

Thus the formula (83) for CNs in the case of several intervals of integration coincides with the formula (63) obtained for the OVF with one interval in which the inner product (77) is used. For sufficiently small w, there exists a root λ_ν^* of equation (83) which can be obtained a segment of an asymptotic series in powers of β; the resulting expressions are similar to (71) and (72) (when $c_\nu \neq 0$ and when $c_\nu = 0$).

The case of two intervals. Consider the integral OVF (73) defined on two intervals of integration ($n = 2$ in (74) given componentwise by (22). In order to determine its CNs in the vicinity of poles when the diameters of the integration intervals are small, we will use representation (79) with a separated pole term.

Transform OVF (73) to the equivalent form with the integration in each operator K_{ij} (22) over $(-1, 1)$. Introducing two-dimensional vector-functions (75) and (76) and applying the procedure developed above for the case of n intervals, we obtain the equation (83) for the sought-for CNs of $K(\lambda)$ in which the inner product is defined according to (45). Using the asymptotic representation (55) we obtain, discarding the terms of the order $O(q)$,

$$(K_\nu^{-1}(\lambda)\mathbf{f}_\nu, \mathbf{f}_\nu) = \sum_{j=1}^{2}(\widetilde{L}_j^{-1}\mathbf{f}_\nu, \varphi_\nu^{(j)}) = G_{\mathbf{f}_\nu}, \tag{84}$$

where

$$G_{\mathbf{f}_\nu} = \frac{|c_\nu|^2}{\alpha}\frac{\pi}{\ln 2}\left(1 + \frac{\pi}{\ln 2}\hat{B}_\nu\right). \tag{85}$$

The expansion of $G_{\mathbf{f}_\nu}$ in powers of small parameters β_i ($i = 1, 2$) is given by (57);

$$|c_\nu|^2 = \sum_{j=1}^{2}|c_{j,\nu}|^2, \quad c_{j,\nu} = \varphi_\nu^{(j)}(0), \quad j = 1, 2; \tag{86}$$

$\varphi_\nu^{(j)}(t) = w_j\varphi_\nu(w_j t + d_j)$, $j = 1, 2$; and $\mathbf{f}_\nu = (\varphi_\nu^{(1)}, \varphi_\nu^{(2)})^T$. Equation (83) takes the form

$$\lambda = F_\nu(\lambda) \tag{87}$$

with

$$F_\nu(\lambda) \equiv \lambda_\nu + m_\nu G_{\mathbf{f}_\nu}. \tag{88}$$

Here

$$\hat{B}_\nu = \frac{c_{1,\nu}B_{1,\nu} + c_{2,\nu}B_{2,\nu}}{|c_\nu|^2}, \tag{89}$$

$B_{j,\nu}$ are given by (28), (33), (35), (37), and (56), where M_{ij} defined by (28) should be replaced by the corresponding quantities $M_{ij,\nu}$ minus the singularity at λ_ν:

$$M_{ij,\nu} = M_{ij,\nu}(\lambda) = \frac{\alpha}{\pi}\ln\frac{1}{|d_i - d_j|} + N(d_i, d_j, \lambda) - \frac{m_\nu c_{i,\nu}}{\lambda_\nu - \lambda}, \quad (90)$$

$$i \neq j, \quad i, j = 1, 2,$$

$$M_{ii,\nu} = M_{ii,\nu}(\lambda) = N(d_i, d_i, \lambda) - \frac{m_\nu c_{i,\nu}}{\lambda_\nu - \lambda}, \quad i = 1, 2. \quad (91)$$

Expanding in powers of small parameters β_i and retaining the first- and second-order terms involving β_i^n, $n = 0, 1, 2$, $i = 1, 2$, and $\beta_1\beta_2$ and discarding the third-order terms involving β_i^n with $n \geq 3$ and $\beta_1^k\beta_2^m$ with $k + m \geq 3$, we obtain the expansion of $F_\nu(\lambda)$ in powers of small parameters β_i (57), which yields

$$F_\nu(\lambda) = \lambda_\nu + \frac{m_\nu}{\alpha}\widetilde{\beta} + O(\widetilde{\beta}^2), \quad \widetilde{\beta} = \frac{\sum_{j=1}^{2}|c_{j,\nu}|^2\beta_j}{|c_\nu|^2}. \quad (92)$$

It is easy to see that $F_\nu(\lambda)$ is a contraction mapping. Indeed, according to formulas (28), (33), (35), (37), (89), and (90) that constitute the explicit definitions of all quantities contained in the expression for $F_\nu(\lambda)$ there exists an $r > 0$ such that $F_\nu(\lambda)$ is a differentiable function in a ball $B_r = \{\lambda : |\lambda - \lambda_\nu| < r\}$, and the derivative

$$\frac{dF_\nu(\lambda)}{d\lambda} = O(\hat{\beta}^2), \quad (93)$$

so that the inequality

$$\left|\frac{dF_\nu(\lambda)}{d\lambda}\right| \leqslant A < 1 \quad (94)$$

holds uniformly for sufficiently small $\hat{\beta}$ in the ball B_r. Therefore, there exists a root $\lambda_\nu = \lambda_\nu^*$ of equation (92) in the vicinity of λ_ν. This root is an approximation to CN of OVF (73).

Substituting into (87), in which the left-hand side is replaced by λ_ν^* and the right-hand side, by (88), where $G_{\mathbf{f}_\nu}$ is expanded using (57) with $M_{ii,\nu} = M_{ii,\nu}(\lambda)$ replaced by their values $\widehat{M}_{ii,\nu} = M_{ii,\nu}(\lambda_\nu)$ at $\lambda = \lambda_\nu$, we obtain the expansion of the root λ_ν^* in powers of small parameters β_i $(i = 1, 2)$ containing all second-order terms (if we take into account that $\lambda - \lambda_\nu = O(\hat{\beta})$ according to (87), (88), and (92)). The resulting formula is similar to the corresponding expansion (71) obtained in the case of one interval of integration.

3. RECTANGULAR SLOTTED CAVITIES

In this section, we will study model problems of finding eigenfrequencies and eigenfields of a family of rectangular slotted cavities using the methods developed above.

3.1. Rectangular Cavity with One Slot

The cross section of this cavity by the plane $x_3 = 0$ (in the Cartesian coordinate system (x_1, x_2, x_3)) is formed by two rectangular domains

$$\Omega^1 = \{\boldsymbol{r} = (x_1, x_2) : 0 < x_1 < a_1;\ 0 < x_2 < b_1\},$$
$$\Omega^2 = \{\boldsymbol{r} = (x_1, x_2) : 0 < x_1 < a_2;\ -b_2 < x_2 < 0\},$$

with the common part of the boundary $\partial\Omega^1 \cap \partial\Omega^2 = \{\boldsymbol{r} : x_2 = 0,\ 0 \leqslant x_1 \leqslant \min(a_1, a_2)\}$, containing the interval $\Gamma = \{\boldsymbol{r} : x_2 = 0,\ d - w < x_1 < d + w\}$ (a slot). The permittivity $\varepsilon = \varepsilon_i(\boldsymbol{r})$, $i = 1, 2$; $\boldsymbol{r} \in \Omega^i$. This geometry (see Fig. 2) corresponds to a slotted cavity with one slot. The squared eigenfrequencies (SEs) are eigenvalues of the boundary eigenvalue problem formulated as follows: It is necessary to determine the values of parameter λ for which there exist nontrivial solutions to the homogeneous Helmholtz equation

$$\Delta u(\boldsymbol{r}) + \lambda\varepsilon u(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \Omega = \Omega^1 \cup \Omega^2, \tag{95}$$

satisfying the homogeneous boundary condition

$$\frac{\partial u}{\partial x_2} = 0 \tag{96}$$

on $(\partial\Omega^1 \setminus \Gamma) \cup (\partial\Omega^2 \setminus \Gamma)$, the conjugation conditions

$$[u^1 - u^2]\big|_\Gamma = 0, \quad u^i = u^i(\boldsymbol{r}), \quad \boldsymbol{r} \in \Omega^i, \quad i = 1, 2, \tag{97}$$

$$\left.\left[\frac{1}{\varepsilon_1}\frac{\partial u^1}{\partial x_2} - \frac{1}{\varepsilon_2}\frac{\partial u^2}{\partial x_2}\right]\right|_\Gamma = 0, \tag{98}$$

and the Meixner condition

$$\iint_{S_\rho} \left(|u|^2 + \left|\frac{\partial u}{\partial x_1}\right|^2 + \left|\frac{\partial u}{\partial x_2}\right|^2\right) d\boldsymbol{r} < \infty; \tag{99}$$

here S_ρ denotes the vicinities of the boundary edges and u is the longitudinal component H_3 of the magnetic field intensity vector $\boldsymbol{H}$.

The Fredholm property and existence of generalized and classical solutions to the corresponding inhomogeneous problem is proved in [5]. Therefore, there may exist only isolated eigenvalues of the problem (95)–(99). According to [5], this problem has a real eigenvalue between every two neighbouring points $\mu_{nm}^{(i)} = \frac{\pi^2}{\varepsilon_i}\left(\frac{n^2}{a_i^2} + \frac{m^2}{b_i^2}\right)$, $n, m = 0, 1, \dots,$ $i = 1, 2$. These eigenvalues coincide with CNs of the integral OVF $K(\lambda)$ that enters the (operator) equation $K(\lambda)\phi = 0$, to which this problem is reduced below.

Introduce Green's functions of the second boundary value problem for the Helmholtz equation in rectangles Ω^i, $i = 1, 2$:

$$G_i(\lambda; \boldsymbol{r}, \boldsymbol{q}) = \frac{4}{a_i b_i} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_n \delta_m \frac{\psi_n(x_1, a_i)\psi_n(x_1^0, a_i)\psi_m(x_2, b_i)\psi_m(x_2^0, b_i)}{\mu_{nm}^{(i)} - \lambda},$$

$$\delta_n = \begin{cases} 1/2, & n = 0, \\ 1, & n \geqslant 1, \end{cases} \quad \psi_n(t, a) = \cos\frac{\pi n t}{a}, \quad \boldsymbol{r} = (x_1, x_2), \quad \boldsymbol{q} = (x_1^0, x_2^0),$$

$$\mu_{nm}^{(i)} = \frac{\tilde{\mu}_{nm}^{(i)}}{\varepsilon_i}, \quad \tilde{\mu}_{nm}^{(i)} = \pi^2\left(\frac{n^2}{a_i^2} + \frac{m^2}{b_i^2}\right), \quad i = 1, 2, \tag{100}$$

where $\tilde{\mu}_{nm}^{(i)}$ are eigenvalues of the second boundary value problem for the Laplace operator in rectangle Ω^i. Green's function $G_i(\lambda; \boldsymbol{r}, \boldsymbol{q})$ is meromorphic with the set of poles

$$\mathfrak{M}^i = \{\mu_{nm}^{(i)}\}_{n,m=0}^{\infty}, \quad i = 1, 2. \tag{101}$$

Representing the solution in the form of Green's potentials, one can reduce the problem (95)–(99) to a homogeneous integral equation with a logarithmic singularity of the kernel [6, 5]

$$\begin{aligned} K(\lambda)\varphi &= \alpha L\varphi + N(\lambda)\varphi \\ &\equiv \int_\Gamma \left[\alpha\frac{1}{\pi}\ln\frac{1}{|t_0 - t|} + N(t_0, t, \lambda)\right]\varphi(t)dt = 0, \quad t_0 = \Gamma, \end{aligned} \tag{102}$$

where $\alpha = \varepsilon_1 + \varepsilon_2$ and the OVF $K(\lambda)$ is as in (73) with

$$N(t_0, t, \lambda) = N_1(t_0, t, \lambda) + g(t_0, t),$$

$$g(t_0, t) = -\sum_{i=1}^{2} \frac{\varepsilon_i}{\pi}\left[\ln\frac{\pi}{a_i} + \ln\frac{2a_i}{\pi}\left|\frac{2\sin\frac{\pi}{2a_i}(t-t_0)}{t-t_0}\right| + \ln\left|2\sin\frac{\pi(t+t_0)}{2a_i}\right|\right],$$

$$N_1(t_0, t, \lambda) = \sum_{i=1}^{2} \left(\gamma_0^{(i)}(\lambda) + \sum_{n=1}^{\infty} \left(\gamma_n^{(i)}(\lambda) - \frac{2\varepsilon_i}{\pi n} \right) \psi_n^{(i)}(t_0) \psi_n^{(i)}(t) \right),$$

$$\gamma_0^{(i)}(\lambda) = \frac{1}{a_i} f(\lambda q_i), \quad \gamma_n^{(i)}(\lambda) = \frac{2}{a_i} \varepsilon_i b_i f(q_i(n^2 r_i - \lambda)),$$

$$q_i = \varepsilon_i b_i^2, \quad r_i = \frac{\pi^2}{\varepsilon_i a_i^2}, \quad i = 1, 2,$$

$$f(z) = \begin{cases} \dfrac{\coth \sqrt{z}}{\sqrt{z}} & z > 0, \\ -\dfrac{\cot \sqrt{-z}}{\sqrt{-z}} & z < 0, \end{cases} \qquad \psi_n^{(i)}(x) = \cos \frac{\pi n x}{a_i}.$$

The asymptotic representation

$$\gamma_n^{(i)}(\lambda) - \frac{2\varepsilon_i}{\pi n} = \frac{2\varepsilon_i}{\pi n^3} + O(n^{-5}), \quad i = 1, 2, \tag{103}$$

which holds uniformly for sufficiently large n in every closed domain that does not contain poles, enables one to prove that $K(\lambda)$ is a meromorphic function of λ with the set of poles $\mathfrak{M}^1 \cup \mathfrak{M}^2$.

