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MAGNETIC FORCE BETWEEN INCLINED CIRCULAR LOOPS (LORENTZ APPROACH)

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Abstract—This paper presents a new general formula for calculating the magnetic force between inclined circular loops placed in any desired position. This formula has been derived from the Lorentz force equation. All mathematical procedures are completely described to define the coil positions that lead to a relatively easy method for calculating the magnetic force between inclined circular loops in any desired position. The presented method is easy to understand, numerically suitable and easily applicable for engineers and physicists. The obtained formula is given in its simplest form from the already existing formulas for calculating the magnetic force between inclined circular loops. We validated the new formula through a series of examples, which are presented here.

1. INTRODUCTION

The calculation of the inductance and magnetic force through analytical or semi-analytical methods has received considerable attention in recent years, and much progress has been made [1–27]. The vast majority of this attention has been given to the purely coaxial geometries [9–25]. More recently the focus has been shifted to calculation of the mutual inductance and magnetic force between circular coils with lateral and angular misalignments [1–15]. Using the powerful numerical methods, such as Finite Element Method (FEM) and Boundary Element Method (BEM) [27, 28], it is possible to calculate accurately and rapidly this important physical quantity. Also, this problem can be tackled by some semi-analytical methods that can considerably reduce the computational time and the enormous mathematical procedures. In this paper we give a new formula of

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the magnetic force between inclined circular loops in any desired positions. This force obtained by using the Lorentz law represents the simplification of the magnetic force between treated coils obtained by the Biot-Savart law and by the approach of the mutual inductance [1]. We treated the most general case using some elementary mathematical transformations to describe coil positions in different planes. The magnetic force has been obtained by a simple integral whose kernel function contains some combinations of the complete elliptic integrals of the first and second kind. The obtained new formula for the magnetic force is very suitable for numerical treatment. The presented formula can be easily used in the calculation of the magnetic force between inclined circular coils of rectangular cross section using the filament method.

2. BASIC EXPRESSIONS

Let's take into consideration two circular loops as showed in Figure 1, where the center of the larger circle (primary coil) of the radius R_P is placed at the plane XOY with the axis of Z along the axis of this circle. The smaller circle (secondary coil) of the radius R_S is placed in an inclined plane (the plane $x'Cy'$) whose general equation is,

$$\lambda \equiv ax + by + cz + D = 0 \tag{1}$$

Also, we define the center of the secondary coil in the plane λ, $C(x_C, y_C, z_C)$ and a point D_S on this circle, $D_S(x_0, y_0, z_0)$. The coordinates of the point D_S can be given by the couple $D_1[x_C - abR_S/(Ll), y_C + (a^2 + c^2)R_S/(Ll), z_C - bcR_S/(Ll)]$ or $D_2[x_C + abR_S/(Ll), y_C - (a^2 + c^2)R_S/(Ll), z_C + bcR_S/(Ll)]$, where $L = (a^2 + b^2 + c^2)^{0.5}$, $l = (a^2 + c^2)^{0.5}$ [9].

Thus, we define the positions of two coils in 3D space that will permit us to calculate all necessary parameters in the calculation of the magnetic force between them.

For coils (see Figure 1) we define:

1) The primary coil of radius R_P is placed in the plane XOY ($Z = 0$) with the center at O (0, 0, 0).

An arbitrary point B_P (x_P, y_P, z_P) of this coil has parametric coordinates,

$$\begin{aligned} x_P &= R_P \cos t \\ y_P &= R_P \sin t \\ z_P &= 0 \qquad t \in (0, 2\pi) \end{aligned} \tag{2}$$

2) The differential element of the primary coil is given by,

$$d\vec{l}_P = R_P \left(-\sin t\vec{i} + \cos t\vec{j} \right) dt, \quad t \in (0, 2\pi) \tag{3}$$

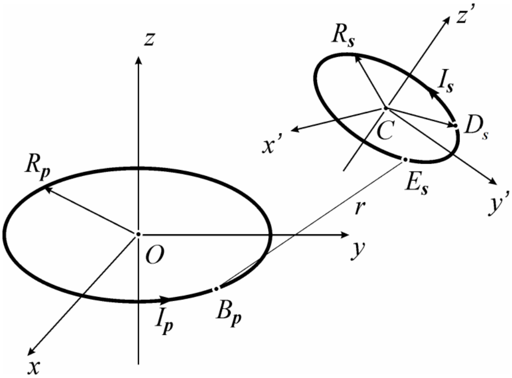

Figure 1. Filamentary circular coils with angular and lateral misalignment.