The SEs of the slotted resonator under study are CNs of $K(\lambda)$ in (102), and the latter are the roots of the transcendental DE $\lambda = F_\nu(\lambda)$ given by (69). In this case (one interval of integration) the right-hand side of (69) written for a chosen pole $\lambda_\nu = \mu_{NM}^{(i_0)} \in \mathfrak{M}_0^{i_0}$ (for a fixed triple N, M, i_0: $N, M = 0, 1, 2, \ldots,$ $i_0 = 1, 2$), takes the form

$$\begin{aligned} F_\nu(\lambda) &\equiv \lambda_\nu + \frac{4}{a_{i_0} b_{i_0}} \cos^2 \frac{\pi N d}{a_{i_0}} \frac{\beta}{\alpha + \beta \alpha \frac{\ln 2}{\pi} + \beta M_{0,\nu}}, \\ \beta &= \left(\frac{1}{\pi} \ln \frac{1}{w} \right)^{-1}, \end{aligned} \tag{104}$$

where $M_{0,\nu}(\lambda)$ is given in Appendix A. In the case of a narrow slot, a root λ_ν^* of (104) determined in the vicinity of a chosen pole $\lambda_\nu = \mu_\nu = \mu_{NM}^{(i_0)}$, gives an approximate value of a SE which is actually the perturbed SE $\mu_{NM}^{(i_0)}$ of the $H_{NM}^{(i_0)}$-oscillation of the (partial) cavity Ω^{i_0} ($N, M = 0, 1, 2, \ldots,$ $i_0 = 1, 2$). This perturbation, which is caused by the presence of a narrow slot, is regular because $\lambda_\nu \to \mu_\nu$ as the diameter of the slot tends to zero ($w \to 0$ and $\beta \to 0$).

3.2. Rectangular Cavity with Two Slots in One Wall

The cross section of this cavity by the plane $x_3 = 0$ is formed by two rectangular domains Ω^1 and Ω^2 described above with the common part of the boundary $\partial\Omega^1 \cap \partial\Omega^2 = \{\boldsymbol{r} : x_2 = 0, 0 \leqslant x_1 \leqslant \min(a_1, a_2)\}$, containing two intervals (slots)

$$\Gamma = \bigcup_{j=1}^{2} \Gamma_i, \quad \Gamma_i = (d_j - w_j, d_j + w_j), \quad j = 1, 2. \tag{105}$$

The permittivity $\varepsilon = \varepsilon_i(\boldsymbol{r})$, $i = 1, 2$; $\boldsymbol{r} \in \Omega^i$. The problem is reduced [5] to the homogeneous integral equation (102) with two intervals of integration Γ. Using the change of variables $t = w_j x + d_j$, $t_0 = w_i x_0 + d_i$, $i, j = 1, 2$, we transform (102) to the equivalent form with the integration over $(-1, 1)$. The smooth part $N(t_0, t, \lambda)$ of the kernel of the resulting OVF can be represented as

$$\begin{aligned} N(w_i x_0 + d_i, w_j x + d_j, \lambda) &= N^{(ij)}(x_0, x, \lambda) \\ = \sum_{p=1}^{2} \varepsilon_p \Bigg\{ &\frac{2}{a_p b_p} D_0^{(p)}(\lambda) + \frac{4}{a_p b_p} \sum_{n=1}^{\infty} D_n^{(p)}(\lambda) \varphi_n^{(p,i)}(x_0) \varphi_n^{(p,j)}(x) \\ &+ \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(i)} \varphi_n^{(p,i)}(x_0) \varphi_n^{(p,j)}(x) \\ &- \frac{1}{\pi} \ln \left| \frac{2 \sin \dfrac{\pi}{2a_p}[(w_j x + d_j) - (w_i x_0 + d_i)]}{\dfrac{\pi}{2a_p}[(w_j x + d_j) - (w_i x_0 + d_i)]} \right| \\ &- \frac{1}{\pi} \ln \left| 2 \sin \frac{\pi}{2a_p}[(w_j x + d_j) + (w_i x + d_i)] \right| \Bigg\}, \end{aligned} \tag{106}$$

where

$$\varphi_n^{(p,j)}(x) = \cos \frac{\pi n (w_j x + d_j)}{a_p}, \quad i, j, p = 1, 2. \tag{107}$$

and quantities $D_n^{(p)}$ and $Q_n^{(i)}$ are specified in Appendix A.

In this case (two intervals of integration), the right-hand side of the transcendental equation of the method in (69) written for a chosen pole $\lambda_\nu = \mu_{NM}^{(i_0)} \in \mathfrak{M}_0^{i_0}$ (for a fixed triple N, M, i_0: $N, M = 0, 1, 2, \ldots$, $i_0 = 1, 2$), takes the form

$$\lambda = F_\nu(\lambda), \quad F_\nu(\lambda) \equiv \lambda_\nu + m_\nu G_{\mathbf{f}_\nu}, \quad m_\nu = \frac{4 \varepsilon_{i_0} b_{i_0}}{a_{i_0}}. \tag{108}$$

Here $G_{\mathbf{f}_\nu}$ is introduced in (85) with $\hat{B}_\nu$ and $B_{j,\nu}$ given by (28), (33), (35), (37), and (89), where M_{ij} defined by (28) should be replaced, by the corresponding quantities $M_{ij,\nu}$ minus the singularity at λ_ν:

$$M_{ij,\nu} = M_{ij,\nu}(\lambda) = \frac{\alpha}{\pi} \ln \frac{1}{|d_i - d_j|} + N(d_i, d_j, \lambda) - \frac{m_\nu c_{i,\nu}}{\lambda_\nu - \lambda}, \tag{109}$$

$$i \neq j, \quad i, j = 1, 2,$$

$$M_{ii,\nu} = M_{ii,\nu}(\lambda) = N(d_i, d_i, \lambda) - \frac{m_\nu c_{i,\nu}}{\lambda_\nu - \lambda}, \quad i = 1, 2, \tag{110}$$

where

$$|c_\nu|^2 = \sum_{j=1}^{2} |c_{j,\nu}|^2, \quad c_{i,\nu} = \varphi_N^{(i_0,i)}(0) = \cos \frac{\pi N d_i}{a_{i_0}}, \quad i = 1, 2,$$
$$\mathbf{f}_\nu = (\varphi_N^{(i_0,1)}, \varphi_N^{(i_0,2)})^T. \tag{111}$$

and the quantities that enter (109) and (110) are given in Appendix A.

3.3. Rectangular Cavity with Two Slots in Opposite Walls

The cross section of this cavity by the plane $x_3 = 0$ is formed by three rectangular domains

$$\Omega^1 = \{\boldsymbol{r} = (x_1, x_2) : 0 < x_1 < a_1;\ 0 < x_2 < b_1\},$$
$$\Omega^2 = \{\boldsymbol{r} : a_{12} < x_1 < a_{12} + a_2;\ -b < x_2 < 0\},$$
$$\Omega^3 = \{\boldsymbol{r} : a_{13} < x_1 < a_{13} + a_3;\ b_1 < x_2 < b_1 + b_3\},$$

with the common parts of the boundaries $\partial\Omega^1 \cap \partial\Omega^2 \subset \{\boldsymbol{r} : x_2 = 0\}$ and $\partial\Omega^1 \cap \partial\Omega^3 \subset \{\boldsymbol{r} : x_2 = b_1\}$ containing the intervals (slots)

$$\Gamma_1 = \{\boldsymbol{r} : x_2 = 0,\ d_1 - w_1 < x_1 < d_1 + w_1\} \subset \partial\Omega^1 \cap \partial\Omega^2,$$
$$\Gamma_2 = \{\boldsymbol{r} : x_2 = b_1,\ d_2 - w_2 < x_1 < d_2 + w_2\} \subset \partial\Omega^1 \cap \partial\Omega^3.$$

The permittivity $\varepsilon = \varepsilon(\boldsymbol{r}) = \varepsilon_i$, $\boldsymbol{r} \in \Omega^i$, $i = 1, 2, 3$, is a piecewise constant function of $\boldsymbol{r}$.

The statement of the corresponding boundary eigenvalue problem is similar to (95)–(99). It is necessary to determine the values of parameter λ for which there exist nontrivial solutions to the homogeneous Helmholtz equation

$$\Delta u(\boldsymbol{r}) + \lambda \varepsilon u(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \Omega = \Omega^1 \cup \Omega^2 \cup \Omega^3, \tag{112}$$

satisfying the homogeneous boundary condition

$$\frac{\partial u}{\partial x_2} = 0 \tag{113}$$

on $(\partial\Omega^3\backslash\Gamma_2)\cup(\partial\Omega^2\backslash\Gamma_1)\cup(\partial\Omega^2\backslash\Gamma)$, where $\Gamma = \Gamma_1\cup\Gamma_2$, the conjugation conditions

$$[u^1 - u^2]\big|_{\Gamma_1} = 0, \quad [u^1 - u^3]\big|_{\Gamma_2} = 0, u^i = u^i(\boldsymbol{r}), \quad \boldsymbol{r}\in\Omega^i, \quad i = 1,2,3, \tag{114}$$

$$\left[\frac{1}{\varepsilon_1}\frac{\partial u^1}{\partial x_2} - \frac{1}{\varepsilon_2}\frac{\partial u^2}{\partial x_2}\right]\Bigg|_{\Gamma_1} = 0, \quad \left[\frac{1}{\varepsilon_1}\frac{\partial u^1}{\partial x_2} - \frac{1}{\varepsilon_3}\frac{\partial u^3}{\partial x_2}\right]\Bigg|_{\Gamma_2} = 0, \tag{115}$$

and the Meixner condition (99).

The Fredholm property and existence of generalized and classical solutions to the corresponding inhomogeneous problem is proved in [5]. Therefore, there may exist only isolated eigenvalues of this problem.

Introduce Green's functions of the second boundary value problem for the Helmholtz equation (112) in rectangles Ω^i, $i = 1,2,3$:

$$G_1(\lambda;\boldsymbol{r},\boldsymbol{q}) = \frac{4}{a_1b_1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\delta_n\delta_m\frac{\psi_n(x_1,a_1)\psi_n(x_1^0,a_1)\psi_m(x_2,b_1)\psi_m(x_2^0,b_1)}{\mu_{nm}^{(1)}-\lambda}, \tag{116}$$

$$G_2(\lambda;\boldsymbol{r},\boldsymbol{q}) = \frac{4}{a_2b_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\delta_n\delta_m \cdot\frac{\psi_n(x_1-a_{12},a_2)\psi_n(x_1^0-a_{12},a_2)\psi_m(x_2,b_2)\psi_m(x_2^0,b_2)}{\mu_{nm}^{(2)}-\lambda}, \tag{117}$$

$$G_3(\lambda;\boldsymbol{r},\boldsymbol{q}) = \frac{4}{a_3b_3}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\delta_n\delta_m \cdot\frac{\psi_n(x_1-a_{13},a_3)\psi_n(x_1^0-a_{13},a_3)\psi_m(x_2-b_1,b_3)\psi_m(x_2^0-b_1,b_3)}{\mu_{nm}^{(3)}-\lambda}. \tag{118}$$

Green's functions $G_i(\lambda;\boldsymbol{r},\boldsymbol{q})$ are meromorphic with the sets of poles $\mathfrak{M}^i = \{\mu_{nm}^{(i)}\}_{n,m=0}^{\infty}$, $i = 1,2,3$.

Applying Green's identity to $u^i = u^i(\boldsymbol{r})$, $\boldsymbol{r}\in\Omega^i$, and Green's function G_i in every domain Ω^i $(i = 1,2,3)$, and denoting

$$\boldsymbol{q}_1^0 = (s,0)\in\Gamma_1, \qquad \boldsymbol{q}_2^0 = (s,b_1)\in\Gamma_2,$$
$$\boldsymbol{r}_1^0 = (t,0)\in\Gamma_1, \qquad \boldsymbol{r}_2^0 = (t,b_1)\in\Gamma_2,$$

we obtain the representation of u in the form of generalized potentials (Green's potentials):

$$u^1(\boldsymbol{r}) = -\int_{\Gamma_1} G_1(\lambda; \boldsymbol{r}, \boldsymbol{q}_1^0) \frac{\partial u^1}{\partial x_2}\bigg|_{\Gamma_1}(s)ds + \int_{\Gamma_2} G_1(\lambda; \boldsymbol{r}, \boldsymbol{q}_2^0) \frac{\partial u^1}{\partial x_2}\bigg|_{\Gamma_2}(s)ds, \ \ \boldsymbol{r} \in \Omega^1,$$

$$u^i(\boldsymbol{r}) = (-1)^i \int_{\Gamma_{i-1}} G_i(\lambda; \boldsymbol{r}, \boldsymbol{q}_i^0) \frac{\partial u^i}{\partial x_2}\bigg|_{\Gamma_{i-1}}(s)ds, \quad \boldsymbol{r} \in \Omega^i, \quad i = 2, 3. \qquad (119)$$

Introduce the notations for the unknown densities according to (115)

$$\varphi_1(s) = \frac{\partial u^1}{\partial x_2}\bigg|_{\Gamma_1}(s) = \frac{\varepsilon_1}{\varepsilon_2} \frac{\partial u^2}{\partial x_2}\bigg|_{\Gamma_1}(s), \quad d_1 - w_1 < s < d_1 + w_1,$$

$$\varphi_2(s) = \frac{\partial u^1}{\partial x_2}\bigg|_{\Gamma_2}(s) = \frac{\varepsilon_1}{\varepsilon_3} \frac{\partial u^3}{\partial x_2}\bigg|_{\Gamma_2}(s), \quad d_2 - w_2 < s < d_2 + w_2.$$

Separate slowly converging parts in the series for kernels of Green's potentials (119) defined using (116)–(118) by performing the summation in m with the help of the formulas

$$\sum_{m=0}^{\infty} \delta_m \frac{\psi_m(t, b_i)}{\mu_{nm}^{(i)} - \lambda} = \frac{b_i \varepsilon_i}{2} f_n^{(i)}(b_i - t, \lambda),$$

$$\sum_{m=0}^{\infty} \delta_m (-1)^m \frac{\psi_m(t, b_i)}{\mu_{nm}^{(i)} - \lambda} = \frac{b_i \varepsilon_i}{2} f_n^{(i)}(t, \lambda),$$

where

$$f_n^{(i)}(u, \lambda) = \begin{cases} \dfrac{\cosh \gamma_n^{(i)} u}{\gamma_n^{(i)} \sinh \gamma_n^{(i)} b_i}, & \dfrac{\pi^2 n^2}{a_i^2} > \varepsilon_i \lambda, \\ -\dfrac{\cos \gamma_n^{(i)} u}{\gamma_n^{(i)} \sin \gamma_n^{(i)} b_i}, & \dfrac{\pi^2 n^2}{a_i^2} < \varepsilon_i \lambda, \end{cases} \qquad (120)$$

$$\gamma_n^{(i)} = \sqrt{\left| \frac{\pi^2 n^2}{a_i^2} - \varepsilon_i \lambda \right|}, \quad i = 1, 2, 3.$$

As a result, we obtain

$$G_1(\lambda; \boldsymbol{r}, \boldsymbol{q}_1^0) = \frac{2\varepsilon_1}{a_1} \sum_{n=0}^{\infty} \delta_n \psi_n(x_1, a_1) \psi_n(s, a_1) f_n^{(1)}(x_2, \lambda), \qquad (121)$$

$$G_1(\lambda;\boldsymbol{r},\boldsymbol{q}_2^0)=\frac{4}{a_1b_1}\sum_{n=0}^{\infty}\delta_n\psi_n(x_1,a_1)\psi_n(s,a_1)\sum_{m=0}^{\infty}\delta_m(-1)^m\frac{\psi_m(x_2,b_1)}{\mu_{nm}^{(1)}-\lambda}$$
$$=\frac{2\varepsilon_1}{a_1}\sum_{n=0}^{\infty}\delta_n\psi_n(x_1,a_1)\psi_n(s,a_1)f_n^{(1)}(x_2,\lambda),\tag{122}$$

$$G_2(\lambda;\boldsymbol{r},\boldsymbol{q}_1^0)=\frac{2\varepsilon_2}{a_2}\sum_{n=0}^{\infty}\delta_n\psi_n(x_1-a_{12},a_2)\psi_n(s-a_{12},a_2)f_n^{(2)}(x_2,\lambda),\tag{123}$$

$$G_3(\lambda;\boldsymbol{r},\boldsymbol{q}_2^0)=\frac{2\varepsilon_3}{a_3}\sum_{n=0}^{\infty}\delta_n\psi_n(x_1-a_{13},a_3)\psi_n(s-a_{13},a_3)f_n^{(3)}(x_2-b_1,\lambda).\tag{124}$$

To represent the solution in the form of Green's potentials (119) in the case of narrow slots, we take w_j, $j=1,2$, as small parameters, make use of the fact that

$$\varphi_j(s)=\frac{\varphi_i^*(s)}{\sqrt{[s-(w_j-d_j)][(w_j+d_j)-s]}},\quad w_j-d_j<s<w_j+d_j\tag{125}$$

according to the edge condition (99), where $\varphi_i^*(s)$ is a differentiable function, change variables by $s=w_jx+d_j$, $j=1,2$ $(-1<x<1)$, and evaluate the integrals involving $\varphi_j(s)$ to obtain, discarding the terms of order $O(w_j^2)$,

$$u^1(\boldsymbol{r})=\frac{2\pi\varepsilon_1}{a_1}\sum_{n=0}^{\infty}\delta_n[\varphi_2^*(d_1)\psi_n(d_2,a_1)-\varphi_1^*(d_1)\psi_n(d_1,a_1)]$$
$$\cdot\,\psi_n(x_1,a_1)f_n^{(1)}(x_2,\lambda),\quad \boldsymbol{r}\in\Omega^1,\tag{126}$$

$$u^2(\boldsymbol{r})=\frac{2\varphi_1^*(d_1)\pi\varepsilon_2^2}{a_2\varepsilon_1}\sum_{n=0}^{\infty}\delta_n\psi_n(x_1-a_{12},a_2)\psi_n(d_1-a_{12},a_2)$$
$$\cdot\, f_n^{(2)}(x_2,\lambda),\quad \boldsymbol{r}\in\Omega^2,\tag{127}$$

$$u^3(\boldsymbol{r})=-\frac{2\varphi_2^*(d_2)\pi\varepsilon_3^2}{a_3\varepsilon_1}\sum_{n=0}^{\infty}\delta_n\psi_n(x_1-a_{13},a_3)\psi_n(d_2-a_{13},a_3)$$
$$\cdot\, f_n^{(3)}(x_2-b_1,\lambda),\quad \boldsymbol{r}\in\Omega^3.\tag{128}$$

Note that for every fixed $x_2\neq 0$ $f_n^{(3)}(x_2-b_1,\lambda)$ and $f_n^{(3)}(x_2,\lambda)$ are exponentially decaying functions of x_2. Setting formally $\varphi_1^*(d_1)=1$

and $\varphi_2^*(d_2) = 0$ in (126)–(128) we obtain the representations for the solution to (95)–(99) in the case of the resonator with one slot formed by two rectangular domains.

Applying conjugation condition (114) to the potentials (119) and using the definitions of the unknown densities, we obtain the system of integral equations

$$\begin{aligned} &\int_{\Gamma_1} [\varepsilon_1 G_1(\lambda; \boldsymbol{r}_1^0, \boldsymbol{q}_1^0) + \varepsilon_2 G_2(\lambda; \boldsymbol{r}_1^0, \boldsymbol{q}_1^0)]\varphi_1(s)ds \\ &\qquad\qquad - \int_{\Gamma_2} \varepsilon_1 G_1(\lambda; \boldsymbol{r}_1^0, \boldsymbol{q}_2^0)\varphi_2(s)ds = 0, \quad t \in \Gamma_1, \\ &-\int_{\Gamma_1} \varepsilon_1 G_1(\lambda; \boldsymbol{r}_2^0, \boldsymbol{q}_1^0)\varphi_1(s)ds \\ &+\int_{\Gamma_2} [\varepsilon_1 G_1(\lambda; \boldsymbol{r}_2^0, \boldsymbol{q}_2^0) + \varepsilon_3 G_3(\lambda; \boldsymbol{r}_2^0, \boldsymbol{q}_2^0)]\varphi_2(s)ds = 0, \quad t \in \Gamma_2, \end{aligned} \tag{129}$$

with respect to $\varphi_1(s)$ and $\varphi_2(s)$. We consider the operator $K(\lambda)$ of the system (129) as an integral OVF (73) defined on two intervals of integration Γ_1 and Γ_2 (see (74) with $n = 2$) given componentwise by (22). Write $K(\lambda)$ in the matrix form (21) using the variables $t, t_0 \in \Gamma_i$, $i, j = 1, 2$

$$K(\lambda) = \begin{pmatrix} K_{11}(\lambda) & K_{12}(\lambda) \\ K_{21}(\lambda) & K_{22}(\lambda) \end{pmatrix}, \tag{130}$$

where

$$\begin{aligned} K_{ij}(\lambda)\varphi_j &= [\alpha_{ij} L_{ij}\varphi_j + N_{ij}(\lambda)]\varphi_j \\ &= \int_{\Gamma_j} \left[\frac{\alpha_{ij}}{\pi} \ln \frac{1}{|t_0 - t|} + N(t_0, t, \lambda) \right] \varphi_j(t)dt, \quad t_0 \in \Gamma_i, \end{aligned} \tag{131}$$

$$i, j = 1{,}2, \quad \alpha_{11} = \varepsilon_1 + \varepsilon_2, \quad \alpha_{22} = \varepsilon_1 + \varepsilon_3, \quad \alpha_{12} = \alpha_{21} = -\varepsilon_1.$$

Transform OVF given by (130) to the equivalent form with the integration in each operator K_{ij} (131) over $(-1, 1)$. The smooth part $N(t_0, t, \lambda)$ of the kernel of $K(\lambda)$ can be represented componentwise using (25) in the form similar to (106). The corresponding formulas are given in Appendix A.

In order to determine the CNs of $K(\lambda)$ in the vicinity of poles when the diameters of the integration intervals are small, we will use representation (79) with a separated pole term and the methods developed in Section 2.

Introducing two-dimensional vector-functions (75) and (76) and applying the procedure developed above for the case of two intervals, we obtain the equation (83) for the sought-for CNs of $K(\lambda)$ in which the inner product is defined according to (45). Using the asymptotic representation (55) we obtain, discarding the terms of the order $O(q)$, the equation $\lambda = F_\nu(\lambda)$ given by (84)–(89). The quantities $B_{j,\nu}$ herein are given by (28), (33), (35), (37), and (89), where M_{ij} defined by (28) should be replaced, in line with (109) and (110), by the quantities $M_{ij,\nu}$ (90) and (91) minus the singularity at λ_ν. The resulting formulas are similar to (108)–(144); the right-hand side of the transcendental equation of the method in (69) $\lambda = F_\nu(\lambda)$ written for a chosen pole $\lambda_\nu = \mu_{NM}^{(i_0)} \in \mathfrak{M}_0^{i_0}$, i.e., for a chosen triple N, M, i_0: $N, M = 0, 1, 2, \ldots$, $i_0 = 1, 2, 3$ has the form (108). The function $G_{\mathbf{f}_\nu}$ is introduced in (85), and $M_{ij,\nu}$ are given by (109) and (110) where $c_{i,\nu} = \varphi_N^{(i_0,i)}(0)$, $\alpha = 1$; the representations for $N(d_i, d_j, \lambda) - m_\nu c_{i,\nu}/(\lambda_\nu - \lambda)$, $i, j = 1, 2$, are given in Appendix A.

Expanding in powers of small parameters β_i and retaining the first- and second-order terms involving β_i^n, $n = 0, 1, 2$, $i = 1, 2$, and $\beta_1\beta_2$ and discarding the third-order terms involving β_i^n with $n \geq 3$ and $\beta_1^k \beta_2^m$ with $k + m \geq 3$, we obtain the expansion of $F_\nu(\lambda)$ in powers of small parameters β_i given by (57) and (92). It is easy to see that $F_\nu(\lambda)$ thus obtained is a contraction mapping.

Substituting into the equation $\lambda = F_\nu(\lambda)$, in which the left-hand side is replaced by λ_ν^* and the right-hand side, by (88), where $G_{\mathbf{f}_\nu}$ is expanded using (57) with $M_{ii,\nu} = M_{ii,\nu}(\lambda)$ replaced by their values $\widehat{M}_{ii,\nu} = M_{ii,\nu}(\lambda_\nu)$ at $\lambda = \lambda_\nu$, we obtain the expansion of the root λ_ν^* in powers of small parameters β_i $(i = 1, 2)$ containing all second-order terms (if we take into account that $\lambda - \lambda_\nu = O(\hat{\beta})$ according to (87), (88), and (92)). The resulting formula is similar to the corresponding expansion (71) obtained in the case of one interval of integration.

3.4. Critical Points and Mode Coupling

The essence of the mathematical model investigated in [7] and developed in [5] for slotted cavities is to consider homogeneous integral equations (102) and (129) as operator DEs, called *the generalized DE*, and determine CNs of the corresponding OVFs $K(\lambda) = K(\lambda, \bar{\eta})$ as implicit functions $\lambda(\bar{\eta})$ of the vector of nonspectral parameters $\bar{\eta} = (a_1, a_2, b_1, b_2, w, \ldots)$; these implicit functions were called *generalized dispersion curves* (GDCs). It is reasonable to determine $\lambda = \lambda(\xi)$ with respect to one particular parameter $\xi \in \bar{\eta}$ (other parameters being fixed), and to analyze one particular GDC on the (λ, ξ)-plane for different values of another parameter $\zeta \in \bar{\eta}$ or the behaviour of several GDCs on the same parameter plane in the vicinities of certain (critical) points. Sometimes GDCs exhibit a behaviour [5] that reveals the presence of different singular (e.g., saddle) points of OVF $K(\lambda, \bar{\eta})$ considered as a two-parameter mapping. Consequently, the interaction can be simulated and explained as a phenomenon inherent to slotted structures.

However, it is virtually impossible to analyze the properties of multi-parameter operators $K(\lambda, \bar{\eta})$ and implicit dependences $\lambda(\bar{\eta})$ as OVFs proper. In order to make this analysis efficient we have reduced in this paper the determination of SEs of slotted cavities under study, e.g., the generalized (operator) DEs $K(\lambda, \bar{\eta}) = 0$, to functional DEs (69), (104), and (108) of the form $\Phi_\nu(\lambda) = 0$, where $\Phi_\nu(\lambda) \equiv \lambda_\nu - F_\nu(\lambda)$ is a complicated multi-parameter mapping, $\Phi_\nu(\lambda) = \Phi_\nu(\lambda, \bar{\eta})$.

It is known [8] that various interaction phenomena and mode coupling is observed in the vicinity of different CPs $P^* = (\lambda^*, \xi^*)$ of $\Phi_\nu(\lambda)$ in the (λ, ξ)-plane. A complete study of all such CPs goes beyond the scope of this paper. Here we mention only certain types of CPs that can be naturally distinguished: (i) the DPs at which two or several poles of $K(\lambda, \bar{\eta})$ given by (101) merge; (ii) certain extrema points of $\Phi_\nu(\lambda)$; and (iii) singularities of $\Phi_\nu(\lambda)$ itself; note, in particular, that its poles are CNs of the principal-part OVFs $L_1(\lambda)$ (11) (zeros of $\alpha + \tilde{g}(\lambda)$ in (14)) and $L_P(\lambda)$ (29).