3) The unit vector N (the unit vector of the axis z') at the point C (center of the secondary coil) laying in the plane λ, is defined by,

$$\vec{N} = \{n_x, n_y, n_z\} = \left\{\frac{a}{|\vec{n}|}, \frac{b}{|\vec{n}|}, \frac{c}{|\vec{n}|}\right\}$$
$$|\vec{n}| = \left(a^2 + b^2 + c^2\right)^{\frac{1}{2}} = L \tag{4}$$

4) The unit vector between two points C and D_S they are placed in the plane λ is,

$$\vec{u} = \{u_x, u_y, u_z\} = \left\{-\frac{ab}{lL}, \frac{l}{L}, -\frac{bc}{lL}\right\}$$
$$l = \left(a^2 + c^2\right)^{\frac{1}{2}}, \quad L = \left(a^2 + b^2 + c^2\right)^{\frac{1}{2}} \tag{5}$$

5) We define the unit vector v as the cross product of the unit vectors N and u as follows,

$$\vec{v} = \vec{N} \times \vec{u} = \{v_x, v_y, v_z\} = \left\{-\frac{c}{l}, 0, \frac{a}{l}\right\}$$
$$l = \left(a^2 + c^2\right)^{\frac{1}{2}} \tag{6}$$

6) An arbitrary point E_S (x_S, y_S, z_S) of the secondary coil has parametric coordinates,

$$\begin{aligned} x_S &= x_C + R_S u_x \cos\phi + R_S v_x \sin\phi \\ y_S &= y_C + R_S u_y \cos\phi + R_S v_y \sin\phi \quad \phi \in (0, 2\pi) \\ z_S &= z_C + R_S u_z \cos\phi + R_S v_z \sin\phi \end{aligned} \tag{7}$$

This is well-known the parametric equation of circle in 3D space.

7) The differential element of the secondary coil is given by,

$$\begin{aligned} d\vec{l}_S &= R_S \left[l_{Sx}\vec{i} + l_{Sz}\vec{j} + l_{Sz}\vec{k} \right] d\phi \quad \phi \in (0, 2\pi) \\ l_{Sx} &= -u_x \sin\phi + v_x \cos\phi \\ l_{Sy} &= -u_y \sin\phi + v_y \cos\phi \\ l_{Sz} &= -u_z \sin\phi + v_z \cos\phi \end{aligned} \tag{8}$$

3. CALCULATION METHOD

From the Lorentz low the magnetic force between two line elements can be calculated by [3],

$$d\vec{F} = I_S d\vec{l}_S \times \vec{B}_P \left(\vec{l}_S \right) \tag{9}$$

where dF is the force acting on the line element dl_S due to an externally imposed magnetic field $B_P(l_S)$. Integrating Equation (9) around a closed circuit C gives the force on the circuit due to the magnetic field $B_P(l_S)$ as,

$$\vec{F} = I_S \oint_C d\vec{l}_S \times \vec{B}_P \left(\vec{l}_S \right) \tag{10}$$

The magnetic field $B_P(l_S)$ produced by the primary coil of the radius R_P carrying the current I_P can be calculated in an arbitrary point D_S (x_S, y_S, z_S) by,

$$\vec{B}_P \left(\vec{l}_S \right) = \frac{\mu_0 I_p}{4\pi} \oint_{l_P} \frac{d\vec{l}_P \times \vec{r}_{SP}}{r_{SP}^3} \tag{11}$$

where

$$\begin{aligned} r_{SP}^2 &= (x_S - x_P)^2 + (y_S - y_P)^2 + (z_S - z_P)^2 \\ &= R_P^2 + x_S^2 + y_S^2 + z_S^2 - 2R_P \sqrt{x_S^2 + y_S^2} \cos(t - \gamma) \end{aligned}$$

$$\cos\gamma = \frac{x_S}{\sqrt{x_S^2 + y_S^2}}, \quad \sin\gamma = \frac{y_S}{\sqrt{x_S^2 + y_S^2}}$$

x_S, y_S, z_S are given by (7) and dl_P by (3).