Consider CPs (i). The set (101) is a multi-parameter family of points $\{\mu_{nm}^{(i)}\}_{n,m=0}^{\infty}$ $(i = 1, 2)$ that can merge at different combinations of parameters. Let us analyze here one particular situation of the interaction in a rectangular cavity with one slot formed by two rectangular domains $\Omega^1 = \{0 < x_1 < a_1,\ 0 < x_2 < b_1\}$ and $\Omega^2 = \{0 < x_1 < a_2,\ -b_2 < x_2 < 0\}$ (problem (95)–(99)): two points of the set (101) of the same index i merge when one parameter $\xi = a_i$ is varied (a_i is the width of a partial rectangular domain) and other parameters are fixed. More specifically, we determine the values of ξ at which two eigenvalues

$$\lambda_\nu = \frac{\pi^2}{\varepsilon_\nu}\left(\frac{n_\nu^2}{\xi^2} + \frac{m_\nu^2}{b_\nu^2}\right), \quad \lambda_\mu = \frac{\pi^2}{\varepsilon_\mu}\left(\frac{n_\mu^2}{\xi^2} + \frac{m_\mu^2}{b_\mu^2}\right)$$

of the Laplacian of one and the same partial rectangular domain coincide. Below we will make use of the fact that there is a one-to-one correspondence between every ν and μ that number λ_ν and λ_μ and the triples of integers (i, N_i, M_i), $i = 1, 2$. Note also that the points of the set (101) are poles of the OVF $K(\lambda)$ in (102) to which problem (95)–(99) is reduced. Considering $\mu_{nm}^{(i)} = \mu_{nm}^{(i)}(\xi)$ as functions of ξ, it

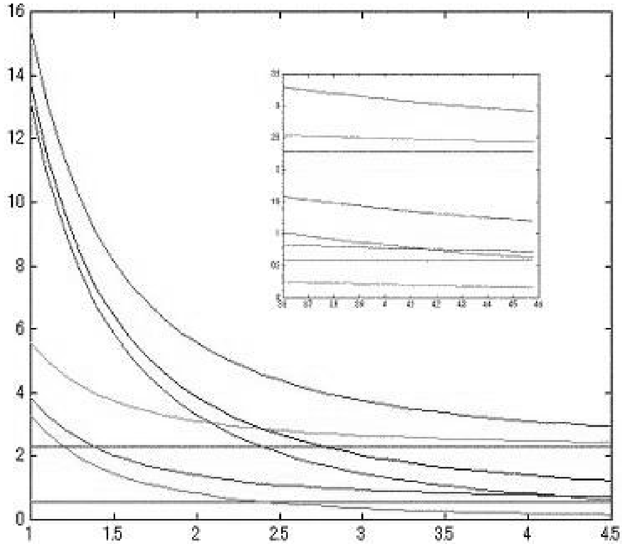

Figure 1. The eigenvalue curves $\mu_{nm}^{(1)} = \mu_{nm}^{(1)}(a_1)$ $(n, m = 0, 1, 2)$ of three first fundamental oscillations of a rectangular resonator with the cross section $\Omega^1 = \{0 < x_1 < a_1;\ 0 < x_2 < b_1\}$, $b_1 = 2.4$. The insert shows the enlarged fragment containing the (extreme right) point of intersection of the $\mu_{11}^{(1)}(a_1)$ and $\mu_{20}^{(1)}(a_1)$ curves, the DP $(\lambda^*, a^*) \approx (0.762, 4.126)$.

is easy to verify that $\mu_{n_1m_1}^{(i)}(\xi) = \mu_{n_2m_2}^{(i)}(\xi) = \lambda^*$ at

$$a_i = b_i\sqrt{\frac{n_1^2 - n_2^2}{m_2^2 - m_1^2}}, \tag{132}$$

when

$$n_1 > n_2 \quad \text{and} \quad m_2 > m_1 \quad \text{or} \quad n_1 < n_2 \quad \text{and} \quad m_2 < m_1. \tag{133}$$

Figure 1 shows $\mu_{nm}^{(1)} = \mu_{nm}^{(1)}(a_1)$ with $n, m = 0, 1, 2$ for a rectangular resonator with the cross section $\Omega^1 = \{0 < x_1 < a_1;\ 0 < x_2 < b_1\}$, $b_1 = 2.4$. One observes eight points of intersection; i.e., there are eight DPs in the range $0.1 < \lambda < 16$, $1 < a_1 < 4.5$.

Condition (132) and (133) specify the infinite family of DPs at which oscillations $H_{n_1m_1}^i$ and $H_{n_2m_2}^i$ interact. Note that (132) gives additionally characteristic dimensions of the partial cavity i at which this interaction takes place.

The interaction of oscillations in this rectangular cavity with one slot was described in [5] close to a DP $P^* = (\lambda^*, a^*)$ (in the (λ, ξ)-plane), at which two poles merge,

$$\lambda_\nu(a^*) = \lambda_\mu(a^*) = \lambda^*; \tag{134}$$

$\xi = a_1$ was taken as the width of a partial (upper) rectangular domain.

Representing the OVF $K(\lambda)$ in (102) as

$$K(\lambda)\varphi = K_{\nu\mu}(\lambda)\varphi + \frac{m_\nu}{\lambda_\nu - \lambda}(\varphi, \varphi_\nu)\varphi_\nu + \frac{m_\mu}{\lambda_\mu - \lambda}(\varphi, \varphi_\mu)\varphi_\mu$$

and performing the approximate semi-inversion of $K(\lambda)$ in the vicinity of $P^* = (\lambda^*, a^*)$, one reduces evaluation of its CNs (the sought-for SEs of the slotted cavity) to the determination of roots $\lambda_\nu^* = \lambda_\nu^*(a_1)$ of the equation $\lambda = F_{\nu,\mu}(\lambda)$ of the form (104). The function $\lambda - F_{\nu,\mu}(\lambda)$ can be reduced in the vicinity of P^* to a canonical quadratic form [5] which exhibits a characteristic saddle-type behavior in the (λ, ξ)-plane (close to (λ^*, a^*)) corresponding to a typical case of interaction. The roots λ_ν^* were determined in the vicinity of $\lambda_\nu = \mu_{11}^{(1)}$ ($N = M = 1$, $i_0 = 1$) and $\lambda_\nu = \mu_{20}^{(1)}$ ($N = 2$, $M = 0$, $i_0 = 1$) which merge at a DP $P^* = (\lambda^*, a^*)$ determined from (132) at $n_1 = m_1 = 1$ and $n_2 = 2$, $m_2 = 0$ ($i = 1$):

$$\mu_{11}^{(1)} = \mu_{20}^{(1)} = \lambda^* \quad \text{at} \quad \lambda^* = \frac{2\pi^2}{\varepsilon_1 a^{*2}}, \quad a^* = b_1\sqrt{3} \quad (b_1 \text{ is fixed}). \tag{135}$$

In the vicinity of this DP, oscillations H_{11}^1 and H_{20}^1 interact and exchange their types.

In [5], the following set of (normalized to the free-space wavelength) parameters was considered: $b_1 = a_2 = b_2 = 2.4$, $d = 1.4$, $\varepsilon_1 = 3.0$, $\varepsilon_2 = 1.0$, and $w = 0.0005$ ($\beta = 0.4133\ldots$). For these values, the DP $(\lambda^*, a^*) \approx (0.762, 4.126)$ is situated in the domain of the $H_{11}^1 \leftrightarrows H_{20}^1$ interaction $3.7 < a_1 < 4.2$, $0.6 < \lambda < 1$. Let us analyze properties of the right-hand side of the equation (104) in this domain using the formulas (153) and (155) given in Appendix B. For the chosen values of parameters, when $b_1, a_2, b_2, d, \varepsilon_1, \varepsilon_2$, and w are fixed and a_1 is varied in the interval $(3.7, 4.2)$, only two poles of the set (101) lie in the interval $0.6 < \lambda < 1$ which contains the DP, $0.509 < \mu_2 < 0.806$ and $0.733 < \mu_1 < 0.839$, and only μ_2 is a pole of $M_{0,\nu}(\lambda)$ and a removable singularity of $F_\nu(\lambda)$; in a vicinity of μ_2, the function $F_\nu(\lambda)$ is calculated according to (104) and formulas (153)–(155) given in Appendix B:

$$F_\nu(\lambda) = \lambda_\nu + \frac{f_\nu(\mu_p - \lambda)}{c_p + (\mu_p - \lambda)M_{\nu,p}}, \quad |\lambda - \mu_p| < \Delta, \quad p = 1, 2. \tag{136}$$

It is easy to see that there are no poles of $M_{0,\nu}(\lambda)$ that merge in the interval $0.6 < \lambda < 1$ except for μ_1 and μ_2.

4. NUMERICAL AND DISCUSSION

Eigenfrequencies of the cylindrical slotted cavities with one and two slots were calculated as roots of the DEs (69), (104), and (108). For a sufficiently small $\beta < \beta_0$ (for practical computations one may choose as large as $\beta_0 \sim 0.9$) the equations $\lambda = F_\nu(\lambda)$ can be solved numerically in a vicinity of a chosen $\lambda_\nu = \mu_{NM}^{(i_0)} \in \mathfrak{M}_0^{i_0}$ (a chosen triple N, M, i_0: $N, M = 0, 1, 2$, $i_0 = 1, 2, 3$) using the fixed-point iterations

$$z_{k+1} = F_\nu(z_k), \quad k = 0, 1, 2, \dots, \quad z_0 = \lambda_\nu. \tag{137}$$

The method converges fast because we have, denoting by $s = F_\nu(s)$ the exact root,

$$\begin{aligned} |z_{k+1} - s| &= |F_\nu(z_k) - F_\nu(s)| \\ &= \left|\frac{dF_\nu(\xi)}{d\lambda}\right| |z_k - s| < \beta A |z_k - s| < \dots < (\beta A)^{k+1} |z_0 - s|, \end{aligned} \tag{138}$$

where ξ is a point in the vicinity of s and A is a constant given by (94), so that the rate of convergence is estimated by const $\cdot\, \beta^{k+1}$.

The SEs calculated vs. the width a_1 of the central rectangular cavity of a two-cavity resonator with one slot graphically coincide with the eigenvalue curves in Fig. 1 (with respect to the scale used in this figure) in the range of slotwidths approximately $10^{-7} < 2w < 10^{-2}$. The same holds for two- and three-cavity resonators.

For exponentially narrow slots ($10^{-8} < 2w < 10^{-4}$), the field distributions in partial domains were calculated using Green's potentials (119) and (126)–(128). For wider slots ($10^{-4} < 2w < 10^{-1}$), calculations were performed using semi-inversion and also the reduction to infinite-matrix (summation) equations as in [5].

4.1. Regular Regimes

The interaction phenomena and mode coupling, when small changes of the structure parameter(s) close to certain critical values cause abrupt (discontinuous) changes in the field distributions, may be generally called *critical regimes*. They have a local character and are observed in small vicinities of certain frequency values, in particular those corresponding to DPs. Other (noncritical) regimes, when the field distributions undergo continuous variations in a wide range of the structure parameter(s) (e.g., in an interval of variation of a chosen parameter ξ that does not contain the critical values of ξ) may be conventionally called *regular regimes*.

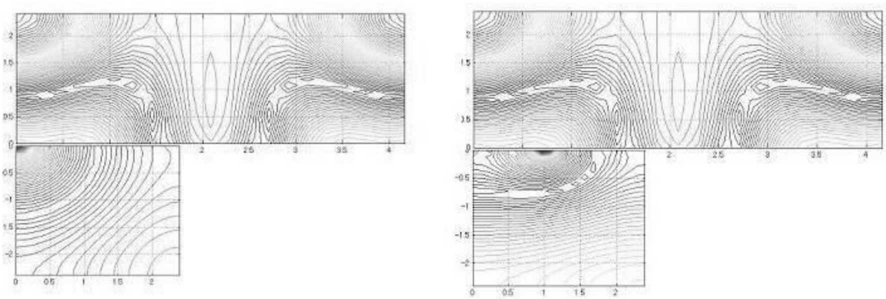

Figure 2. The H_{11}^{1} field distribution in a two-cavity resonator with one exponentially narrow slot ($\beta = \left(\frac{1}{\pi}\ln\frac{1}{w}\right)^{-1} = 0.2$, which corresponds to the slotwidth $2w = 3\cdot 10^{-7}$) cut in the left-middle and middle positions in the wall separating the cavities (the widths and heights of the upper and lower cavities 1 and 2 are, respectively, $a_1 = 4.2$, $b_1 = 2.4$, $a_2 = 2.4$, $b_2 = 2.4$; all dimensions are normalized to the free-space wavelength).

We calculated the distributions of eigenfields of the first three fundamental eigenoscillations H_{nm}^{i} ($n, m = 0, 1, 2,\ i = 1, 2, 3$) in the three types of slotted cavities under study in several regular regimes. The results of computations enable us to formulate several conclusions.

(i) The $H_{nm}^{i_0}$ field distributions in the cavity i with $i \neq i_0$ strongly depend on the slot positions d_j, while in the cavity i_0, the fields are virtually independent on d_j ($j = 1, 2$, Figs. 4, 9). However, we cannot state that there are certain critical slotwidths or slot positions: the variations in the field structure with respect to the latter have a continuous character, which is illustrated by Fig. 9 and also by Figs. 10, 11, and 18–20. Note that in the latter case (Figs. 18–20), an increase of the slotwidth in the range of larger values causes gradual changes in the field structure. The same holds when, e.g., the width of one of the cavities is increased outside the intervals containing its critical values (see Fig. 5 and Figs. 10–12 and 13–16).

(ii) The field amplitude in the three-cavity resonator is always higher in the cavity filled with a dielectric having larger permittivity; this is clearly seen in all figures representing these resonators (e.g., in Figs. 6–8), where in the central cavity 1 the permittivity $\epsilon_1 = 3$ and in upper and lower cavities 2 and 3, $\epsilon_2 = \epsilon_3 = 1$.

(iii) Several oscillation types of equal indices and, first of all, the fundamental H_{11}^{1} (in the three-cavity resonator with two slots composed of three equal cavities) exhibit a stable symmetric

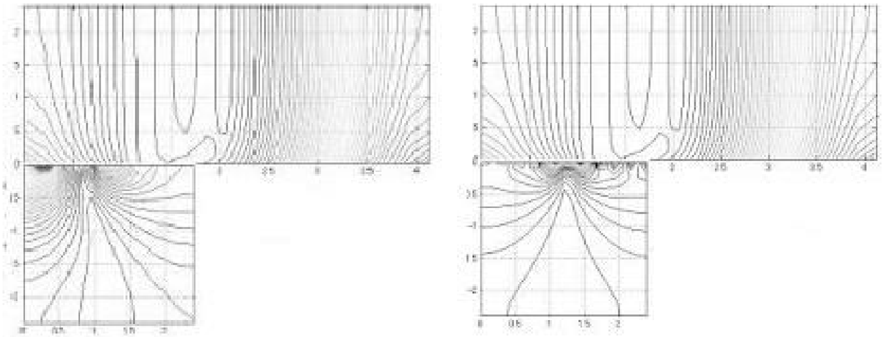

Figure 3. The H^1_{10} field distribution in a two-cavity resonator with two exponentially narrow slots ($2w_1 = 2w_2 = 3 \cdot 10^{-7}$, $\beta = 0.2$; $a_1 = 4.2$, $b_1 = 2.4$, $a_2 = 2.4$, $b_2 = 2.4$). The slot positions can be clearly seen.