$\mu_0 = 4\pi \times 10^{-7}$ H/m — the permeability of free space (vacuum).

Equation (11) can be written by three components as follows,

$$B_x(x_S, y_S, z_S) = \frac{\mu_0 I_P R_P z_S}{4\pi} \int_0^{2\pi} \frac{\cos t}{r_{SP}^3} dt$$

$$
\begin{aligned}
B_y(x_S, y_S, z_S) &= \frac{\mu_0 I_P R_P z_S}{4\pi} \int_0^{2\pi} \frac{\sin t}{r_{SP}^3} dt \\
B_z(x_S, y_S, z_S) &= \frac{\mu_0 I_P R_P}{4\pi} \int_0^{2\pi} \frac{\left[R_P - \sqrt{x_S^2 + y_S^2}\cos(t-\gamma)\right]}{r_{SP}^3} dt
\end{aligned} \tag{12}
$$

Introducing the substitution $t - \gamma = \pi - 2\beta$ expression (2) can be obtained in the form,

$$
\begin{aligned}
B_x(x_S, y_S, z_S) &= T_0 z_S \cos\gamma \int_0^{2\pi} \frac{\left(1 - 2\cos^2\beta\right)}{\Delta^3} d\beta \\
B_y(x_S, y_S, z_S) &= T_0 z_S \sin\gamma \int_0^{2\pi} \frac{\left(1 - 2\cos^2\beta\right)}{\Delta^3} d\beta \\
B_z(x_S, y_S, z_S) &= -T_0 \int_0^{2\pi} \frac{\left(\sqrt{x_S^2+y_S^2} - R_P - 2\sqrt{x_S^2+y_S^2}\cos^2\beta\right)}{\Delta^3} d\beta \\
T_0 &= \frac{\mu_0 I_P R_P}{\pi\left[(R_P + \sqrt{x_S^2+y_S^2})^2 + z_S^2\right]^{3/2}} \\
\Delta &= \sqrt{1 - k^2\sin^2\beta} \\
k^2 &= \frac{4R_P\sqrt{x_S^2+y_S^2}}{\left(R_P + \sqrt{x_S^2+y_S^2}\right)^2 + z_S^2}
\end{aligned} \tag{13}
$$

Resolving integrals in (13) we obtained the final form of the magnetic field in an arbitrary point D_S (x_S, y_S, z_S) produced by the primary coil of the radius R_P carrying the current I_P,

$$
\begin{aligned}
B_x(x_S, y_S, z_S) &= -\frac{\mu_0 I_P z_S x_S k}{8\pi\sqrt{R_P}\left(x_S^2+y_S^2\right)^{5/4}} L_0 \\
B_y(x_S, y_S, z_S) &= -\frac{\mu_0 I_P z_S y_S k}{8\pi\sqrt{R_P}\left(x_S^2+y_S^2\right)^{5/4}} L_0 \\
B_z(x_S, y_S, z_S) &= \frac{\mu_0 I_P k}{8\pi\sqrt{R_P}\left(x_S^2+y_S^2\right)^{3/4}} S_0
\end{aligned} \tag{14}
$$

where

$$L_0 = 2K(k) - \frac{2-k^2}{1-k^2}E(k)$$

$$S_0 = 2\sqrt{x_S^2+y_S^2}K(k) - \frac{2\sqrt{x_S^2+y_S^2} - \left(R_P + \sqrt{x_S^2+y_S^2}\right)k^2}{1-k^2}E(k)$$

$K(k)$ — complete elliptic integral of the first kind [29, 30];

$E(k)$ — complete elliptic integral of the second kind [29, 30].