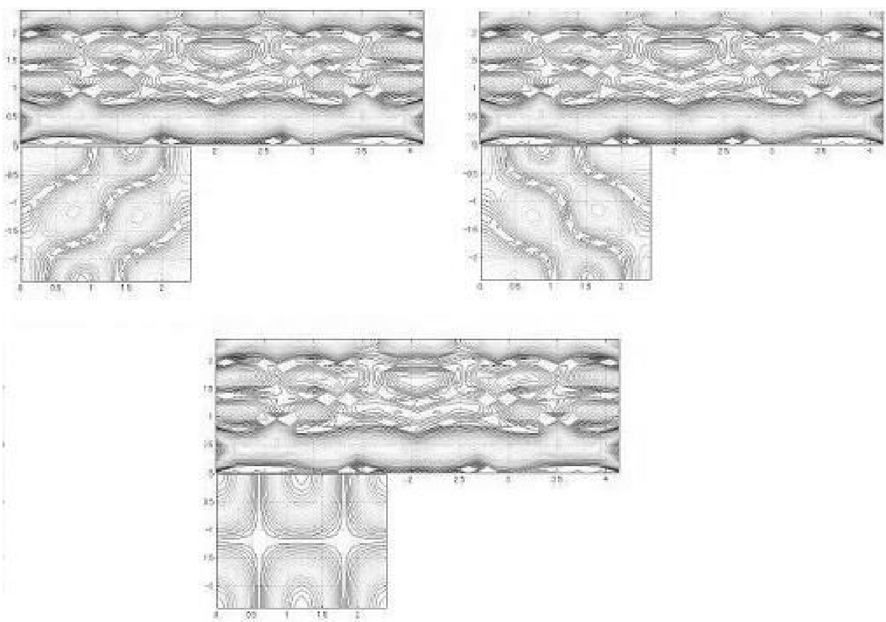

Figure 4. The H^1_{21} field distribution in a two-cavity resonator with one slot, which occupies the right-middle, middle, and left-middle positions. The geometric parameters are given in the caption to Fig. 2.

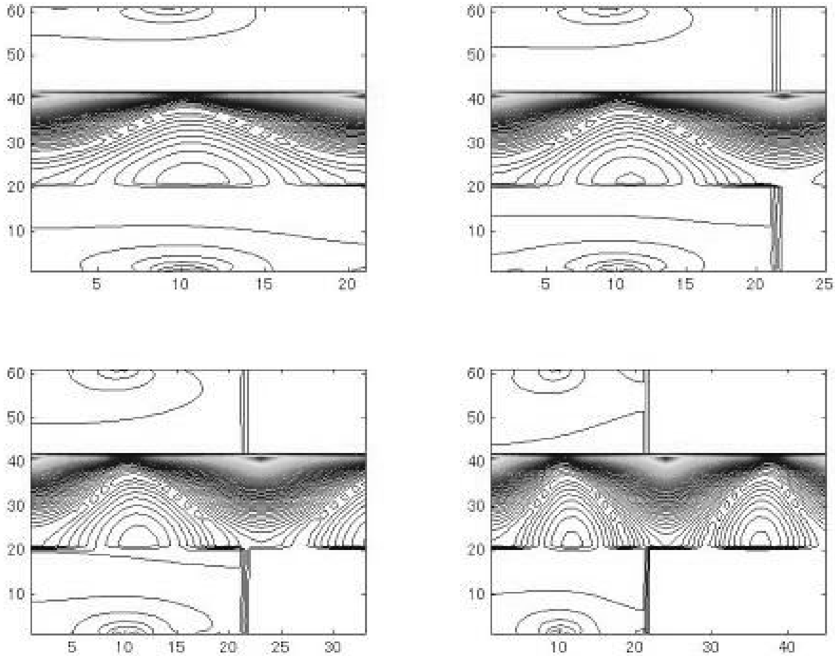

Figure 5. The H^1_{11} field distribution in a three-cavity resonator with two exponentially narrow slots ($2w_1 = 2w_2 = 3 \cdot 10^{-7}$, $\beta = 0.2$) cut symmetrically in the opposite walls of the central cavity 1 (with regard to the initial geometry in the upper left corner) as the width a_1 of the central cavity 1 increases from 2.4 to 4.2; other geometric parameters are $b_1 = a_2 = b_2 = a_3 = b_3 = 2.4$.

behavior (Fig. 8); on the other hand, the oscillations of unequal indices, like H^1_{10} and H^2_{20} may be nonsymmetric with respect to the horizontal symmetry line of the resonator (Figs. 4, 5, 9).

The H^1_{11} field distributions in two-cavity resonators with wider slots (the case of one slot) are shown in Figs. 10–20. The chains of pictures show, respectively, the effects of the increasing height b_2 of the lower cavity Ω^2 (Figs. 10, 11, 13–17) and of the displacement of the slot from the right central to the extreme right position (Fig. 12). The latter definitely causes the field concentration in the partial domain Ω^1 close to the symmetry lines of the rectangle. One can see also how the next oscillation type is established as b_2 increases. Figures 18–20 show how the increasing width of the slot influences the H^1_{11} field structure.

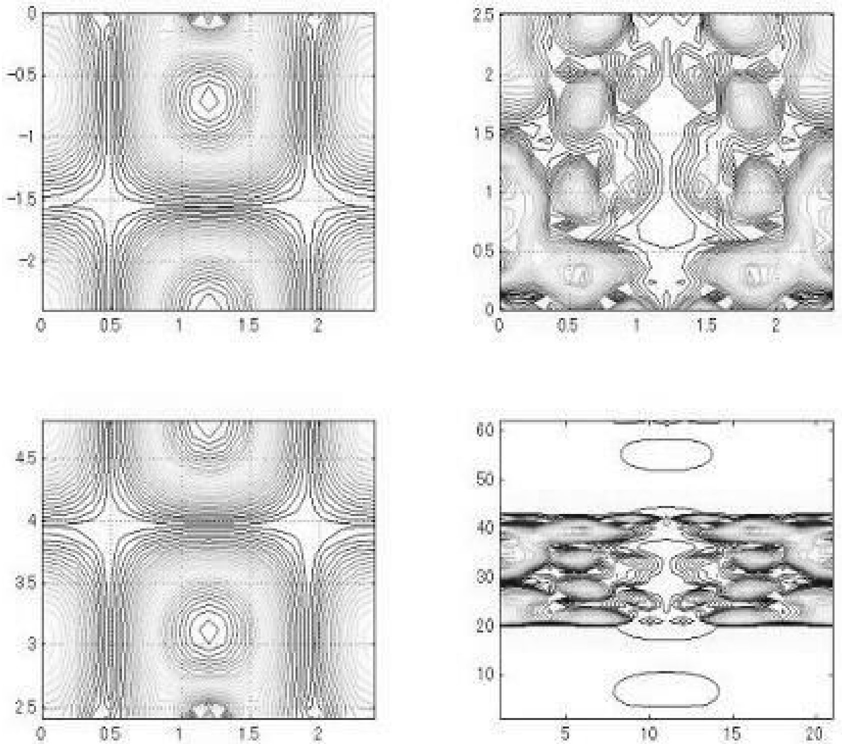

Figure 6. The H_{22}^2 field distribution in a three-cavity resonator with two exponentially narrow slots cut symmetrically in opposite walls. The geometric parameters are given in the caption to Fig. 5 with $a_1 = 2.4$. The bottom-right picture shows the entire resonator (when the same scale is used for all three partial cavities); the left pictures show the fields in the upper and lower cavities, and the top-right picture shows the central cavity (a particular scale is applied in each partial cavity to display the details). The field amplitude in the central cavity is by several orders of magnitude greater than in other cavities.

4.2. Critical Regimes: Interaction of Oscillations

The locations of several 'zero-order' DPs at which the eigenvalue curves $\mu_{nm}^{(1)} = \mu_{nm}^{(1)}(a_1)$ $(n, m = 0, 1, 2)$ of three first fundamental oscillations intersect can be seen from Fig. 1. The interaction may occur in the vicinity of every such point. However, its character and intensity depend on the cavity parameters, as follows from Fig. 23.

A numerical analysis of the typical example of interaction consid-

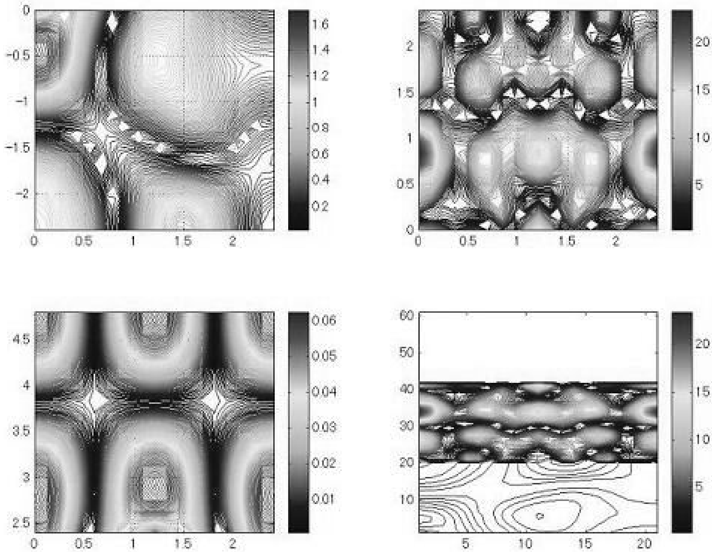

Figure 7. The H_{22}^2 field distribution in a three-cavity resonator with two exponentially narrow slots; the lower slot is cut (in the wall separating the central (1) and lower (2) cavities) close to the left corner and the upper slot is cut in the middle (of the wall separating the middle and upper cavities). The bottom-right picture shows the entire resonator (the same scale is used for all three partial cavities); the left pictures show the fields in the upper and lower cavities, and the top-right picture shows the central cavity (a particular scale is applied in each partial cavity to display details). The field amplitude in the central cavity is by several orders of magnitude greater than in other cavities. The geometric parameters are as in Fig. 6.

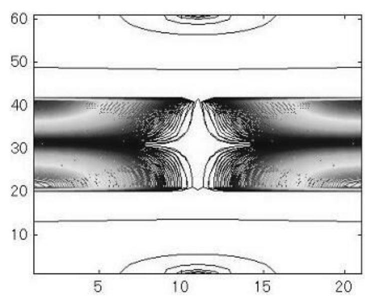

Figure 8. The H_{11}^1 filed distribution in a three-cavity resonator with two exponentially narrow slots cut symmetrically in opposite walls. The geometric parameters are as in Fig. 6.

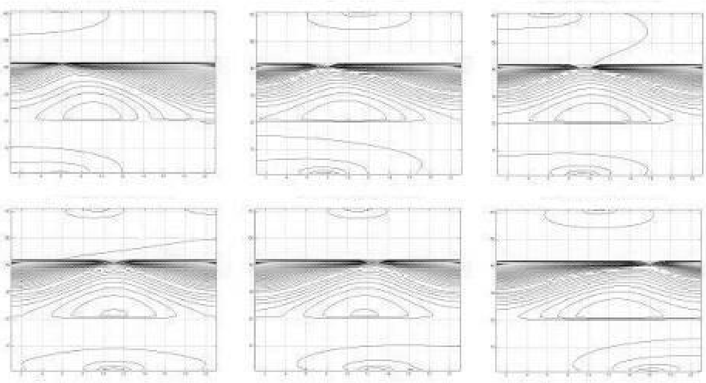

Figure 9. The variation of the H^1_{10} field with respect to the upper slot position in a three-cavity resonator with two exponentially narrow slots cut symmetrically in opposite walls. The geometric parameters are as in Fig. 6.

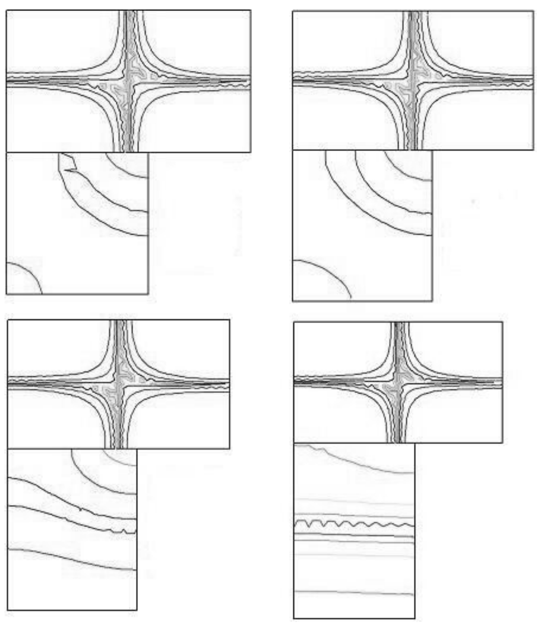

Figure 10. The variation of the H^1_{11} field with respect to the height of the lower cavity increasing from $b_2 = 2.6$ (top-left) to $b_2 = 3.4$ (bottom-right) in a two-cavity resonator with one narrow slot ($2w = 0.001$, $\beta = 0.4548$); the slot position $d = 2$, $a_1 = 4.126$, and $b_1 = a_2 = 2.4$.