Applying Equation (10) magnetic force components are,

$$\begin{aligned} F_x &= I_S R_S \int_0^{2\pi} \left[B_z(x_S, y_S, z_S) l_{Sy} - B_y(x_S, y_S, z_S) l_{Sz}\right] d\phi \\ F_y &= I_S R_S \int_0^{2\pi} \left[B_x(x_S, y_S, z_S) l_{Sz} - B_z(x_S, y_S, z_S) l_{Sx}\right] d\phi \\ F_z &= I_S R_S \int_0^{2\pi} \left[B_y(x_S, y_S, z_S) l_{Sx} - B_x(x_S, y_S, z_S) l_{Sy}\right] d\phi \end{aligned} \tag{15}$$

where B_x, B_y and B_z are given by (14) and l_{Sx}, l_{Sy} and l_{Sz} by (8).

After some transformations we obtain from (10) the final form of magnetic force components,

$$\begin{aligned} F_x &= \frac{\mu_0 I_P I_S R_S}{8\pi\sqrt{R_P}} \int_0^{2\pi} I_x d\phi \\ F_y &= \frac{\mu_0 I_P I_S R_S}{8\pi\sqrt{R_P}} \int_0^{2\pi} I_y d\phi \\ F_z &= \frac{\mu_0 I_P I_S R_S}{8\pi\sqrt{R_P}} \int_0^{2\pi} I_z d\phi \end{aligned} \tag{16}$$

where

$$\begin{aligned} I_x &= \frac{k}{(x_S^2+y_S^2)^{3/4}} \left[\frac{z_S y_S}{\sqrt{x_S^2+y_S^2}} L_0 l_{Sz} + S_0 l_{Sy}\right] \\ &= \frac{k}{\left(x_S^2+y_S^2\right)^{5/4}} \left[z_S y_S l_{Sz} L_0 + \sqrt{x_S^2+y_S^2} l_{Sy} S_0\right] \end{aligned}$$

$$
\begin{aligned}
I_y &= -\frac{k}{\left(x_S^2+y_S^2\right)^{3/4}}\left[\frac{z_S x_S}{\sqrt{x_S^2+y_S^2}}L_0 l_{Sz}+S_0 l_{Sx}\right] \\
&= -\frac{k}{\left(x_S^2+y_S^2\right)^{5/4}}\left[z_S x_S l_{Sz} L_0+\sqrt{x_S^2+y_S^2}\, l_{Sx} S_0\right] \\
I_z &= \frac{k z_S L_0}{\left(x_S^2+y_S^2\right)^{5/4}}\left[x_S l_{Sy}-y_S l_{Sx}\right] \\
&= \frac{k}{\left(x_S^2+y_S^2\right)^{5/4}} z_S\left[x_S l_{Sy}-y_S l_{Sx}\right] L_0 \\
k^2 &= \frac{4R_P\sqrt{x_S^2+y_S^2}}{\left(R_P+\sqrt{x_S^2+y_S^2}\right)^2+z_S^2} \\
L_0 &= 2K(k)-\frac{2-k^2}{1-k^2}E(k) \\
S_0 &= 2\sqrt{x_S^2+y_S^2}K(k)-\frac{2\sqrt{x_S^2+y_S^2}-\left(R_P+\sqrt{x_S^2+y_S^2}\right)k^2}{1-k^2}E(k)
\end{aligned}
$$

$(u_x,\ u_y,\ u_z)$; $(v_x,\ v_y,\ v_z)$; $(x_S,\ y_S,\ z_S)$ and $(l_{Sx},\ l_{Sy},\ l_{Sz})$ are given by (5), (6), (7) and (8) respectively.

3.1. Special Cases

a) $b = a = 0$, $c = 1$ (The secondary coil is parallel to the plane XOY). Even though this case is directly covered by the general case (16) we give the complete expressions of the magnetic force because this case is involved in many technical applications. For this case,

$$
\begin{aligned}
& x_S = x_C - R_S\sin\phi \quad l_{Sx} = -\cos\phi \\
& y_S = y_C + R_S\cos\phi \quad l_{Sy} = -\sin\phi \\
& z_S = z_C \quad l_{Sz} = 0 \\
& I_x = -\frac{k\sin\phi}{\left(x_S^2+y_S^2\right)^{3/4}}S_0 \\
& I_y = \frac{k\cos\phi}{\left(x_S^2+y_S^2\right)^{3/4}}S_0 \\
& I_z = \frac{k z_C\left[R_S - x_C\sin\phi - y_C\cos\phi\right]L_0}{\left(x_S^2+y_S^2\right)^{5/4}}
\end{aligned}
\tag{17}
$$

$$k^2 = \frac{4R_P\sqrt{x_S^2 + y_S^2}}{\left(R_P + \sqrt{x_S^2 + y_S^2}\right)^2 + z_C^2}$$

In the case of two coaxial coils (the center of the secondary coil is laying on the Z axis, $x_C = y_C = 0$) the magnetic force components are,

$$\begin{aligned} F_x &= 0 \\ F_y &= 0 \\ F_z &= \frac{\mu_0 I_P I_S k z_C}{4\sqrt{R_P R_S}} L_0 \\ k^2 &= \frac{4R_P R_S}{(R_P + R_S)^2 + z_C^2} \end{aligned} \tag{18}$$

b) $b = c = 0$, $a = 1$ (The secondary coil is parallel to the plane YOZ). This case is also directly covered by the general case (16). For this case we have,

$$\begin{aligned} x_S &= x_C \quad l_{Sx} = 0 \\ y_S &= y_C + R_S \cos\phi \quad l_{Sy} = -\sin\phi \\ z_S &= z_C + R_S \sin\phi \quad l_{Sz} = \cos\phi \\ I_x &= \frac{k}{\left(x_C^2 + y_S^2\right)^{3/4}} \left[\frac{z_S y_S \cos\phi}{\sqrt{x_C^2 + y_S^2}} L_0 - \sin\phi S_0\right] \\ I_y &= -\frac{k z_S x_C \cos\phi}{\left(x_C^2 + y_S^2\right)^{5/4}} L_0 \\ I_z &= -\frac{k z_S x_C \sin\phi L_0}{\left(x_C^2 + y_S^2\right)^{5/4}} \\ k^2 &= \frac{4R_P\sqrt{x_C^2 + y_S^2}}{\left(R_P + \sqrt{x_C^2 + y_S^2}\right)^2 + z_S^2} \end{aligned} \tag{19}$$

c) $a = c = 0$, $b = 1$ (The secondary coil is parallel to the plane XOZ).

This special case is not covered by the general case (17) and it is necessary to carefully solve it. For this special case we keep (16) with changes,

$$\begin{aligned} x_S &= x_C + R_S \sin\phi \quad l_{Sx} = \cos\phi \\ y_S &= y_C \quad l_{Sy} = 0 \\ z_S &= z_C + R_S \cos\phi \quad l_{Sz} = -\sin\phi \end{aligned}$$

$$
\begin{aligned}
I_x &= -\frac{k z_S y_C \sin\phi}{\left(x_S^2 + y_C^2\right)^{5/4}} L_0 \\
I_y &= -\frac{k}{\left(x_S^2 + y_C^2\right)^{3/4}} \left[-\frac{z_S x_S \sin\phi}{\sqrt{x_S^2 + y_S^2}} L_0 + S_0 \cos\phi \right] \\
I_z &= -\frac{k \cos\phi z_S y_C}{\left(x_S^2 + y_C^2\right)^{5/4}} L_0 \\
k^2 &= \frac{4 R_P \sqrt{x_S^2 + y_C^2}}{\left(R_P + \sqrt{x_S^2 + y_C^2}\right)^2 + z_S^2}
\end{aligned} \tag{20}
$$

4. EXAMPLES

To verify the validity of the presented formula for the magnetic force, we present the following set of examples.

Example 1.

The center of the primary coil of the radius $R_P = 0.2\,\mathrm{m}$ is O (0; 0; 0) and the center of the secondary coil of the radius $R_S = 0.1\,\mathrm{m}$ is C ($x_C = 0.1\,\mathrm{m}$; $y_C = 0.1\,\mathrm{m}$; $z_C = 0.1\,\mathrm{m}$). The secondary coil is located in the plane $x + y + z = 0.3$. Calculate the magnetic force between coils. All currents are unit.