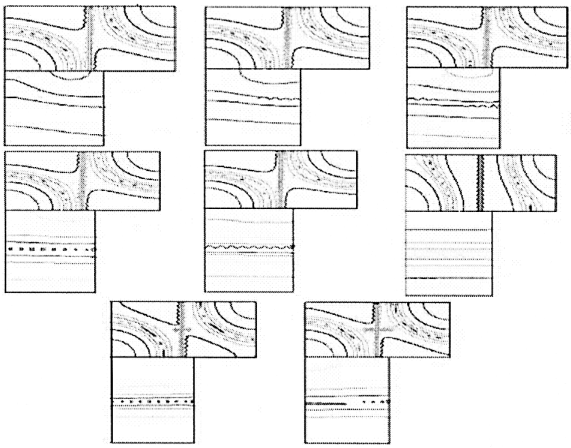

Figure 11. The variation of the H^1_{11} field with respect to the height of the lower cavity increasing from $b_2 = 2.6$ (top-left) to $b_2 = 3.4$ (bottom-right) in a two-cavity resonator with one narrow slot ($2w = 0.001$, $\beta = 0.4548$); the slot position $d = 1.5$, $a_1 = 4.126$, and $b_1 = a_2 = 2.4$.

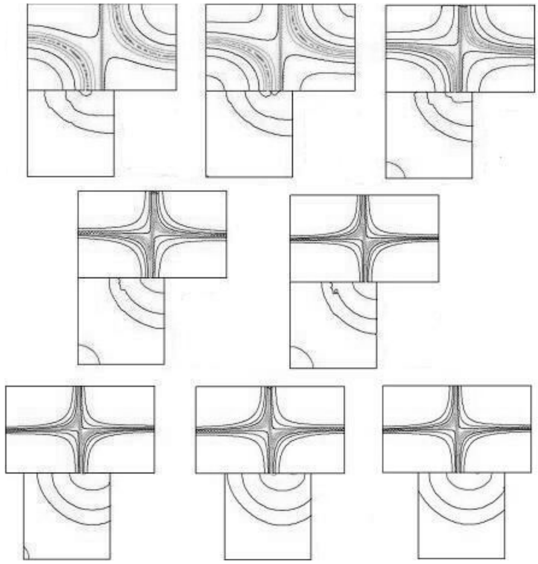

Figure 12. The variations of the H^1_{11} field in a two-cavity resonator with respect to (i) the slot position (the first five pictures) changing from $d = 1.5$ (left in the first line) to $d = 2$ (right in the second line) and (ii) position of the lower cavity 2. The geometric parameters are as in Fig. 10 with $b_2 = 2.6$.

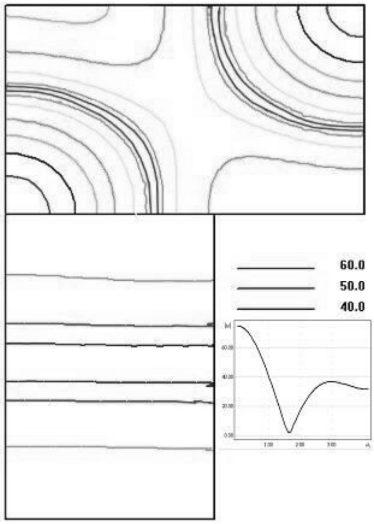

Figure 13. The H_{11}^1 field in a two-cavity resonator with one slot; the slot position $d = 1.5$, the slotwidth $2w = 0.001$ ($\beta = 0.4548$), and the width of the lower cavity $2\ b_2 = 2.351$; other geometric parameters are $a_1 = 4.126$ and $b_1 = a_2 = 2.4$. The insert shows the current on the wall separating the cavities. The horizontal lines with figures indicate the characteristic (maximum) scale of the field amplitude.

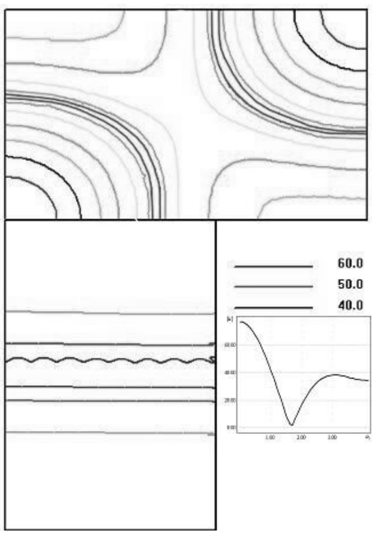

Figure 14. The H_{11}^1 field in a two-cavity resonator with one slot; $b_2 = 2.353$; for other parameters see Fig. 13.

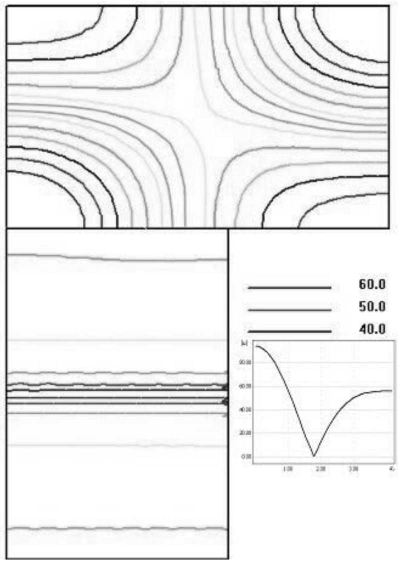

Figure 15. The H_{11}^1 field in a two-cavity resonator with one slot; $b_2 = 2.357$; for other parameters see Fig. 13.

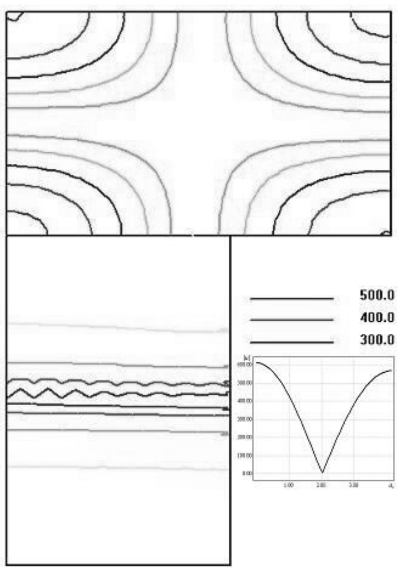

Figure 16. The H_{11}^1 field in a two-cavity resonator with one slot; the slot position $d = 2.0$, $2w = 0.001$ ($\beta = 0.4548$), and the width of the lower cavity 2 $b_2 = 2.353$; for other parameters see Fig. 13.

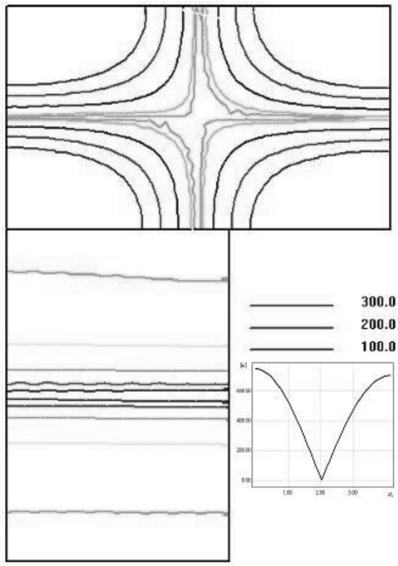

Figure 17. The H_{11}^1 field in a two-cavity resonator with one slot; $d = 2.0$, and $b_2 = 2.357$; for other parameters see Fig. 13.

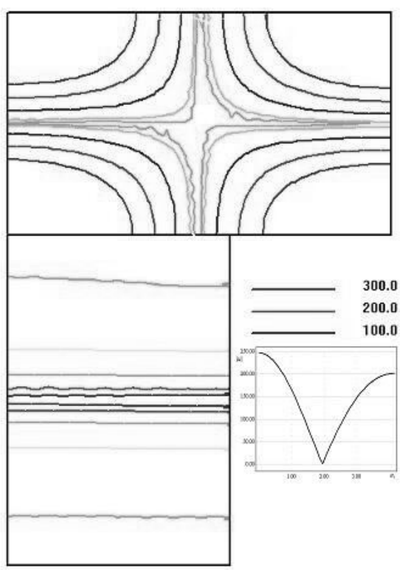

Figure 18. The H_{11}^1 field in a two-cavity resonator with one slot; $d = 2.0$, the slotwidth $2w = 0.02$ ($\beta = 0.6822$), and $b_2 = 2.357$; for other parameters see Fig. 13.

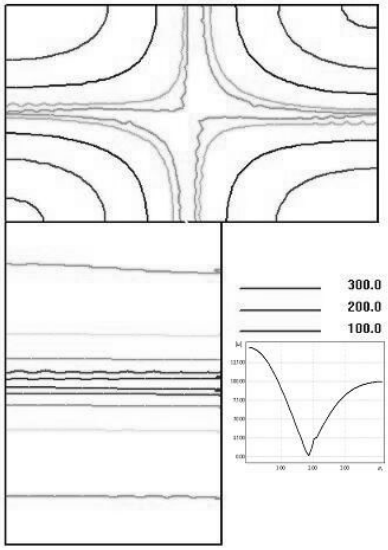

Figure 19. The H_{11}^1 field in a two-cavity resonator with one slot; $d = 2.0$, $2w = 0.03$ ($\beta = 0.7481$), and $b_2 = 2.357$; for other parameters see Fig. 13.

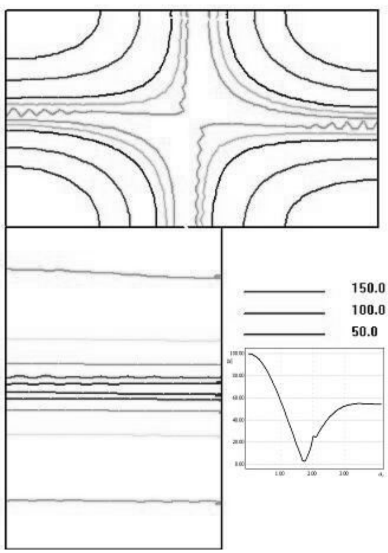

Figure 20. The H_{11}^1 field in a two-cavity resonator with one slot; $d = 2.0$, $2w = 0.04$ ($\beta = 0.8031$), and $b_2 = 2.357$; for other parameters see Fig. 13.

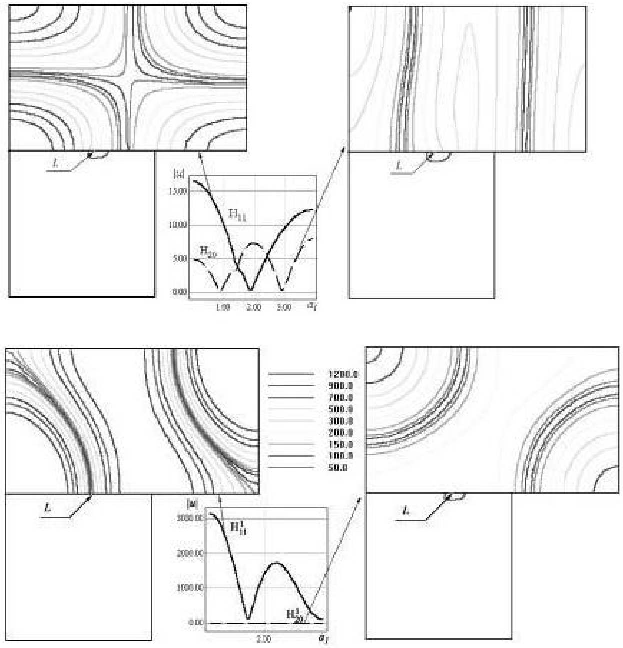

Figure 21. The dynamics of the $H_{11}^1 \leftrightarrows H_{20}^1$ interaction in the left vicinity of the DP $(\lambda^*, a^*) \approx (0.762, 4.126)$ shown in Fig. 1. The geometric parameters are $2w = 0.001$ $(\beta = 0.4548)$, $d = 2$, and $b_1 = a_2 = a_2 = 2.4$.

ered in Section 3.4 is based on the results of calculations performed for the rectangular cavity with one slot in the domain $3.7 < a_1 < 4.2$, $0.6 < \lambda < 1$ of the $H_{11}^1 \leftrightarrows H_{20}^1$ interaction. According to the insert in Fig. 1, there is only one DP in this domain. We consider the influence of the width of rectangular domain Ω^1 (parameter a_1) on the structure of H_{11}^1- and H_{20}^1-types of oscillations, other geometrical parameters and material parameters being fixed.

Figure 23 shows the dependences of $\mu_{11}^{(1)}$ and $\mu_{20}^{(1)}$ and the corresponding SEs. The curves are plotted in the (λ, a_1)-plane, and have the point of intersection when the width of the domain $a_1 = a^* \approx 4.156$. At this point SEs are $\lambda_{11}^{(1)}(a^*) = 0.7616$ and $\lambda_{20}^{(1)}(a^*) = 0.7618$.

Figures 21 and 22 show how the H_{11}^1 and H_{20}^1 oscillations exchange their types as the width a_1 of the upper cavity passes the critical value a^*.

It can be seen however that when the resonator is 'tuned' to an $H_{nm}^{i_0}$ oscillation, strong local changes and the field restructuring occur

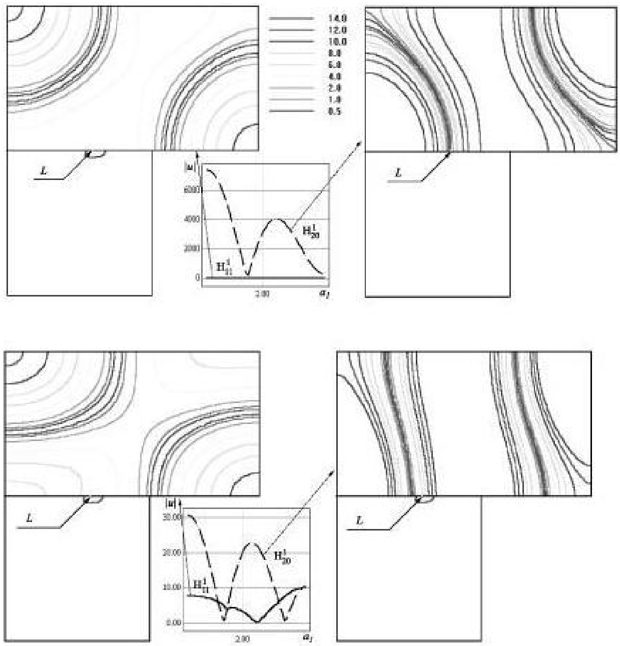

Figure 22. The dynamics of the $H^1_{11} \leftrightarrows H^1_{20}$ interaction in the right vicinity of the DP $(\lambda^*, a^*) \approx (0.762, 4.126)$. The geometric parameters are as in Fig. 21.