The magnetic force obtained by [1] is,

$$
\begin{aligned}
F_x &= -0.1080729656128444\,\mu\mathrm{N} \\
F_y &= -0.1080729656128444\,\mu\mathrm{N} \\
F_z &= -1.407372060313650\,\mu\mathrm{N} \\
F &= 1.415646724272760\,\mu\mathrm{N}
\end{aligned}
$$

The magnetic force obtained by (16) (Lorentz approach) is,

$$
\begin{aligned}
F_x &= -0.1080729656128444\,\mu\mathrm{N} \\
F_y &= -0.1080729656128444\,\mu\mathrm{N} \\
F_z &= -1.407372060313650\,\mu\mathrm{N} \\
F &= 1.415646724272760\,\mu\mathrm{N}
\end{aligned}
$$

Example 2.

The center of the primary coil of the radius $R_P = 0.4\,\mathrm{m}$ is O (0; 0; 0) and the center of the secondary coil of the radius $R_S = 0.05\,\mathrm{m}$ is C ($x_C = 0.1\,\mathrm{m}$; $y_C = 0.15\,\mathrm{m}$; $z_C = 0.0\,\mathrm{m}$). The secondary coil is located in the plane $3x + 2y + z = 0.6$. Calculate the magnetic force between coils. All currents are unit.

The magnetic force obtained by [1] is,

$$F_x = 4.171776672650815\,\mathrm{nN}$$
$$F_y = 6.523855691357912\,\mathrm{nN}$$
$$F_z = 27.71549975211961\,\mathrm{nN}$$
$$F = 28.77695849456434\,\mathrm{nN}$$

The magnetic force obtained by (16) (Lorentz approach) is,

$$F_x = 4.171776672650815\,\mathrm{nN}$$
$$F_y = 6.523855691357912\,\mathrm{nN}$$
$$F_z = 27.71549975211961\,\mathrm{nN}$$
$$F = 28.77695849456434\,\mathrm{nN}$$

Example 3.

The center of the primary coil of the radius $R_P = 0.9\,\mathrm{m}$ is O (0; 0; 0) and the center of the secondary coil of the radius $R_S = 0.6\,\mathrm{m}$ is C ($x_C = 0.3\,\mathrm{m}$; $y_C = 0.2\,\mathrm{m}$; $z_C = 0.5\,\mathrm{m}$). The secondary coil is located in the plane $x + y + z = 1$. Calculate the magnetic force between coils. All currents are unit.

The magnetic force obtained by [1] is,

$$F_x = 0.5228604018646984\,\mu\mathrm{N}$$
$$F_y = 0.4983356050923922\,\mu\mathrm{N}$$
$$F_z = -0.6364927281992902\,\mu\mathrm{N}$$
$$F = 0.9627275669635154\,\mu\mathrm{N}$$

The magnetic force obtained by (16) (Lorentz approach) is,

$$F_x = 0.5228604018646984\,\mu\mathrm{N}$$
$$F_y = 0.4983356050923922\,\mu\mathrm{N}$$
$$F_z = -0.6364927281992902\,\mu\mathrm{N}$$
$$F = 0.9627275669635154\,\mu\mathrm{N}$$

Example 4.

The center of the primary coil of the radius $R_P = 0.005\,\mathrm{m}$ is O (0; 0; 0) and the center of the secondary coil of the radius $R_S = 0.001\,\mathrm{m}$ is C ($x_C = 0.003\,\mathrm{m}$; $y_C = 0.001\,\mathrm{m}$; $z_C = 0.0005\,\mathrm{m}$). The secondary coil is located in the plane $3x + y + 2z = 0.011$. Calculate the magnetic force between coils. All currents are unit.

The magnetic force obtained by [1] is,

$$F_x = 0.1370009982312461\,\mu\mathrm{N}$$
$$F_y = 0.04566699941041536\,\mu\mathrm{N}$$
$$F_y = 0.09856738399856347\,\mu\mathrm{N}$$
$$F = 0.1748435802076503\,\mu\mathrm{N}$$

The magnetic force obtained by (16) (Lorentz approach) is,

$$F_x = 0.1370009982312461\,\mu\text{N}$$
$$F_y = 0.04566699941041536\,\mu\text{N}$$
$$F_z = 0.09856738399856347\,\mu\text{N}$$
$$F = 0.1748435802076503\,\mu\text{N}$$

Example 5.

The center of the primary coil of the radius $R_P = 0.3\,\text{m}$ is O (0; 0; 0) and the center of the secondary coil of the radius $R_S = 0.3\,\text{m}$ is C ($x_C = 0.1\,\text{m}$; $y_C = -0.3\,\text{m}$; $z_C = 0.2\,\text{m}$). The secondary coil is located in the plane $x - 2y + z = 0.9$. Calculate the magnetic force between coils. All currents are unit but of the opposite sign.

The magnetic force obtained by [1] is,

$$F_x = 0.2292455704933025\,\mu\text{N}$$
$$F_y = -0.5621415690326643\,\mu\text{N}$$
$$F_z = -0.09249247340323912\,\mu\text{N}$$
$$F = 0.6140940749279009\,\mu\text{N}$$

The magnetic force obtained by (16) (Lorentz approach) is,

$$F_x = 0.2292455704933025\,\mu\text{N}$$
$$F_y = -0.5621415690326643\,\mu\text{N}$$
$$F_z = -0.09249247340323912\,\mu\text{N}$$
$$F = 0.6140940749279009\,\mu\text{N}$$

Example 6.

In the following examples we verify the validity of each special case obtained by two different approaches.

The center of the primary coil of the radius $R_P = 1\,\text{m}$ is O (0; 0; 0) and the center of the secondary coil of the radius $R_S = 0.5\,\text{m}$ is C ($x_C = 2\,\text{m}$; $y_C = 2\,\text{m}$; $z_C = 2\,\text{m}$). The secondary coil is located in the plane $z = 2$. Coils are with parallel axes. Calculate the magnetic force between coils. All currents are unit.

The magnetic force obtained by [1] is,

$$F_x = -2.745371984357346\,\text{nN}$$
$$F_y = -2.745371984357346\,\text{nN}$$
$$F_z = 3.509473102444032\,\text{nN}$$
$$F = 5.233596862748077\,\text{nN}$$

The magnetic force obtained by (17) (Lorentz approach) is,

$$F_x = -2.745371984357346\,\text{nN}$$
$$F_y = -2.745371984357346\,\text{nN}$$
$$F_z = 3.509473102444032\,\text{nN}$$
$$F = 5.233596862748077\,\text{nN}$$

The special case ($a = b = 0$, $c = 1$) is directly included in the general case (16).

Example 7.

The center of the primary coil of the radius $R_P = 1\,\text{m}$ is O (0; 0; 0) and the center of the secondary coil of the radius $R_S = 0.5\,\text{m}$ is C ($x_C = 1\,\text{m}$; $y_C = 2\,\text{m}$; $z_C = 3\,\text{m}$). The secondary coil is located in the plane $x = 1$. Coils are with perpendicular axes. Calculate the magnetic force between coils. All currents are unit.

The magnetic force obtained by [1] is,

$$F_x = 1.939241379554508\,\text{nN}$$
$$F_y = -1.861181718234281\,\text{nN}$$
$$F_z = -2.202382194552672\,\text{nN}$$
$$F = 3.474930480937514\,\text{nN}$$

The magnetic force obtained by (19) (Lorentz approach) is,

$$F_x = 1.939241379554508\,\text{nN}$$
$$F_y = -1.861181718234281\,\text{nN}$$
$$F_z = -2.202382194552672\,\text{nN}$$
$$F = 3.474930480937514\,\text{nN}$$

The special case ($b = c = 0$, $a = 1$) is directly included in the general case (16).

Example 8.