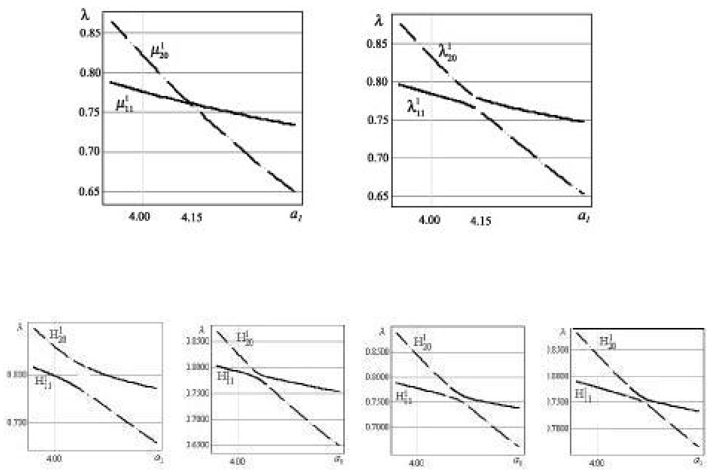

Figure 23. $\mu_{11}^{(1)}$ and $\mu_{20}^{(1)}$ vs. a_1 (top-left graph) and the corresponding SEs in the vicinity of the DP $(\lambda^*, a^*) \approx (0.762, 4.126)$ for different slot positions.

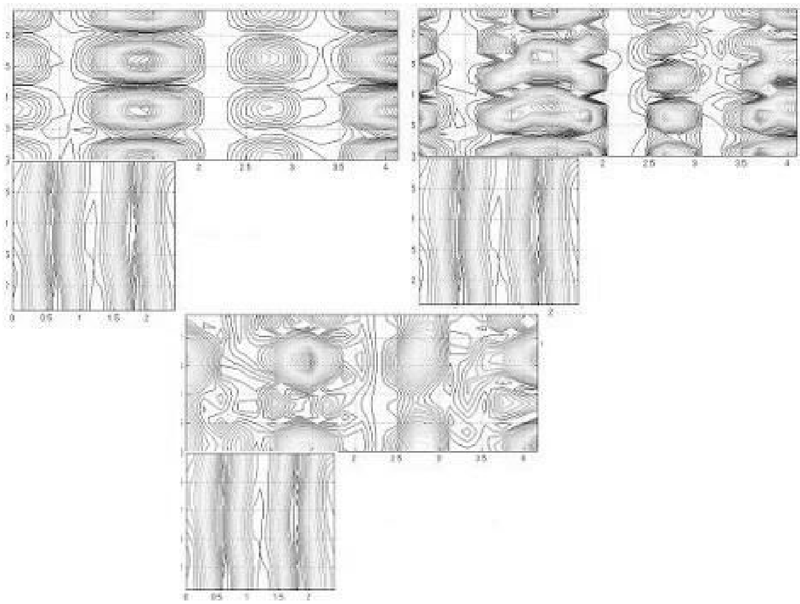

Figure 24. An example of the transition process of interaction in a two-cavity resonator with one exponentially narrow slot.

only in the domain Ω^{i_0}, while in other partial domains Ω^i with $i \neq i_0$ the field distributions remain stable, and also in the vicinity of the critical resonator dimensions. Also, the following conclusions can be formulated.

(i) Obviously, there is no interaction of oscillations in a rectangular cylindrical resonator without a slot and, in particular, in each partial rectangular cavity of the two- and three-cavity resonators under consideration when the slot is absent. On the other hand, a certain form of interaction always takes place in the presence of an arbitrarily narrow slot cut in the wall.

(ii) Interaction takes place when the eigenfrequencies μ^i_{nm} of the same index i merge. As a hypothesis, confirmed by many numerical results, we may state that there is no interaction in the vicinity of the points where $\mu^{i_1}_{n'm'} = \mu^{i_2}_{nm}$ with $i_1 \neq i_2$.

Two more typical examples of critical regimes are shown in Figs. 24 and 25.

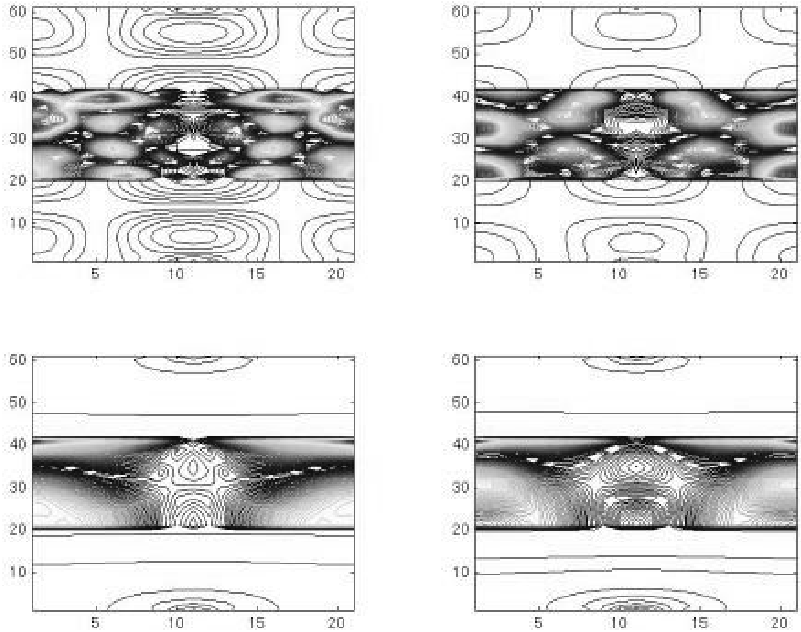

Figure 25. An example of the transition process of interaction in a three-cavity resonator with two exponentially narrow slots cut symmetrically in the lower and upper walls of the central cavity.

5. CONCLUSION

The description of eigenfields in slotted resonators represents a complicated multi-parameter problem where various critical regimes can be described in terms of CPs of the operators of the problem. Different types of CPs lead to different kinds of local effects, such as interaction and mode coupling. It is convenient to study the operator equations using integral OVFs with a logarithmic singularity of the kernel.

In this work, the methods of analytical semi-inversion of these integral OVFs are developed to the case of several intervals of integration. The results are applied to the analysis of a family of slotted resonators. The eigenfrequencies and field distributions are calculated in various critical and regular regimes.

It is shown that the field distributions undergo pronounced variations in type and amplitude in very small vicinities of isolated values of one of the resonator's geometric parameters (e.g., the width (a_1) of one of the partial domains). These values virtually do not depend on the slotwidth (at least when the latter is sufficiently small) and on the position of the lower resonator (in a broad range of variation of the corresponding geometric parameters). Therefore, such characteristic values may be considered, in particular, as constants inherent to the resonator under study (when all other resonator's parameters are fixed). The interaction mode is preserved in a broad range of variation of geometric parameters.

The occurrence of interaction is demonstrated for closed structures as a result of inserting a small inhomogeneity (in the form of a narrow slot).

The results obtained in this study can be used for improving the design of filters and switches on the basis of simple prototype structures.

ACKNOWLEDGMENT

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APPENDIX A.

The coefficients in (57) are determined from the following formulas:

$$d_3(\lambda) = \frac{\ln 2}{\pi}\left(\frac{\ln 2}{\pi} + \frac{M_{11}}{\alpha}\right)\left[\frac{M_{11}}{\alpha} - 1 + \frac{|c_2|^2}{c_0}\frac{\ln 2}{\pi}\right],$$

$$d_4(\lambda) = \frac{\ln 2}{\pi}\left(\frac{\ln 2}{\pi} + \frac{M_{22}}{\alpha}\right)\left[\frac{M_{22}}{\alpha} - 1 + \frac{|c_1|^2}{c_0}\frac{\ln 2}{\pi}\right],$$

$$d_5(\lambda) = \frac{1}{c_0}\left[-\frac{\ln 2}{\pi}(c_0 A_1 A_2 - A_{12}) - \alpha^2\left(\frac{\ln 2}{\pi}\right)^2 \sum_{i=1}^{2}|c_i|^2(A_1 b_2 + A_2 b_{i1} + b_{i12})\right],$$

$$A_i = \frac{\ln 2}{\pi} + \frac{M_{ii}}{\alpha}, \quad A_{12} = \left(\frac{\ln 2}{\pi}\right)^2 + \frac{\ln 2}{\pi\alpha}(M_{11} + M_{22}) + \frac{|M|}{\alpha^2},$$

$$|M| = M_{11}M_{22} - M_{12}M_{21}, \quad b_{ii} = \alpha\frac{\pi}{\ln 2}M_{ii}, \quad i = 1, 2,$$

$$b_{12} = \alpha^2 + b_{22}, \qquad b_{21} = \alpha^2 + b_{11}.$$

The quantity $M_{0,\nu}(\lambda)$ in (104) is represented as

$$M_{0,\nu}(\lambda) = \sum_{i=1}^{2} \varepsilon_i \left\{ \frac{2}{a_i b_i} D_0^{(i)}(\lambda) + \sum_{n=1}^{\infty} \left[\frac{4}{a_i b_i} D_{n,i_0NM}^{(i)}(\lambda) + \frac{2}{\pi} Q_n^{(i)} \right] \right.$$
$$\left. \cdot \cos \frac{\pi n d}{a_i} - \frac{1}{\pi} \ln \left| 2 \sin \frac{\pi d}{a_i} \right| \right\}, \tag{A1}$$

where

$$D_0^{(i)}(\lambda) = \sum_{m=0}^{\infty} \frac{\delta_m}{\mu_{0m}^{(i)} - \lambda}, \quad i = 1,2,$$
$$D_n^{(i)}(\lambda) = \lambda \sum_{m=0}^{\infty} \frac{\delta_m}{\mu_{nm}^{(i)}(\mu_{nm}^{(i)} - \lambda)}, \quad i = 1,2, \quad n = 1,2,\dots,$$
$$D_{n,i_0NM}^{(i)}(\lambda) = \sum_{m=0}^{\infty} \delta_m \left(\frac{1 - \delta_{inm,i_0NM}}{\mu_{nm}^{(i)} - \lambda} - \frac{1}{\mu_{nm}^{(i)}} \right),$$
$$i, i_0 = 1,2, \qquad n = 1,2,\dots, \qquad N, M = 0,1,2,\dots,$$
$$Q_n^{(i)} = Q_n^{(i)}(a_i, b_i) = \frac{e^{-\pi n b_i / a_i}}{\sinh(\pi n b_i / a_i)}, \quad i = 1,2, \quad n = 1,2,\dots. \tag{A2}$$

and δ_{inm,i_0NM} is the triple Kronecker delta which eliminates the term containing the chosen pole λ_ν,

$$\delta_{inm,i_0NM} = \begin{cases} 1, & i = i_0,\ n = N,\ m = M, \\ 0, & \text{otherwise} \end{cases} \tag{A3}$$

so that $M_{0,\nu}(\lambda)$ has no singularity at the separated pole λ_ν in (104). The series for $D_n^{(i)}(\lambda)$, $i = 1,2$, $n = 1,2,\dots$, converge uniformly in every closed domain on the complex λ-plane that does not contain (real) points $\mu_{nm}^{(i)}$ and define therefore meromorphic functions of λ with the respective poles. In addition, one can perform iterated summation in $D_n^{(i)}(\lambda)$ and $D_{n,i_0NM}^{(i)}(\lambda)$ with i, $i_0 = 1,2$ and $n \geq 1$ by virtue of the estimates

$$\left| \frac{1}{\mu_{nm}^{(i)}(\mu_{nm}^{(i)} - \lambda)} \right| \leq \left(\frac{\varepsilon_i a_i^2 b_i^2}{\pi^2} \right)^2 \frac{1}{n^2} \left| \frac{1}{m^2 - (\varepsilon_i b_i^2/\pi^2)\lambda} \right|, \quad i = 1,2, \tag{A4}$$

which hold for sufficiently large n and m.