The center of the primary coil of the radius $R_P = 1\,\text{m}$ is O (0; 0; 0) and the center of the secondary coil of the radius $R_S = 0.5\text{m}$ is C ($x_C = 2\,\text{m}$; $y_C = 2\,\text{m}$; $z_C = 2\,\text{m}$). The secondary coil is located in the plane $y = 2$. Coils are with perpendicular axes. Calculate the magnetic force between coils. All currents are unit.

Applying directly the special case (10(a)) or (15(a)), ($b = 1$; $a = c = 0$) in [1], the components of the magnetic force are:

$$F_x = -4.901398177052345\,\text{nN}$$
$$F_y = -1.984872313200137\,\text{nN}$$
$$F_z = -2.582265710169336\,\text{nN}$$
$$F = 5.884855001411407\,\text{nN}$$

The magnetic force obtained by (20) (Lorentz approach) is,

$$F_x = -4.901398177052345\,\mathrm{nN}$$
$$F_y = -1.984872313200137\,\mathrm{nN}$$
$$F_z = -2.582265710169336\,\mathrm{nN}$$
$$F = 5.884855001411407\,\mathrm{nN}$$

Thus we confirmed the validity of this singular case (20).

Example 9.

In this example we calculated the restoring (radial) magnetic force F_r and the propulsive (axial) magnetic force F_a between the primary circular coil $R_P = 42.5\,\mathrm{mm}$ and the secondary circular coil $R_S = 20\,\mathrm{mm}$ with the axial displacement $d = 3\,\mathrm{mm}$. Filamentary circular coils are with parallel axes and the distance between planes in which they are positioned is $c = 10\,\mathrm{mm}$ [15]. All currents are equal to 1 A.

In [15] we calculated the restoring and the axial magnetic forces for circular loops with parallel axes positioned in different planes.

For previous data we obtained,

$$F_r = 0.3281745285065932 \times 10^{-7}\,\mathrm{N}$$
$$F_a = -3.996817851575968 \times 10^{-7}\,\mathrm{N}$$

Applying the method here presented, the center of the primary loop $R_P = 42.5\,\mathrm{mm}$ is O (0; 0; 0) positioned in the plane $z = 0$ and the center of the secondary loop of the radius $R_S = 20\,\mathrm{mm}$ is C ($x_C = 0.003\,\mathrm{m}$; $y_C = 0\,\mathrm{m}$; $z_C = 0.01\,\mathrm{m}$) positioned in the plane $z = 10\,\mathrm{mm}$ whose equation is defined by parameters $a = 0$, $b = 0$ and $c = 1$. By the presented work the magnetic force components are,

$$F_x = 0.3281745285065932 \times 10^{-7}\,\mathrm{N}$$
$$F_y = 0\,\mathrm{N}$$
$$F_z = -3.996817851575967 \times 10^{-7}\,\mathrm{N}$$

All results are in an excellent agreement as expected and the restoring and axial forces (F_r and F_a) [15] correspond to the components F_x and F_z from this work respectively.

From previous examples we can conclude that all results obtained by this work for calculating the magnetic force between circular loops positioned in any desired position are in an excellent agreement with results given in [1] that confirm the validity of all methods. If we compare two methods presented in [1] and this one presented in this paper obviously that the presented work here is obtained in the simplest form. All calculations were executed using Mathematica programming. A Mathematica implementation of previous formulas is available from the authors on the request.

5. CONCLUSION

In this paper, we give a new formula for calculating the magnetic force between inclined circular loops placed in any desired positions. This formula is derived from the Lorentz force equation. We confirmed already obtained formulas of the magnetic force between inclined circular loops derived by the Biot-Savart law and by the approach of the mutual inductance. In order to use the new formula, whose final expressions are given per (16) or per (17), (18), (19) and (20), one needs to provide the radius of the primary and secondary coils, the position of the center of the secondary coil (the primary coil is assumed to be centered at origin), and the plane equation with all unit vectors in which the secondary coil is located. With these parameters, the problem is completely defined. All possible cases were tested with this formula, and none of them failed. We note that our new formula is general, very suitable and easily applicable for engineers and physicists. In this formula, the kernels are relatively simple and expressed by elliptic integrals of the first and second kind so that their integration using Mathematica or Matlab programming is accurate with significant reduced computational time.

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