The questions in (109) and (110) are as follows:

$$N(d_i, d_j, \lambda) - \frac{m_\nu c_{i,\nu}}{\lambda_\nu - \lambda}$$
$$= \sum_{p=1}^{2} \varepsilon_p \Bigg\{ \frac{2}{a_p b_p} D_0^{(p)}(\lambda) + \frac{4}{a_p b_p} \sum_{n=1}^{\infty} D_{n,i_0NM}^{(p)}(\lambda)\varphi_n^{(p,i)}(0) \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(p)} \varphi_n^{(p,i)}(0)$$
$$- \frac{1}{\pi}\left[\ln\left|\frac{2\sin\frac{\pi}{2a_p}(d_j - d_i)}{\frac{\pi}{2a_p}(d_j - d_i)}\right| + \ln\left|2\sin\frac{\pi}{2a_p}(d_j + d_i)\right|\right]\Bigg\}, \qquad \text{(A5)}$$
$$N(d_i, d_i, \lambda) - \frac{m_\nu c_{i,\nu}}{\lambda_\nu - \lambda}$$
$$= \sum_{p=1}^{2} \varepsilon_p \Bigg\{ \frac{2}{a_p b_p} D_0^{(p)}(\lambda) + \frac{4}{a_p b_p} \sum_{n=1}^{\infty} D_{n,i_0NM}^{(p)}(\lambda)\varphi_n^{(p,i)}(0)$$
$$+ \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(p)} \varphi_n^{(p,i)}(0) - \frac{1}{\pi}\left[\ln\left|4\sin\frac{\pi d_i}{a_p}\right|\right]\Bigg\}. \qquad \text{(A6)}$$

The smooth part $N(t_0, t, \lambda)$ of the kernel of $K(\lambda)$ in (131) are represented componentwise using (25) in the form similar to (106):

$$N(w_1 x_0 + d_1, w_1 x + d_1, \lambda) = N^{(11)}(x_0, x, \lambda)$$
$$= \sum_{p=1}^{2} \varepsilon_p \Bigg\{ \frac{2}{a_p b_p} D_0^{(p)}(\lambda) + \frac{4}{a_p b_p} \sum_{n=1}^{\infty} D_n^{(p)}(\lambda)\varphi_n^{(p,1)}(x_0)\varphi_n^{(p,1)}(x)$$
$$+ \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(p)} \varphi_n^{(p,1)}(x_0)\varphi_n^{(p,1)}(x) - \frac{1}{\pi} \ln\left|\frac{2\sin\frac{\pi w_1}{2a_p}[(x - x_0)]}{\frac{\pi w_1}{2a_p}[(x - x_0)]}\right|$$
$$- \frac{1}{\pi}\ln\left|2\sin\frac{\pi}{2a_p}[w_1(x + x_0) + 2d_1]\right|\Bigg\}; \qquad \text{(A7)}$$

$$N(w_2 x_0 + d_2, w_2 x + d_2, \lambda) = N^{(22)}(x_0, x, \lambda)$$
$$= \varepsilon_1 \Bigg\{ \frac{2}{a_1 b_1} D_0^{(1)}(\lambda) + \frac{4}{a_1 b_1} \sum_{n=1}^{\infty} D_n^{(1)}(\lambda)\varphi_n^{(1,2)}(x_0)\varphi_n^{(1,2)}(x)$$
$$+ \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(1)} \varphi_n^{(1,2)}(x_0)\varphi_n^{(1,2)}(x) - \frac{1}{\pi} \ln\left|\frac{2\sin\frac{\pi w_2}{2a_1}[(x - x_0)]}{\frac{\pi w_2}{2a_1}[(x - x_0)]}\right|$$

$$
\begin{aligned}
&- \frac{1}{\pi} \ln \left| 2 \sin \frac{\pi}{2a_1} [w_2(x + x_0) + 2d_2] \right| \Bigg\} \\
&+ \varepsilon_3 \Bigg\{ \frac{2}{a_3 b_3} D_0^{(3)}(\lambda) + \frac{4}{a_3 b_3} \sum_{n=1}^{\infty} D_n^{(3)}(\lambda) \varphi_n^{(3,2)}(x_0) \varphi_n^{(3,2)}(x) \\
&+ \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(3)} \varphi_n^{(3,2)}(x_0) \varphi_n^{(3,2)}(x) - \frac{1}{\pi} \ln \left| \frac{2 \sin \frac{\pi w_2}{2a_3} [(x - x_0)]}{\frac{\pi w_2}{2a_3} [(x - x_0)]} \right| \\
&- \frac{1}{\pi} \ln \left| 2 \sin \frac{\pi}{2a_3} [w_2(x + x_0) + 2d_2] \right| \Bigg\}, \qquad \text{(A8)}
\end{aligned}
$$

$$
\begin{aligned}
&N(w_1 x_0 + d_1, w_2 x + d_2, \lambda) = N^{(12)}(x_0, x, \lambda) \\
&\quad = -\varepsilon_1 \Bigg\{ \frac{\alpha}{\pi} \ln \frac{1}{|w_1 x_0 - w_2 x + (d_1 - d_2)|} + \frac{2}{a_1 b_1} D_0^{(1)}(\lambda) \\
&\quad + \frac{4}{a_1 b_1} \sum_{n=1}^{\infty} D_n^{(1)}(\lambda) \varphi_n^{(1,1)}(x_0) \varphi_n^{(1,2)}(x) \\
&\quad + \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(1)} \varphi_n^{(1,1)}(x_0) \varphi_n^{(1,2)}(x) \\
&\quad - \frac{1}{\pi} \ln \left| \frac{2 \sin \frac{\pi}{2a_1} [(w_2 x + d_2) - (w_1 x_0 + d_1)]}{\frac{\pi}{2a_1} [(w_2 x + d_2) - (w_1 x_0 + d_1)]} \right| \\
&\quad - \frac{1}{\pi} \ln \left| 2 \sin \frac{\pi}{2a_1} [(w_2 x + d_2) + (w_1 x_0 + d_1)] \right| \Bigg\}, \qquad \text{(A9)}
\end{aligned}
$$

$$
\begin{aligned}
&N(w_2 x_0 + d_2, w_1 x + d_1, \lambda) = N^{(21)}(x_0, x, \lambda) \\
&\quad = -\varepsilon_1 \Bigg\{ \frac{1}{\pi} \ln \frac{1}{|w_1 x_0 - w_2 x + (d_1 - d_2)|} + \frac{2}{a_1 b_1} D_0^{(1)}(\lambda) \\
&\quad - \frac{1}{\pi} \ln \left| \frac{2 \sin \frac{\pi}{2a_1} [(w_2 x + d_2) - (w_1 x_0 + d_1)]}{\frac{\pi}{2a_1} [(w_2 x + d_2) - (w_1 x_0 + d_1)]} \right| \\
&\quad - \frac{1}{\pi} \ln \left| 2 \sin \frac{\pi}{2a_1} [(w_2 x + d_2) + (w_1 x_0 + d_1)] \right| \Bigg\}. \qquad \text{(A10)}
\end{aligned}
$$

Here

$$\varphi_n^{(1,j)}(x) = \cos\frac{\pi n(w_j x + d_j)}{a_1},$$
$$\varphi_n^{(i,j)}(x) = \cos\frac{\pi n(w_j x + d_j - a_{1i})}{a_3}, \quad j = 1,2, \quad i = 2,3, \quad n = 1,2,\dots .$$

The quantities $N(d_i, d_j, \lambda) - m_\nu c_{i,\nu}/(\lambda_\nu - \lambda)$, $i, j = 1, 2$, that enter the DE in the case of a resonator with two slots cut in opposite walls are represented as follows:

$$\begin{aligned} N(d_1, d_1, \lambda) - \frac{m_\nu c_{1,\nu}}{\lambda_\nu - \lambda} &= \sum_{p=1}^{2} \varepsilon_p \Bigg\{ \frac{2}{a_p b_p} D_0^{(p)}(\lambda) + \frac{4}{a_p b_p} \sum_{n=1}^{\infty} D_{n,i_0NM}^{(p)}(\lambda) \varphi_n^{(p,1)}(0) \\ &\quad + \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(p)} \varphi_n^{(p,1)}(0) - \frac{1}{\pi} \ln \left| 4 \sin\frac{\pi d_1}{a_p} \right| \Bigg\}; \end{aligned} \tag{A11}$$

$$\begin{aligned} N(d_2, d_2, \lambda) - \frac{m_\nu c_{2,\nu}}{\lambda_\nu - \lambda} &= \sum_{p=1,3} \varepsilon_p \Bigg\{ \frac{2}{a_p b_p} D_0^{(p)}(\lambda) + \frac{4}{a_p b_p} \sum_{n=1}^{\infty} D_{n,i_0NM}^{(p)}(\lambda) \varphi_n^{(p,2)}(0) \\ &\quad + \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(p)} \varphi_n^{(p,2)}(0) - \frac{1}{\pi} \ln \left| 4 \sin\frac{\pi d_2}{a_p} \right| \Bigg\}; \end{aligned} \tag{A12}$$

$$\begin{aligned} N(d_1, d_2, \lambda) - \frac{m_\nu c_{1,\nu}}{\lambda_\nu - \lambda} &= -\varepsilon_1 \Bigg\{ \frac{2}{a_1 b_1} D_0^{(1)}(\lambda) + \frac{4}{a_1 b_1} \sum_{n=1}^{\infty} D_{n,i_0NM}^{(1)}(\lambda) \varphi_n^{(1,1)}(0) \\ &\quad + \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(1)} \varphi_n^{(1,1)}(0) \\ &\quad - \frac{1}{\pi} \left[\ln \left| \frac{2 \sin\frac{\pi}{2a_p}(d_2 - d_1)}{\frac{\pi}{2a_p}(d_2 - d_1)} \right| + \ln \left| 2 \sin\frac{\pi}{2a_p}(d_2 + d_1) \right| \right] \Bigg\}; \end{aligned} \tag{A13}$$

$$
\begin{aligned}
N(d_2, d_1, \lambda) &- \frac{m_\nu c_{2,\nu}}{\lambda_\nu - \lambda} \\
&= -\varepsilon_1 \left\{ \frac{2}{a_1 b_1} D_0^{(1)}(\lambda) + \frac{4}{a_1 b_1} \sum_{n=1}^{\infty} D_{n,i_0NM}^{(1)}(\lambda) \varphi_n^{(1,2)}(0) \right. \\
&\quad + \frac{2}{\pi} \sum_{n=1}^{\infty} Q_n^{(1)} \varphi_n^{(1,2)}(0) \\
&\quad \left. - \frac{1}{\pi} \left[\ln \left| \frac{2 \sin \frac{\pi}{2a_p}(d_2 - d_1)}{\frac{\pi}{2a_p}(d_2 - d_1)} \right| + \ln \left| 2 \sin \frac{\pi}{2a_p}(d_2 + d_1) \right| \right] \right\}.
\end{aligned}
\tag{A14}
$$

APPENDIX B.

$M_{0,\nu}(\lambda)$ in (104) is a meromorphic function of λ with first-order poles at the points $\mu_{nm}^{(i)}$ ($\{inm\} \neq \{i_0NM\}$, $n, m = 0, 1, 2, \ldots$, $i = 1, 2$). However, the function $F_\nu(\lambda)$ in (104) has removable singularities at these points. Indeed, according to (104), $F_\nu(\lambda)$ can be represented in the vicinity of every chosen first-order pole $\mu_p = \mu_{N_1M_1}^{(I)}$ ($p = 1, 2, \ldots$, $N_1, M_1 = 0, 1, 2, \ldots$, $I = 1, 2$) as

$$
F_\nu(\lambda) = \lambda_\nu + \frac{f_\nu}{c_p(\mu_p - \lambda)^{-1} + M_{\nu,p}(\lambda)} = \lambda_\nu + \frac{f_\nu(\mu_p - \lambda)}{c_p + (\mu_p - \lambda) M_{\nu,p}}, \tag{B1}
$$

where $\lambda \neq \mu_p$,

$$
c_p = c_{N_1M_1}^{(I)} = \delta_{M_1} \frac{4\varepsilon_I}{a_I b_I} \cos \frac{\pi N_1 d}{a_I}, \quad c_p \neq 0, \tag{B2}
$$

and $M_{\nu,p}$ has no singularity at $\lambda = \mu_p$. Representation (B1) enables one to extend the definition of $F_\nu(\lambda)$ by continuity to the point μ_p by setting

$$
F_\nu(\mu_p) = \lambda_\nu, \quad p = 1, 2, \ldots . \tag{B3}
$$

If $c_p = 0$ which holds, for a fixed $N_I = 1, 2, \ldots$, at

$$
d = d_{m,*} = \frac{a_I(m + 0.5)}{N_I}, \quad m + 0.5 < N_I, \quad m = 1, 2, \ldots , \tag{B4}
$$

then $M_{\nu,p}$ and $F_\nu(\lambda)$ has no singularity at $\lambda = \mu_p$.

Consider the situation when two first-order poles μ_1 and μ_2 of $M_{0,\nu}(\lambda)$ merge, $\mu_1 = \mu_2$, at certain values of parameters, e.g., when b_1, a_2, b_2, d, ε_1, ε_2, and w are fixed and a_1 is varied (as in [5]). Then $\mu_1 = \mu_1(a_1)$ and $\mu_2 = \mu_2(a_1)$ and $\mu_1(a_1) = \mu_2(a_1) = \mu^*$ (merge) at a certain critical value $a_1 = a^*$. Represent $F_\nu(\lambda)$ as

$$\begin{aligned} F_\nu(\lambda) &= \lambda_\nu + \frac{f_\nu}{c_1(\mu_1-\lambda)^{-1} + c_2(\mu_2-\lambda)^{-1} + M_{\nu,12}(\lambda)} \\ &= \lambda_\nu + \frac{f_\nu(\mu_1-\lambda)(\mu_2-\lambda)}{c_1(\mu_2-\lambda) + c_2(\mu_1-\lambda) + (\mu_1-\lambda)(\mu_2-\lambda)M_{\nu,12}(\lambda)}, \end{aligned} \tag{B5}$$

where $\lambda \neq \mu_k$, $k = 1, 2$,

$$c_k = c^{(I_k)}_{N_k M_k} = \delta_{M_k} \frac{4\varepsilon_{I_k}}{a_{I_l} b_{I_k}} \cos\frac{\pi N_k d}{a_{I_k}}, \quad k = 1, 2, \tag{B6}$$

and $M_{\nu,12}$ has no singularity at $\lambda = \mu_k$, $k = 1, 2$. In order to extend the definition of $F_\nu(\lambda)$ by continuity to the points $\lambda = \mu_k$, $k = 1, 2$, and $\lambda = \mu^*$, which is valid at $a_1 = a^*$, we have to distinguish two cases (i) $c_1 + c_2 \neq 0$ (regular case) and (ii) $c_1 + c_2 = 0$ (singular case) and use the following representations for $F_\nu(\lambda)$ at $a_1 = a^*$ and $\mu_1(a^*) = \mu_2(a^*) = \mu^*$:

$$F_\nu(\lambda) = \begin{cases} \lambda_\nu + \dfrac{f_\nu(\mu^*-\lambda)}{c_1 + c_2 + (\mu^*-\lambda)M_{\nu,12}(\lambda)}, & c_1 + c_2 \neq 0, \\ \lambda_\nu + \dfrac{f_\nu}{M_{\nu,12}(\mu^*)}, & c_1 + c_2 = 0, \end{cases} \tag{B7}$$

so that

$$F_\nu(\mu^*) = \begin{cases} \lambda_\nu, & c_1 + c_2 \neq 0, \\ \lambda_\nu + \dfrac{f_\nu}{M_{\nu,12}(\mu^*)}, & c_1 + c_2 = 0. \end{cases} \tag{B8}$$

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